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# Vorticity alignment with Lyapunov vectors and rate-of-strain eigenvectors



# A. Encinas-Bartos, G. Haller

Institute for Mechanical Systems, ETH Zurich, Zurich, Switzerland

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# ABSTRACT

We derive asymptotic estimates for the projection of the vorticity onto principal directions of material stretching in 3D flows. In flows with pointwise bounded vorticity, these estimates predict vorticity alignment with Lyapunov vectors along trajectories with positive Lyapunov exponents. Specifically, we find that in inviscid flows with conservative body forces, the vorticity exactly aligns with the intersection of the planes orthogonal to the dominant forward and backward Lyapunov vectors along trajectories with positive Lyapunov exponent. Furthermore, we derive asymptotic estimates for the vorticity alignment with the intermediate eigenvector of the rate-of-strain tensor for viscous flows under general forcing. We illustrate these results on explicit solutions of Euler's equation and on direct numerical simulations of homogeneous isotropic turbulence.

## 1. Introduction

Investigating the distribution of the vorticity in three dimensional (3D) flows is crucial for uncovering the intricate and chaotic dynamics driving the formation of the complex structures in turbulence. Specifically, understanding the stretching and tilting of the vorticity along a Lagrangian particle trajectory sheds light on the underlying mechanisms driving the formation and evolution of these structures.

The geometry of vortex lines is closely linked to the global existence of solutions in 3D Navier stokes flows [1–4] and to the vortex stretching mechanism [5–8]. Pioneering experiments from Taylor [9] showed that vortex stretching is responsible for the vorticity production and dissipation in isotropic turbulence. In the absence of external forcing, the dynamics of the vorticity  $\boldsymbol{\omega}(\boldsymbol{x},t) \in \mathbb{R}^3$  at a point  $\boldsymbol{x} \in \mathbb{R}^3$  and time  $t \in \mathbb{R}$  depend on both the rate-of-strain tensor,  $\mathbf{S}(\boldsymbol{x},t) = \frac{1}{2}(\nabla \boldsymbol{\nu}(\boldsymbol{x},t)^T + \nabla \boldsymbol{\nu}(\boldsymbol{x},t))$  of the velocity field  $\boldsymbol{\nu}(\boldsymbol{x},t)$  and on the viscous term  $\boldsymbol{\nu} \Delta \boldsymbol{\omega}(\boldsymbol{x},t)$ , as seen from the vorticity transport equation

$$\frac{D}{Dt}\boldsymbol{\omega}(\mathbf{x},t) = \mathbf{S}(\mathbf{x},t)\boldsymbol{\omega}(\mathbf{x},t) + v\boldsymbol{\Delta}\boldsymbol{\omega}(\mathbf{x},t).$$
(1)

This equation shows that in Euler flows, i.e. inviscid flows with v = 0, vortex lines evolve as material lines. Similarly, in high-Reynolds number flows, the non-local viscous contribution in the Navier–Stokes equation is negligible over short time scales and the vorticity evolves approximately as a material vector [9]. The instantaneous stretching of the vorticity vector  $\omega$  is therefore closely related to the eigenvectors and eigenvalues of the rate-of-strain tensor **S**. In inviscid flows, the long-term stretching and tilting of the vorticity vector along a Lagrangian particle trajectory is most naturally described in a Lagrangian setting

through the eigenvalues and eigenvectors of the Cauchy–Green strain tensor [10].

Over the last two decades, extensive research has been dedicated to investigate statistical correlations between the vorticity and the rateof-strain tensor. Experimental [11,12] and numerical [13-16] studies in turbulence have suggested that the vorticity tends to align preferentially with the intermediate eigenvector of S(x, t), whose corresponding eigenvalue tends to be positive. Vortex lines in regions of high vorticity tend to be straight and well-aligned, thereby displaying highly organized structures [17]. Regions of high vorticity frequently arise in connection with a preferential alignment with the intermediate eigenvector of the rate-of-strain tensor [13,15,18]. Other studies by Kerr [19] and Jiménez [15] proposed that, in regions of approximately 2D velocity, the vorticity aligns with the intermediate rate-of-strain eigenvector. In that 2D case, the vorticity is neither stretched nor tilted. Rather, the alignment occurs because the vorticity and the intermediate eigenvector of S(x, t) are approximately orthogonal to the plane of the flow in such regions.

In addition to the studies that focus on the instantaneous alignment between the vorticity and the eigenvectors of the rate-of-strain tensor, Xu et al. [20] and Pumir et al. [21] started incorporating dynamical information into the analysis. They considered the evolution of the vorticity along a trajectory and compared the alignment between the vorticity at time  $t + \Delta t$  with the most stretching direction specified by the strain rate at time t. They found that over short times, the vorticity evolving along a trajectory aligns with the most stretching direction of  $\mathbf{S}(\mathbf{x}, t)$ . However, for  $\Delta t > 2\tau_{\eta}$ , where  $\tau_{\eta}$  is the Kolmogorov time scale, the observed alignment breaks down. This is in agreement

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<sup>\*</sup> Corresponding author. E-mail address: georgehaller@ethz.ch (G. Haller).

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with the theory of Objective Eulerian Coherent Structures (OECS) developed by Serra and Haller [22]: over short times, material elements align with the dominant eigenvector of S(x, t). This alignment persists along a trajectory as long as the eigenvectors of the rate-of-strain tensor vary slowly along that trajectory [23,24]. This persistence of the rate-of-strain hypothesis was first proposed by Batchelor [25,26] as a convenient assumption that allows the study of deformation of material elements under a fixed strain rate. Numerical and experimental Lagrangian data was not readily available at that time and hence this assumption could not be verified until the 1980s. Numerical simulations performed by Girimaji and Pope [27,28] finally showed, however, that the persistence of strain assumption remains valid only over times of the order of  $2\tau_n$ .

As an alternative, Hamlington et al. [29,30] proposed to split the rate-of-strain tensor into a local and non-local component  $S = S^L + S^B$ . Here  $S^L$  is the local rate-of-strain induced by a vortical structure in a spherical neighborhood, and  $S^B$  is the non-local background rate-of-strain induced in the vicinity of the structure by all the remaining vorticity. This decomposition is non-unique since it depends on the radius of the local neighborhood. Furthermore, the non-local rate-of-strain  $S^B$  is not easily computable in practice since it relies on higher-order spatial derivatives of the velocity field. Hamlington et al. [29,30] showed through numerical simulations that the vorticity preferentially aligns both with the most stretching eigenvector of the non-local (background) rate-of-strain  $S^B$  and with the intermediate eigenvector of the regular rate-of-strain S.

The evolution of the vorticity can also be studied in a Lagrangian setting, i.e., along a trajectory. Ni et al. [10] compared numerically the evolution of the vorticity with that of thin rods (which approximate material fibers) dispersed in a fluid. Both the vorticity and the rods were found to align with the most stretching direction of the Cauchy– Green strain tensor under forward advection. The alignment of the vorticity was argued to be weaker compared to that of the thin rods, since the viscous term was expected to cause its dynamics to divert from purely material evolution. We will further discuss the validity of these findings in view of the results of this paper (see the discussion after Remark 1).

Finally in further related work, Tsinober [31], Guala et al. [32] and Lüthi et al. [33] studied the evolution of the vorticity along a Lagrangian particle trajectory with respect to the rate-of-strain eigenvectors for non-zero viscosity. They showed numerically that over longer time intervals, the viscous term enhances the preferential alignment of the vorticity with the intermediate eigenvector of the rate-of-strain tensor.

A statistical description of the Navier–Stokes equation allows modeling the evolution of the vorticity dynamics using the Fokker–Planck equation [34–36]. By decoupling the evolution of the vorticity vector from the dynamics of the rate-of-strain tensor and hence treating the vorticity as a passive vector, Johnson and Meneveau [37] established a probabilistic model that captures the temporal correlations of rate-of-strain tensor and vorticity along a Lagrangian fluid trajectory. Additionally, Carbone and Wilczek [38] derived a model for the vorticity dynamics in low-Reynolds number flows that takes into account the first-order contribution of the pressure in the velocity gradient dynamics. The model is asymptotically exact at small Reynolds numbers and correctly predicts the preferential alignment between the vorticity and the intermediate eigenvector of the rate-of-strain tensor up to intermediate Reynolds numbers. For high Reynolds number, however, the modeling assumptions break down.

Here we derive precise estimates on the vorticity evolution in a Lagrangian frame that imply various alignment results for incompressible flows with bounded vorticity. First, for inviscid flows with potential body forces, we obtain an exact alignment between the vorticity and the normal of the plane of the forward and backward Lyapunov vectors (i.e. asymptotic limits of the forward and backward material stretching exponents). This normal coincides with the intermediate Lyapunov vector along trajectories whose forward and backward dynamics are identical. We similarly provide estimates for the alignment of the vorticity with principal stretching directions of the rate-of-strain tensor in general forced viscous flows. These estimates depend on the principal strains, the variation of the rate-of-strain tensor along a trajectory and the curl of the external (viscous and non-potential) forces. We illustrate all these results on laminar and turbulent flow data.

## 2. Lagrangian setup

Consider a 3D incompressible velocity field

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t), \quad \nabla \cdot \mathbf{v} = 0, \tag{2}$$

on the spatial domain  $\mathbf{x} \in U \subset \mathbb{R}^3$ . A trajectory of this velocity field that starts from the position  $\mathbf{x}_0$  at time  $t_0$  is denoted by  $\mathbf{x}(t; t_0, \mathbf{x}_0)$ . The flow map  $\mathbf{F}_{t_0}^t : \mathbf{x}_0 \mapsto \mathbf{x}(t; t_0, \mathbf{x}_0)$  is then defined as the mapping from initial positions into current positions along trajectories. The equation of variations (or linearized flow) along  $\mathbf{x}(t; t_0, \mathbf{x}_0)$  is a homogeneous system of non-autonomous linear ordinary differential equations (ODEs) of the form

$$\dot{\boldsymbol{\xi}}(t) = \left[ \boldsymbol{\nabla} \boldsymbol{\nu} \left( \boldsymbol{x}(t; t_0, \boldsymbol{x}_0), t \right) \right] \boldsymbol{\xi}(t), \tag{3}$$

where  $\nabla v \left( \mathbf{x}(t; t_0, \mathbf{x}_0), t \right)$  indicates the gradient of the velocity field and  $\xi(t) := \xi(t; t_0, \mathbf{x}_0)$ . This system of ODEs describes the material evolution of a perturbation  $\xi(t)$  along  $\mathbf{x}(t; t_0, \mathbf{x}_0)$ , starting from an infinitesimally small perturbation  $\xi_0$  to the initial condition  $\xi_0$ . The general solution of this system is

$$\boldsymbol{\xi}(t) = \boldsymbol{\nabla} \mathbf{F}_{t_0}^t \left( \boldsymbol{x}_0 \right) \boldsymbol{\xi}_0. \tag{4}$$

Taking the inner product of (3) with  $\xi(t)$  and integrating in time gives that the length of  $\xi(t)$  evolves as

$$\left|\boldsymbol{\xi}(t)\right|^{2} = \left\langle \boldsymbol{\xi}_{0}, \mathbf{C}_{t_{0}}^{t}\left(\boldsymbol{x}_{0}\right) \boldsymbol{\xi}_{0} \right\rangle, \qquad \mathbf{C}_{t_{0}}^{t}\left(\boldsymbol{x}_{0}\right) = \left[\boldsymbol{\nabla} \mathbf{F}_{t_{0}}^{t}\left(\boldsymbol{x}_{0}\right)\right]^{1} \boldsymbol{\nabla} \mathbf{F}_{t_{0}}^{t}\left(\boldsymbol{x}_{0}\right),$$

with  $C_{t_0}^t(\mathbf{x}_0)$  denoting the right Cauchy–Green strain tensor. This symmetric, positive definite tensor has three real, positive eigenvalues that can be ordered as

$$0 < \lambda_1(t, t_0, \mathbf{x}_0) \le \lambda_2(t, t_0, \mathbf{x}_0) \le \lambda_3(t, t_0, \mathbf{x}_0),$$
  

$$\lambda_1(t; t_0, \mathbf{x}_0) \lambda_2(t; t_0, \mathbf{x}_0) \lambda_3(t; t_0, \mathbf{x}_0) = 1,$$
(5)

where the second relation follows from the incompressibility of  $v(\mathbf{x}, t)$ . In the turbulence literature, the eigenvalues and eigenvectors are frequently written in descending order, whereas in the literature on Lagrangian Coherent Structures (LCSs) they are listed in ascending order. In this work, we follow the LCS literature (see, e.g., [39], for a recent review) to remain consistent with the cited references. The corresponding unit eigenvectors of  $\mathbf{C}_{t_0}^{t}(\mathbf{x}_0)$  satisfy

$$\mathbf{C}_{t_0}^t \left( \mathbf{x}_0 \right) \boldsymbol{\xi}_i(t; t_0, \mathbf{x}_0) = \lambda_i(t; t_0, \mathbf{x}_0) \boldsymbol{\xi}_i(t; t_0, \mathbf{x}_0), \tag{6}$$

$$\left\langle \xi_{i},\xi_{i}\right\rangle =\delta_{ii},\quad i,j=1,2,3. \tag{7}$$

The vectors  $\boldsymbol{\xi}_{j}(t;t_{0},\boldsymbol{x}_{0})$  are also the right singular vectors of  $\boldsymbol{\nabla}\mathbf{F}_{t_{0}}^{t}(\boldsymbol{x}_{0})$ . A useful identity is

$$\nabla \mathbf{F}_{t_0}^t \left( \mathbf{x}_0 \right) \boldsymbol{\xi}_j(t; t_0, \mathbf{x}_0) = \sqrt{\lambda_j(t; t_0, \mathbf{x}_0)} \boldsymbol{\eta}_j(t; t_0, \mathbf{x}_0), \quad j = 1, 2, 3,$$
(8)

where the orthonormal vectors  $\boldsymbol{\eta}_j(t; t_0, \boldsymbol{x}_0)$  are the left singular vectors of  $\boldsymbol{\nabla} \mathbf{F}_{t_0}^t(\boldsymbol{x}_0)$  corresponding to the singular values  $\sqrt{\lambda_j(t; t_0, \boldsymbol{x}_0)}$ , respectively (see, e.g., Haller [39]). Based on formula (8), the finite time Lyapunov exponent (FTLE) is often used to identify an overall growth exponent in the temporal evolution of  $\lambda_3(t, t_0, \boldsymbol{x}_0)$  over a time interval  $[t_0, t_1]$  via the formula

$$\text{FTLE}_{t_0}^{t_1}(\mathbf{x}_0) = \frac{1}{|t_1 - t_0|} \log \sqrt{\lambda_3(t_1; t_0, \mathbf{x}_0)}.$$
(9)

A well-defined asymptotic growth exponent for  $\sqrt{\lambda_3(t_1; t_0, \mathbf{x}_0)}$  may continue to exist as  $t_1 \to \pm \infty$ . In such cases, the maximal forward- and

backward Lyapunov exponents  $\mu_3^{\pm}(t_0, \mathbf{x}_0)$  of the trajectory  $\mathbf{x}(t; t_0, \mathbf{x}_0)$  can be defined as the limiting singular values

$$\mu_{3}^{\pm} \left( \mathbf{x}_{0}, t_{0} \right) = \lim_{t \to \pm \infty} \frac{1}{|t - t_{0}|} \log \left( \frac{\left| \nabla \mathbf{F}_{t_{0}}^{t} \left( \mathbf{x}_{0} \right) \boldsymbol{\xi}_{3}(t; t_{0}, \mathbf{x}_{0}) \right|}{|\boldsymbol{\xi}_{3}(t; t_{0}, \mathbf{x}_{0})|} \right)$$
$$= \lim_{t \to \pm \infty} \frac{1}{|t - t_{0}|} \log \sqrt{\lambda_{3}(t, t_{0}, \mathbf{x}_{0})}. \tag{10}$$

Similarly, the dominant forward and backward Lyapunov vectors can be defined as the limiting dominant singular vectors:

$$\boldsymbol{\zeta}_{3}^{\pm}(\boldsymbol{x}_{0}, t_{0}) = \lim_{t \to +\infty} \boldsymbol{\xi}_{3}(t; t_{0}, \boldsymbol{x}_{0}).$$
(11)

The existence of the limits (10) and (11) cannot be *a priori* guaranteed for trajectories of a general velocity field (2). In unsteady flows, continuous Lyapunov spectra take over the role of discrete Lyapunov exponents [40]. Flows restricted to compact regions are generally expected to have bounded Lyapunov spectra for most initial conditions but the conditions guaranteeing this are technically involved and lie beyond the scope of this paper.

If, however, the incompressible velocity field  $v(x, t) \equiv v(x)$  is steady and the domain *U* is compact and invariant under the flow, then the limits (10) and (11) (along with further Lyapunov vectors and exponents corresponding to the remaining singular values of  $\nabla \mathbf{F}_{t_0}^t(x_0)$ ) exist for almost all  $x_0$  by the multiplicative ergodic theorem of Oseledets [41]. This theorem is originally stated for maps, which in turn implies the existence of these limits for time periodic flows as well via the Poincaré maps associated with those flows. We note that Eqs. (10) and (11) refer to the theoretical Lyapunov exponents and vectors defined for infinitesimally small initial perturbations to the underlying trajectories [41,42]. Alternative definitions for Lyapunov vectors are also available and generally give different results (see, e.g., Wolfe and Samelson [43]).

#### 3. Eulerian setup

We assume now that in an inertial frame, the velocity field v(x, t) in Eq. (2) is a solution of the 3D, incompressible Navier–Stokes equation

$$\frac{D\boldsymbol{v}(\boldsymbol{x},t)}{Dt} = -\frac{1}{\rho}\nabla p(\boldsymbol{x},t) + v\Delta \boldsymbol{v}(\boldsymbol{x},t) + \boldsymbol{f}(\boldsymbol{x},t), \qquad (12)$$

with the constant density  $\rho$ , the pressure field  $p(\mathbf{x}, t)$  and the body force vector  $f(\mathbf{x}, t)$ . Taking the curl of this equation gives the incompressible vorticity transport equation

$$\frac{D\omega}{Dt} = [\nabla v] \omega + v\Delta\omega + \nabla \times f, \qquad (13)$$

for the vorticity  $\omega = \nabla \times v$ . We note that the last term on the right-hand side of Eq. (13) vanishes if all body forces acting on the fluid are potential forces.

For the remainder of this work, we assume that the velocity field remains regular, i.e., the vorticity does not blow up in finite time. We are aware that there are degenerate initial conditions for which the 3D inviscid ( $\nu = 0$ ) vorticity transport equation (13) may yield unbounded vorticity growth [44–46]. We exclude, however, such degenerate cases from our discussion and emphasize that in experimentally observable flows a finite time blow up of the vorticity is not possible.

Analogously to the maximal material stretching exponent quantified by the  $\text{FTLE}_{t_0}^{t_1}(\mathbf{x}_0)$  field defined in (9), we can define the vorticity stretching exponent (VSE) field [37] as

$$\text{VSE}_{t_0}^{t_1}\left(\mathbf{x}_0\right) = \frac{1}{\left|t_1 - t_0\right|} \log \frac{\left|\boldsymbol{\omega}\left(\mathbf{x}(t_1; t_0, \mathbf{x}_0), t_1\right)\right|}{\left|\boldsymbol{\omega}\left(\mathbf{x}_0, t_0\right)\right|},\tag{14}$$

which we will use in our later formulas and numerical experiments to characterize the long–term evolution of the vorticity magnitude. Vorticity alignment is often studied with respect to the eigenvectors  $e_i(\mathbf{x}, t)$  of the rate–of–strain tensor

$$\mathbf{S}(\mathbf{x},t) = \frac{1}{2} \left[ \left[ \nabla \boldsymbol{v} \left( \mathbf{x}, t \right) \right]^{\mathrm{T}} + \nabla \boldsymbol{v} \left( \mathbf{x}, t \right) \right].$$

This symmetric tensor is objective, i.e., it transforms properly as a linear operator under any Euclidean frame change of the form  $\mathbf{x} = \mathbf{Q}(t)\mathbf{y} + \mathbf{b}(t)$ , where  $\mathbf{Q}(t) \in SO(3)$  is an arbitrary time–dependent rotation tensor and  $\mathbf{b}(t) \in \mathbb{R}^3$  is an arbitrary translation vector. The eigenvectors  $\mathbf{e}_j(\mathbf{x}, t)$  of  $\mathbf{S}(\mathbf{x}, t)$  are also objective, because the strain eigenvectors  $\tilde{\mathbf{e}}_j(\mathbf{y}, t)$  in the  $\mathbf{y}$ -frame come out to be

$$\tilde{\boldsymbol{e}}_{j}(\boldsymbol{y},t) = \boldsymbol{Q}^{\mathrm{T}}(t)\boldsymbol{e}_{j}(\boldsymbol{x},t).$$
(15)

In contrast, the vorticity  $\tilde{\omega}$  of the transformed velocity field obeys the formula

$$\tilde{\boldsymbol{\omega}}(\boldsymbol{y},t) = \boldsymbol{Q}^{\mathrm{T}}(t) \left[ \boldsymbol{\omega}(\boldsymbol{x},t) - \dot{\boldsymbol{q}}(t) \right], \tag{16}$$

where  $\dot{q}(t)$  is the vorticity associated with the frame change (see Haller [39]). This shows that unlike  $e_j(x, t)$ , the vorticity field  $\omega(x, t)$  is not objective. A comparison of formulas (15) and (16) implies that any possible alignment (or lack thereof) observed between the vorticity and the rate–of–strain eigenvectors is inherently frame–dependent due to the non–preservation of the inner product under a general frame change:

$$\left\langle \tilde{\boldsymbol{\omega}}(\mathbf{y},t), \tilde{\mathbf{e}}_{j}(\mathbf{y},t) \right\rangle = \left\langle \boldsymbol{\omega}(\mathbf{x},t), \boldsymbol{e}_{j}(\mathbf{x},t) \right\rangle + \left\langle \dot{\boldsymbol{q}}(t), \boldsymbol{e}_{j}(\mathbf{x},t) \right\rangle$$

This is also true for our upcoming alignment results, which will specifically be valid in inertial frames. In such frames, the fundamental form of Eq. (12) remains unchanged.

#### 4. Vorticity alignment for Euler flows in the Lagrangian frame

Under potential body forces, the inviscid limit of Eq. (13), restricted to a trajectory  $\mathbf{x}(t; t_0, \mathbf{x}_0)$  of the velocity field  $\mathbf{v}(\mathbf{x}, t)$ , is a non-autonomous, homogeneous linear system of ODEs of the form

$$\frac{D\omega}{Dt} = \left[ \nabla \boldsymbol{\nu} \left( \boldsymbol{x}(t;t_0,\boldsymbol{x}_0), t \right) \right] \boldsymbol{\omega}.$$
(17)

This linear system formally coincides with the equation of variations, and hence by Eq. (4), its solution can be written as

$$\boldsymbol{\omega}\left(\boldsymbol{x}(t;t_{0},\boldsymbol{x}_{0}),t\right) = \boldsymbol{\nabla} \mathbf{F}_{t_{0}}^{t}\left(\boldsymbol{x}_{0}\right)\boldsymbol{\omega}_{0}\left(\boldsymbol{x}_{0}\right).$$
(18)

Using the shorthand notation

$$\boldsymbol{\omega}_t(\boldsymbol{x}_0) := \boldsymbol{\omega} \left( \boldsymbol{x}(t; t_0, \boldsymbol{x}_0), t \right)$$

we square both sides of Eq. (18) to obtain

$$\left|\boldsymbol{\omega}_{t}(\boldsymbol{x}_{0})\right|^{2} = \left|\boldsymbol{\nabla}\mathbf{F}_{t_{0}}^{t}\left(\boldsymbol{x}_{0}\right)\boldsymbol{\omega}_{t_{0}}\left(\boldsymbol{x}_{0}\right)\right|^{2},\tag{19}$$

whose right-hand side can be upper estimated as

$$\left| \nabla \mathbf{F}_{t_0}^t \left( \mathbf{x}_0 \right) \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right) \right|^2 = \left\langle \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right), \mathbf{C}_{t_0}^t \left( \mathbf{x}_0 \right) \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right) \right\rangle$$
(20)

$$\leq \lambda_3(t; t_0, \mathbf{x}_0) \left| \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right) \right| \quad . \tag{21}$$

At the same time, using the identity (8), we obtain the lower estimate

$$\left| \nabla \mathbf{F}_{t_0}^t \left( \mathbf{x}_0 \right) \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right) \right|^2 = \left| \nabla \mathbf{F}_{t_0}^t \left( \mathbf{x}_0 \right) \sum_{j=1}^3 \left\langle \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right), \boldsymbol{\xi}_j(t; t_0, \mathbf{x}_0) \right\rangle \right.$$
$$\left. \cdot \boldsymbol{\xi}_j(t; t_0, \mathbf{x}_0) \right|^2$$
$$= \left| \sum_{j=1}^3 \left\langle \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right), \boldsymbol{\xi}_j(t; t_0, \mathbf{x}_0) \right\rangle \right.$$
$$\left. \cdot \sqrt{\lambda_j(t; t_0, \mathbf{x}_0)} \boldsymbol{\eta}_j(t; t_0, \mathbf{x}_0) \right|^2$$
$$= \sum_{j=1}^3 \left\langle \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right), \boldsymbol{\xi}_j(t; t_0, \mathbf{x}_0) \right\rangle^2 \lambda_j(t; t_0, \mathbf{x}_0)$$
$$\geq \left\langle \boldsymbol{\omega}_{t_0} \left( \mathbf{x}_0 \right), \boldsymbol{\xi}_j(t; t_0, \mathbf{x}_0) \right\rangle^2 \lambda_j(t; t_0, \mathbf{x}_0), \quad (22)$$
$$j = 1, 2, 3.$$

Combining the expressions (19) and (22) with j = 3, we obtain the estimate

$$\left\langle \boldsymbol{\omega}_{t_0} \left( \boldsymbol{x}_0 \right), \boldsymbol{\xi}_3(t; t_0, \boldsymbol{x}_0) \right\rangle^2 \lambda_3(t; t_0, \boldsymbol{x}_0) \le \left| \boldsymbol{\omega}_t(\boldsymbol{x}_0) \right|^2,$$
(23)

or, equivalently,

$$\left\langle \boldsymbol{\omega}_{t_0} \left( \boldsymbol{x}_0 \right), \boldsymbol{\xi}_3(t; t_0, \boldsymbol{x}_0) \right\rangle^2 \le \frac{\left| \boldsymbol{\omega}_t(\boldsymbol{x}_0) \right|^2}{\lambda_3(t; t_0, \boldsymbol{x}_0)}.$$
(24)

Using the unit vector

 $\mathbf{e}_{\boldsymbol{\omega}_{0}}\left(\boldsymbol{x}_{0}\right) = \frac{\boldsymbol{\omega}_{t_{0}}\left(\boldsymbol{x}_{0}\right)}{\left|\boldsymbol{\omega}_{t_{0}}\left(\boldsymbol{x}_{0}\right)\right|},$ 

along with Eq. (24), we obtain the upper estimate

$$\left\langle \mathbf{e}_{\boldsymbol{\omega}_{0}}\left(\mathbf{x}_{0}\right), \boldsymbol{\xi}_{3}(t;t_{0},\mathbf{x}_{0})\right\rangle^{2} \leq \frac{1}{\lambda_{3}(t;t_{0},\mathbf{x}_{0})} \frac{\left|\boldsymbol{\omega}_{t}\left(\mathbf{x}_{0}\right)\right|^{2}}{\left|\boldsymbol{\omega}_{t_{0}}\left(\mathbf{x}_{0}\right)\right|^{2}},\tag{25}$$

which can be rewritten as

$$\left|\left\langle \mathbf{e}_{\boldsymbol{\omega}_{0}}\left(\mathbf{x}_{0}\right),\boldsymbol{\xi}_{3}(t;t_{0},\mathbf{x}_{0})\right\rangle\right| \leq B_{t_{0}}^{t}\left(\mathbf{x}_{0}\right) := e^{\left[\operatorname{VSE}_{t_{0}}^{t}\left(\mathbf{x}_{0}\right) - \operatorname{FTLE}_{t_{0}}^{t}\left(\mathbf{x}_{0}\right)\right](t-t_{0})}.$$
 (26)

This last inequality suggests that in incompressible Euler flows, material stretching drives the vorticity vector away from aligning with  $\xi_3$ , whereas vorticity stretching has the opposite effect.

Assume now that the largest forward and backward Lyapunov exponents  $\mu_3^{\pm}(t_0, \mathbf{x}_0)$  along the trajectory  $\mathbf{x}(t; t_0, \mathbf{x}_0)$  exist and are positive, i.e.,

$$0 < \mu_3^{\pm} \left( \mathbf{x}_0, t_0 \right) := \lim_{t \to \pm \infty} \frac{1}{2 \left( t - t_0 \right)} \log \lambda_3(t; t_0, \mathbf{x}_0) < \infty,$$
(27)

which implies that

$$\lim_{t \to \pm\infty} \text{FTLE}_{t_0}^t(\mathbf{x}_0) = \mu_3^{\pm}(\mathbf{x}_0), \tag{28}$$

$$\lim_{t \to \pm \infty} \xi_3(t; t_0, \mathbf{x}_0) = \zeta_3^{\pm}(\mathbf{x}_0, t_0).$$
<sup>(29)</sup>

For flows with pointwise uniformly bounded vorticity, the first term on the right-hand side of the inequality (26) vanishes in the limits  $t \to \pm \infty$ . Using the asymptotic estimates we then deduce the following alignment results for incompressible Euler flows defined on a compact invariant domain *U* subject to potential body forces.

**Theorem 1.** Consider an incompressible Euler velocity field  $v(\mathbf{x}, t)$ , defined on the invariant flow region U with pointwise uniformly bounded vorticity, subject only to potential forces. Assume that along a trajectory  $\mathbf{x}(t; t_0, \mathbf{x}_0)$ , the maximal Lyapunov exponents  $\mu_3^{\pm}(\mathbf{x}_0, t_0)$  are well defined and positive, and the corresponding dominant Lyapunov vectors  $\zeta_3^{\pm}(\mathbf{x}_0, t_0)$  satisfy the non-degeneracy condition

 $\boldsymbol{\zeta}_{3}^{+}(\boldsymbol{x}_{0},t_{0}) \not\parallel \boldsymbol{\zeta}_{3}^{-}(\boldsymbol{x}_{0},t_{0}).$ 

Define then

$$\boldsymbol{n}(\boldsymbol{x}_{0}, t_{0}) = \boldsymbol{\zeta}_{3}^{+}(\boldsymbol{x}_{0}, t_{0}) \times \boldsymbol{\zeta}_{3}^{-}(\boldsymbol{x}_{0}, t_{0})$$

as the vector along the intersection of the two planes orthogonal to  $\zeta_3^{\pm}(\mathbf{x}_0)$ . Then, the vorticity  $\boldsymbol{\omega}(\mathbf{x}_0, t_0)$  exactly aligns with  $n(\mathbf{x}_0, t_0)$ , i.e.,

$$\boldsymbol{\omega}\left(\boldsymbol{x}_{0},t_{0}\right) \parallel \boldsymbol{n}(\boldsymbol{x}_{0},t_{0}). \tag{30}$$

Theorem 1 states that in order for the vorticity to remain bounded along a trajectory with positive Lyapunov exponents, it must be orthogonal to the most stretching directions in both forward and backward time. This direction is well defined as long as the forward and backward most stretching directions are distinct. If, in addition, the flow domain is compact and the velocity field is steady or time periodic, then the Lyapunov vectors and exponents are guaranteed to exist for almost all initial conditions  $x_0$  according to the multiplicative ergodic theory of Oseledets [41]. We, therefore, obtain the following strengthened result in that case. **Corollary 1.** In a steady or time periodic, incompressible Euler velocity field  $v(\mathbf{x}, t)$ , defined on a compact invariant domain U with pointwise uniformly bounded vorticity subject only to potential body forces, we have, at almost all initial points  $\mathbf{x}_0$ , the exact alignment

 $\boldsymbol{\omega}(\boldsymbol{x}_0) \parallel \boldsymbol{n}(\boldsymbol{x}_0).$ 

**Remark 1.** We now discuss a special setting in which the vorticity  $\omega(\mathbf{x}_0, t_0)$  specifically aligns with the asymptotic limit of the intermediate left Cauchy–Green eigenvector,  $\xi_2(t; t_0, \mathbf{x}_0)$ . We first note that a total of three pairs of Lyapunov exponents and vectors can be defined via the limits

$$\mu_{j}^{\pm}(\mathbf{x}_{0}, t_{0}) = \lim_{t \to \pm \infty} \frac{1}{|t - t_{0}|} \log \sqrt{\lambda_{j}(t, t_{0}, \mathbf{x}_{0})},$$
(31)

$$\boldsymbol{\zeta}_{j}^{\pm}(\boldsymbol{x}_{0}, t_{0}) = \lim_{t \to \pm \infty} \boldsymbol{\xi}_{j}(t; t_{0}, \boldsymbol{x}_{0}), \quad j = 1, 2, 3.$$
(32)

If these limits exist for a given  $(\mathbf{x}_0, t_0)$  pair and the forward- and backward-asymptotic behaviors of a trajectory  $\mathbf{x}(t; t_0, \mathbf{x}_0)$  are the same (e.g.,  $\mathbf{x}(t; t_0, \mathbf{x}_0)$  is a periodic orbit, a quasiperiodic torus or a chaotic invariant set), then we have

$$\mu_{3}^{+}(\mathbf{x}_{0},t_{0}) = \frac{1}{\mu_{1}^{-}(\mathbf{x}_{0},t_{0})}, \quad \mu_{2}^{+}(\mathbf{x}_{0},t_{0}) = \frac{1}{\mu_{2}^{-}(\mathbf{x}_{0},t_{0})}, \\ \mu_{1}^{+}(\mathbf{x}_{0},t_{0}) = \frac{1}{\mu_{3}^{-}(\mathbf{x}_{0},t_{0})},$$
(33)

$$\begin{aligned} \boldsymbol{\zeta}_{3}^{+}(\mathbf{x}_{0},t_{0}) &= \boldsymbol{\zeta}_{1}^{-}(\mathbf{x}_{0},t_{0}), \quad \boldsymbol{\zeta}_{2}^{+}(\mathbf{x}_{0},t_{0}) = \boldsymbol{\zeta}_{2}^{-}(\mathbf{x}_{0},t_{0}), \\ \boldsymbol{\zeta}_{1}^{+}(\mathbf{x}_{0},t_{0}) &= \boldsymbol{\zeta}_{3}^{-}(\mathbf{x}_{0},t_{0}). \end{aligned}$$
(34)

In this case, we have

$$\begin{aligned} \boldsymbol{\zeta}_{3}^{+}(\boldsymbol{x}_{0}, t_{0}) \times \boldsymbol{\zeta}_{3}^{-}(\boldsymbol{x}_{0}, t_{0}) &= \boldsymbol{\zeta}_{3}^{+}(\boldsymbol{x}_{0}, t_{0}) \times \boldsymbol{\zeta}_{1}^{+}(\boldsymbol{x}_{0}, t_{0}) \\ &= \boldsymbol{\zeta}_{3}^{-}(\boldsymbol{x}_{0}, t_{0}) \times \boldsymbol{\zeta}_{1}^{-}(\boldsymbol{x}_{0}, t_{0}) \end{aligned} \tag{35}$$

As a consequence, if the forward- and backward-asymptotic behaviors along a trajectory are the same, then the vorticity vector  $\boldsymbol{\omega}_{t_0}(\boldsymbol{x}_0)$  aligns exactly with the single intermediate Lyapunov vector  $\boldsymbol{\zeta}_2^{\pm}(\boldsymbol{x}_0, t_0)$  in flows satisfying the conditions of Corollary 1.

Remark 2. We now contrast our conclusions with previous vorticity alignment results in the Lagrangian frame. Ni et al. [10] propose that the evolving vorticity  $\boldsymbol{\omega}(\boldsymbol{x}(t;t_0,\boldsymbol{x}_0),t)$  tends to align with the dominant left singular vector  $\eta_3(t; t_0, \mathbf{x}_0)$  of the deformation gradient  $\nabla \mathbf{F}_{t_0}^t(\mathbf{x}_0)$ . Their argument is based on the fact that a generic initial condition of the inviscid vorticity transport equation (17) will align with  $\nabla \mathbf{F}_{t_0}^t(\mathbf{x}_0) \boldsymbol{\xi}_3(t; t_0, \mathbf{x}_0)$ , given that the initial condition  $\boldsymbol{\xi}_3(t; t_0, \mathbf{x}_0)$ stretches the most over the time interval  $[t_0, t]$ . Ni et al. [10] attribute the lack of a perfect alignment between  $\boldsymbol{\omega}\left(\boldsymbol{x}(t;t_0,\boldsymbol{x}_0),t\right)$  and  $\boldsymbol{\eta}_3(t;t_0,\boldsymbol{x}_0)$ to viscous effects. Note, however, that such a perfect alignment cannot take place in inviscid flows (under potential body forces) along trajectories with pointwise uniformly bounded vorticity and positive maximal Lyapunov exponent. The reason is that such an alignment between  $\boldsymbol{\omega}(\mathbf{x}(t;t_0,\mathbf{x}_0),t)$  and  $\boldsymbol{\eta}_3(t;t_0,\mathbf{x}_0)$  would lead to unbounded vorticity growth. Indeed, by Eq. (22), the vorticity would grow unbounded in forward time unless  $\langle \boldsymbol{\omega}_{t_0}(\mathbf{x}_0), \boldsymbol{\xi}_3(t;t_0,\mathbf{x}_0) \rangle = 0$ , i.e., unless the vorticity is exactly normal to the dominant stretching direction at all points along a trajectory with positive Lyapunov exponents. This initial alignment then implies that, by the identity (8), the evolving vorticity and the left singular vector must, in fact, satisfy

$$\boldsymbol{\omega}\left(\boldsymbol{x}(t;t_0,\boldsymbol{x}_0),t\right) \perp \boldsymbol{\eta}_3(t;t_0,\boldsymbol{x}_0) \tag{36}$$

in an inviscid flow with bounded vorticity, as opposed to  $\omega(\mathbf{x}(t; t_0, \mathbf{x}_0), t) \parallel \boldsymbol{\eta}_3(t; t_0, \mathbf{x}_0).$ 

As most trajectories in a turbulent flow have positive Lyapunov exponents, we conclude for near-inviscid, turbulent flows under potential external forces the following: The alignment of the vorticity over asymptotically long time intervals should be the weakest with the dominant left singular eigenvector  $\eta_3(t; t_0, \mathbf{x}_0)$  in comparison with the other two singular vectors. Numerical simulations, such as those of Ni et al. [10], may well show substantial alignment with  $\eta_3(t; t_0, \mathbf{x}_0)$ , but this can only be the result of numerical instability. Indeed, the slightest inaccuracy in computing  $\omega_{t_0}(\mathbf{x}_0)$  and the inevitably growing inaccuracies in computing  $\eta_3(t; t_0, \mathbf{x}_0)$  could result in a non-zero vorticity component along  $\eta_3(t; t_0, \mathbf{x}_0)$ , signaling an alignment with  $\eta_3(t; t_0, \mathbf{x}_0)$ . Such an observed alignment, however, may only be real under more substantial viscous forces for trajectories that in fact enhance, rather than impede, the alignment.

# 5. Vorticity alignment for Navier-Stokes flows in the Eulerian frame

We first recall the identity

$$[\nabla \mathbf{v}]\boldsymbol{\omega} = (\mathbf{S} + \mathbf{W})\boldsymbol{\omega} = \mathbf{S}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{S}\boldsymbol{\omega},$$

where  $\mathbf{W}(\mathbf{x},t) = \frac{1}{2} \left[ \left[ \nabla \mathbf{v}(\mathbf{x},t) \right]^{\mathrm{T}} - \nabla \mathbf{v}(\mathbf{x},t) \right]$  is the spin tensor. Based on this identity, we can re-write the homogeneous linear system of ODEs (17) as

$$\frac{D\boldsymbol{\omega}}{Dt} = \left[ \mathbf{S} \left( \mathbf{x}(t; t_0, \mathbf{x}_0), t \right) \right] \boldsymbol{\omega}.$$
(37)

We assume now that along a trajectory,  $\mathbf{x}(t; t_0, \mathbf{x}_0)$ , the rate-of-strain tensor has a uniformly bounded variation from its initial value  $\mathbf{S}_0(\mathbf{x}_0, t_0) := \mathbf{S}(\mathbf{x}(t_0; t_0, \mathbf{x}_0), t_0)$ , i.e., there exists a function  $\Gamma(\mathbf{x}_0, t)$  such that

$$\mathbf{S}\left(\mathbf{x}(t;t_{0},\mathbf{x}_{0}),t\right) = \mathbf{S}_{0}(\mathbf{x}_{0},t_{0}) + \boldsymbol{\Gamma}\left(\mathbf{x}_{0},t\right),$$
$$\left|\boldsymbol{\Gamma}\left(\mathbf{x}_{0},t\right)\right| \leq \gamma\left(\mathbf{x}_{0}\right) < \infty, \quad t \in \mathbb{R}.$$
(38)

The eigenvalues  $\sigma_{0j}(\mathbf{x}_0, t_0)$  and corresponding eigenvectors  $\mathbf{e}_{0j}(\mathbf{x}_0, t_0)$  of  $\mathbf{S}_0(\mathbf{x}_0, t_0)$  then satisfy

$$\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})\mathbf{e}_{0j}(\mathbf{x}_{0}, t_{0}) = \sigma_{0j}(\mathbf{x}_{0}, t_{0})\mathbf{e}_{0j}(\mathbf{x}_{0}, t_{0}),$$
(39)

$$\sigma_{01}(\mathbf{x}_0, t_0) \le \sigma_{02}(\mathbf{x}_0, t_0) \le \sigma_{03}(\mathbf{x}_0, t_0), \qquad \sum_{j=1}^3 \sigma_{0j}(\mathbf{x}_0, t_0) = 0.$$
(40)

Along such a trajectory, the vorticity transport equation (13) can be re-written as

$$\frac{D\boldsymbol{\omega}}{Dt} = \mathbf{S}_0(\mathbf{x}_0, t_0)\boldsymbol{\omega} + \nu \Delta \boldsymbol{\omega} + \nabla \times \mathbf{f} + \boldsymbol{\Gamma} \left( \mathbf{x}_0, t \right) \boldsymbol{\omega}, \tag{41}$$

whose integral form becomes

$$\boldsymbol{\omega}_{t}\left(\mathbf{x}_{0}\right) = e^{\mathbf{S}_{0}\left(\mathbf{x}_{0},t_{0}\right)\left(t-t_{0}\right)}\boldsymbol{\omega}_{t_{0}}\left(\mathbf{x}_{0}\right)$$

$$+ \int_{t_{0}}^{t} e^{\mathbf{S}_{0}\left(\mathbf{x}_{0},t_{0}\right)\left(t-s\right)}\left[\nu\Delta\boldsymbol{\omega}_{s}\left(\mathbf{x}_{0}\right)\right]$$

$$+ \nabla \times \boldsymbol{f}\left(\mathbf{x}\left(s;t_{0},\mathbf{x}_{0}\right),s\right) + \boldsymbol{\Gamma}\left(\mathbf{x}_{0},s\right)\boldsymbol{\omega}_{s}\left(\mathbf{x}_{0}\right)\right] ds.$$
(42)

Using this equation in a derivation outlined in the Appendix A, we obtain the following forward-and backward-asymptotic estimates for the alignment between the vorticity and the principal Eulerian stretching directions:

$$\left\langle \mathbf{e}_{\boldsymbol{\omega}_{0}}, \mathbf{e}_{03}(\boldsymbol{x}_{0}, t_{0}) \right\rangle^{2} \leq 1 - \left( 1 - \left| \int_{t_{0}}^{\infty} e^{\sigma_{03}(\boldsymbol{x}_{0}, t_{0})(t_{0} - s)} \left\langle \mathbf{e}_{03}(\boldsymbol{x}_{0}, t_{0}), \boldsymbol{h}_{s}(\boldsymbol{x}_{0}; \boldsymbol{\nu}, \boldsymbol{f}) \right\rangle ds \right| \right)^{2},$$

$$(43)$$

$$\left\langle \mathbf{e}_{\omega_{0}}, \mathbf{e}_{01}(\mathbf{x}_{0}, t_{0}) \right\rangle \leq 1 - \left( 1 - \left| \int_{t_{0}}^{-\infty} e^{\sigma_{01}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \left\langle \mathbf{e}_{01}(\mathbf{x}_{0}, t_{0}), \mathbf{h}_{s}(\mathbf{x}_{0}; \nu, \mathbf{f}) \right\rangle ds \right| \right)^{2},$$
(44)

with

$$h_t(\mathbf{x}_0; v, f) := \frac{v \Delta \boldsymbol{\omega}_t(\mathbf{x}_0) + \nabla \times f(\mathbf{x}(t; t_0, \mathbf{x}_0), t) + \Gamma(\mathbf{x}_0, t) \boldsymbol{\omega}_t(\mathbf{x}_0)}{\left|\boldsymbol{\omega}_{t_0}(\mathbf{x}_0)\right|}.$$

The eigenvectors  $\mathbf{e}_{0i}(\mathbf{x}_0, t_0)$  are mutually orthogonal and hence

$$\left\langle \mathbf{e}_{01}(\mathbf{x}_{0},t_{0}),\mathbf{e}_{\omega_{t_{0}}}\right\rangle^{2} + \left\langle \mathbf{e}_{02}(\mathbf{x}_{0},t_{0}),\mathbf{e}_{\omega_{t_{0}}}\right\rangle^{2} + \left\langle \mathbf{e}_{03}(\mathbf{x}_{0},t_{0}),\mathbf{e}_{\omega_{t_{0}}}\right\rangle^{2} = 1.$$
 (45)

The angle  $\beta(\mathbf{x}_0, t_0)$  between the vorticity  $\omega_0$  and the intermediate eigenvector  $\mathbf{e}_{02}(\mathbf{x}_0, t_0)$  of the rate-of-strain tensor then satisfies

$$\sin^2 \beta(\mathbf{x}_0, t_0) = \left\langle \mathbf{e}_{01}(\mathbf{x}_0, t_0), \mathbf{e}_{\omega_{t_0}} \right\rangle^2 + \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{e}_{\omega_{t_0}} \right\rangle^2.$$
(46)

Combining the inequalities (43)–(44) with Eq. (46) then gives the following strengthened version of Theorem 1 with respect to the eigenvectors of the rate-of-strain tensor.

**Theorem 2.** Consider an incompressible Navier–Stokes velocity field  $v(\mathbf{x}, t)$ , defined on the invariant flow region U with pointwise uniformly bounded vorticity. Assume that along a trajectory  $\mathbf{x}(t; t_0, \mathbf{x}_0)$ , the rate-of-strain tensor has a uniformly bounded variation  $\Gamma(\mathbf{x}_0, t)$  around its initial value  $\mathbf{S}_0(\mathbf{x}_0, t_0)$ , as in Eq. (38). Finally, assume that  $\mathbf{S}_0(\mathbf{x}_0, t_0)$  has no repeated eigenvalues. Then the angle  $\beta(\mathbf{x}_0, t_0)$  between the vorticity  $\boldsymbol{\omega}(\mathbf{x}_0, t_0)$  and the intermediate rate-of-strain eigenvector  $\mathbf{e}_{02}(\mathbf{x}_0, t_0)$  satisfies

$$\sin^{2} \beta(\mathbf{x}_{0}, t_{0}) \leq 2 - \left(1 - \left|\int_{t_{0}}^{\infty} e^{\sigma_{03}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \langle \mathbf{e}_{03}(\mathbf{x}_{0}, t_{0}), \mathbf{h}_{s}(\mathbf{x}_{0}; \mathbf{v}, \mathbf{f}) \rangle \, ds\right|\right)^{2} - \left(1 - \left|\int_{t_{0}}^{-\infty} e^{\sigma_{01}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \langle \mathbf{e}_{01}(\mathbf{x}_{0}, t_{0}), \mathbf{h}_{s}(\mathbf{x}_{0}; \mathbf{v}, \mathbf{f}) \rangle \, ds\right|\right)^{2}.$$
(47)

Note that both the forward and backward integrals from Theorem 2 are guaranteed to converge provided that the viscous and external body forces are uniformly bounded. The pointwise convergence proof of the upper bound (47) is given in the Appendix B. We consider this to be an important result as it guarantees the existence of the asymptotic limits in the upper bound of Theorem 2.

By writing out the right-hand side of inequality (47), we can further upper estimate  $\sin^2 \beta(\mathbf{x}_0, t_0)$  as

$$\sin^{2} \beta(\mathbf{x}_{0}, t_{0}) \leq 2\left(\left|\int_{t_{0}}^{\infty} e^{\sigma_{03}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \langle \mathbf{e}_{03}(\mathbf{x}_{0}, t_{0}), \mathbf{h}_{s}(\mathbf{x}_{0}; \mathbf{v}, \mathbf{f}) \rangle ds\right| + \left|\int_{t_{0}}^{-\infty} e^{\sigma_{01}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \langle \mathbf{e}_{01}(\mathbf{x}_{0}, t_{0}), \mathbf{h}_{s}(\mathbf{x}_{0}; \mathbf{v}, \mathbf{f}) \rangle ds\right|\right)$$
(48)

Therefore, the upper bound from inequality (47) becomes tight if the forward and backward integrals are sufficiently small

$$\left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \ll 1,$$
(49)

$$\left| \int_{t_0}^{-\infty} e^{\sigma_{01}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{01}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \nu, \mathbf{f}) \right\rangle ds \right| \ll 1.$$
 (50)

Physically, the integrals from Theorem 2 represent a weighted average, i.e. convolution, between the exponential of the principal strain rates  $(\sigma_{03}(\mathbf{x}_0) > 0, \sigma_{01}(\mathbf{x}_0) < 0)$  and the sum of the strain rate variation times the vorticity and the curl of the viscous and non-conservative forces normalized by the initial vorticity magnitude. The exponential functions respectively vanish in the asymptotic limits  $t \to \pm \infty$  and the exponential decay is determined by the exponents  $\sigma_{03}(\mathbf{x}_0)$  and  $\sigma_{01}(\mathbf{x}_0)$ . The asymptotic estimates in Eqs. (43)-(44) are therefore monotonically decreasing functions of  $\sigma_{03}(\mathbf{x}_0)$  and  $-\sigma_{01}(\mathbf{x}_0)$ . High stretching and compression in the  $\mathbf{e}_{03}(\mathbf{x}_0) - \mathbf{e}_{01}(\mathbf{x}_0)$  plane results in a quick exponential decay of the upper bounds (43)-(44). Under the conditions of Theorem 2, we have therefore obtained that the vorticity vector approximately aligns with the intermediate rate-of-strain eigenvector in high strain regions provided that the curl of the viscous and external forces and the strain rate variation along a short trajectory segment are small relative to the initial vorticity magnitude. We expect these conditions to hold for a large number of trajectories in isotropic turbulent flows characterized by high Reynolds numbers [27].

Note that Theorem 2 is also valid for low Reynolds number flows. In such flows, however, viscous forces are high and the upper bound (47) is not guaranteed to be tight. The alignment between  $\omega_0$  and  $e_{02}(x_0, t_0)$  generally depends on the interplay between the rate-of-strain tensor, the viscous forces, and body forces. For sufficiently high Reynolds numbers, Theorem 2 provides mathematical evidence for the preferential alignment of the vorticity with the intermediate eigenvector of the rate-of-strain tensor. This strong preferential alignment increases for high Reynolds numbers provided that the curl of the body forces and the variation of the rate-of-strain tensor remain small.

For inviscid flows under potential forces, we then obtain the following strengthened version of Theorem 2.

**Corollary 2.** In inviscid flows defined on a compact invariant domain *U* with pointwise uniformly bounded vorticity and potential forces, the angle  $\beta(\mathbf{x}_0, t_0)$  between the vorticity  $\boldsymbol{\omega}_0$  and the intermediate eigenvector  $\mathbf{e}_{02}(\mathbf{x}_0, t_0)$  of the initial rate-of-strain tensor  $\mathbf{S}_0(\mathbf{x}_0, t_0)$  satisfies

$$\sin^{2} \beta(\mathbf{x}_{0}, t_{0}) \leq 2 - \left(1 - \left|\int_{t_{0}}^{\infty} e^{\sigma_{03}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_{0}, t_{0}), \mathbf{\Gamma}(\mathbf{x}_{0}, s) \frac{\omega_{s}(\mathbf{x}_{0})}{\left|\omega_{t_{0}}(\mathbf{x}_{0})\right|} \right\rangle ds \right|\right)^{2} - \left(1 - \left|\int_{t_{0}}^{-\infty} e^{\sigma_{01}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \left\langle \mathbf{e}_{01}(\mathbf{x}_{0}, t_{0}), \mathbf{\Gamma}(\mathbf{x}_{0}, s) \frac{\omega_{s}(\mathbf{x}_{0})}{\left|\omega_{t_{0}}(\mathbf{x}_{0})\right|} \right\rangle ds \right|\right)^{2}. \quad (51)$$

The upper bound (51) increases as a function of  $\sigma_{03}$  and  $-\sigma_{01}$ , and decreases as a function of  $\Gamma$ . Therefore, for inviscid flows with arbitrary time dependence and uniformly bounded vorticity, we expect strong preferential alignment between the vorticity and the intermediate eigenvector of the rate-of-strain tensor in high strain regions with small variations of  $\Gamma$ . We expect this preferential alignment to increase on average if the fluid is dominated by areas of such behavior.

In steady flows, the rate-of-strain tensor has no explicit time dependence and the variation of the rate-of-strain tensor is identically zero at fixed points. For such flows, therefore, Corollary 2 implies that at fixed points, the vorticity aligns exactly with the intermediate eigenvector of the rate-of-strain tensor  $\mathbf{S}_0(\mathbf{x}_0)$  since the right-hand side vanishes at such points. At fixed points we thus have  $\sin^2 \beta(\mathbf{x}_0, t_0) \equiv 0$ . Additionally,  $\sin^2 \beta(\mathbf{x}_0, t_0)$  depends continuously on the initial conditions  $\mathbf{x}_0$ . Therefore, the preferential alignment between  $\boldsymbol{\omega}_{t_0}$  and  $\mathbf{e}_{02}(\mathbf{x}_0)$  extends to a sufficiently small neighborhood around the fixed point.

#### 6. Numerical examples

We first investigate the alignment properties of the vorticity in an analytic non-integrable 3D Euler flow with chaotic trajectories. We then proceed to analyze vorticity alignment in 3D homogenous isotropic turbulence data from the Johns Hopkins Turbulence Database [47].

## 6.1. Steady ABC-flow

As a first example, we consider the fully 3D steady Arnold–Beltrami– Childress (ABC) flow [48], given by the velocity field

$$\boldsymbol{v}(\boldsymbol{x}) = \begin{pmatrix} C\sin(z) + B\cos(y) \\ A\sin(x) + C\cos(z) \\ B\sin(y) + A\cos(x) \end{pmatrix},$$
(52)

defined on the 3D toroidal domain  $U \in [0, 2\pi]^3$  with  $A = \sqrt{2}$ , B = 1,  $C = \sqrt{3}$ . This velocity field is an exact solution of the incompressible Euler equation. The flow shows a very rich behavior and admits a variety of features analogous to KAM tori and hyperbolic structures [49,50]. The ABC-flow is a Beltrami flow that satisfies:

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\omega}(\boldsymbol{x}),\tag{53}$$

and hence the vorticity is globally bounded. Since  $v(\mathbf{x})$  is a steady velocity field defined on a compact domain given by the three-dimensional torus, Oseledets multiplicative ergodic theorem guarantees the existence of the mathematically exact Lyapunov exponents  $\mu_3^{\pm}(\mathbf{x}_0)$  and the Lyapunov vectors  $\zeta_3^{\pm}(\mathbf{x}_0)$  for almost all initial conditions  $\mathbf{x}_0$ . The finite time Lyapunov exponent and the right singular vectors  $\xi_3(\mathbf{x}_0)$  of the linearized flow map are thus guaranteed to converge for nearly all initial conditions to  $\mu_3^{\pm}(\mathbf{x}_0)$  and  $\zeta_3^{\pm}(\mathbf{x}_0)$ . We will compute the linearized flow map  $\nabla \mathbf{F}_{t_0}^{t_1}(\mathbf{x}_0)$  along a trajectory launched from  $\mathbf{x}_0$  over a sufficiently long time interval  $t \in [t_0, t_1]$  and then obtain  $\text{FTLE}_{t_0}^{t_1}(\mathbf{x}_0)$  and  $\xi_3(t_1; t_0, \mathbf{x}_0)$  from a singular value decomposition (SVD) of  $\nabla \mathbf{F}_{t_0}^{t_1}(\mathbf{x}_0)$ , as done by Oettinger and Haller [51].

Figs. 1a-2a respectively show the  $\mathcal{B}_{t_0}^{t_1}$  field (defined in Eq. (26)), computed over the time intervals [0, 1] and [0, 10]. Features of the  $\mathcal{B}_{t_0}^{t_1}$ field are related to those of the  $FTLE_{t_0}^{t_1}$  field for sufficiently small  $VSE_{t_0}^{t_1}$ , as seen from the formula (26). Negative trenches of the  $VSE_{t_0}^{t_1}$  field indicate areas where the vorticity vector is compressed, whereas positive ridges of the  $VSE_{t_0}^{t_1}$  field indicate vortex stretching. Minima in the  $\mathcal{B}_{t_0}^{t_1}$  field reveal areas where the vorticity is preferentially orthogonal to the most stretching direction given by  $\xi_3(t_1; t_0, \mathbf{x}_0)$ . The zoomed insets in Fig. 1 show that for short intervals, vortex stretching (VSE > 0) is caused by the initial alignment of  $\omega_0$  with the most stretching direction  $\xi_3$ . On the contrary compression of the vorticity vector (VSE < 0) is associated with orthogonality between  $\omega_0$  and  $\xi_3$ . The contribution of  $VSE_{t_0}^{t_1}$  to the  $\mathcal{B}_{t_0}^{t_1}$  field becomes negligible as time increases because the vorticity remains bounded (see also zoomed insets of Fig. 2). Taking the asymptotic limits  $t_1 \rightarrow \pm \infty$ , we therefore conclude the following from Corollary 1: For almost all initial conditions with positive Lyapunov exponents, the vorticity exactly aligns with

$$\boldsymbol{n}(\boldsymbol{x}_0) = \boldsymbol{\zeta}_3^+(\boldsymbol{x}_0) \times \boldsymbol{\zeta}_3^-(\boldsymbol{x}_0),$$

i.e., with the direction along the intersection of the two planes orthogonal to the forward and backward dominant Lyapunov vectors  $\zeta_{\pm}^{\pm}(\mathbf{x}_0)$ .

If the forward and backward most stretching directions are collinear  $(\zeta_3^+(x_0) \| \zeta_3^-(x_0))$ , then there is no unique intersection line between the two planes. In that case the angle  $\alpha(x_0)$  between  $\omega_0$  and  $n(x_0)$  is not well defined. Away from such degenerate points,  $\sin^2 \alpha(x_0)$  is given by

$$\sin^{2} \alpha(\mathbf{x}_{0}) = \frac{\left|\mathbf{e}_{\omega_{t_{0}}} \times \mathbf{n}(\mathbf{x}_{0})\right|^{2}}{\left|\mathbf{n}(\mathbf{x}_{0})\right|^{2}}$$

$$= \frac{\left|\mathbf{e}_{\omega_{t_{0}}} \times \left(\boldsymbol{\zeta}_{3}^{+}(\mathbf{x}_{0}) \times \boldsymbol{\zeta}_{3}^{-}(\mathbf{x}_{0})\right)\right|^{2}}{\left|\boldsymbol{\zeta}_{3}^{+}(\mathbf{x}_{0}) \times \boldsymbol{\zeta}_{3}^{-}(\mathbf{x}_{0})\right|^{2}}$$

$$= \frac{\left(\left\langle \mathbf{e}_{\omega_{t_{0}}}, \boldsymbol{\zeta}_{3}^{+}(\mathbf{x}_{0})\right\rangle \boldsymbol{\zeta}_{3}^{-}(\mathbf{x}_{0}) - \left\langle \mathbf{e}_{\omega_{t_{0}}}, \boldsymbol{\zeta}_{3}^{-}(\mathbf{x}_{0})\right\rangle \boldsymbol{\zeta}_{3}^{+}(\mathbf{x}_{0})\right)^{2}}{\left|\boldsymbol{\zeta}_{3}^{+}(\mathbf{x}_{0}) \times \boldsymbol{\zeta}_{3}^{-}(\mathbf{x}_{0})\right|^{2}}$$

$$= \frac{\left\langle \mathbf{e}_{\omega_{t_{0}}}, \boldsymbol{\zeta}_{3}^{+}(\mathbf{x}_{0})\right\rangle^{2} + \left\langle \mathbf{e}_{\omega_{t_{0}}}, \boldsymbol{\zeta}_{3}^{-}(\mathbf{x}_{0})\right\rangle^{2}}{\left|\mathbf{n}(\mathbf{x}_{0})\right|^{2}}$$
(54)

This formula shows  $\sin^2 \alpha(\mathbf{x}_0)$  to be inversely proportional to  $|\mathbf{n}(\mathbf{x}_0)|^2$ . Therefore, in regions where the forward and backward dominant Lyapunov vectors are nearly collinear  $(|\mathbf{n}(\mathbf{x}_0)|^2 \ll 1)$ , the numerical computation of  $\sin^2 \alpha(\mathbf{x}_0)$  will be noisy.

Fig. 3 displays  $\sin^2 \alpha(\mathbf{x}_0)$  together with the FTLE and the Lagrangian Averaged Vorticity Deviation (LAVD) given by:

$$LAVD_{t_0}^t(\mathbf{x}_0) = \frac{1}{|t - t_0|} \int_{t_0}^t \left| \boldsymbol{\omega}_s(\mathbf{x}_0) - \overline{\boldsymbol{\omega}_s(\mathbf{x}_0)} \right| ds,$$
(55)

where  $\overline{\omega_s(\mathbf{x}_0)}$  denotes the spatially averaged vorticity. The LAVD is an objective diagnostic used to visualize vortical flow structures from



Fig. 1. Vorticity alignment and Lagrangian diagnostics for the ABC-flow for  $t \in [0, 1]$ . The Lagrangian diagnostic  $B_{l_0}^t(\mathbf{x}_0) := e^{\left[\operatorname{VSE}_{l_0}^t(\mathbf{x}_0) - \operatorname{FTLE}_{l_0}^t(\mathbf{x}_0)\right](t-t_0)}$  (panel a) is an upper bound to  $\left|\left\langle \mathbf{e}_{a_0}(\mathbf{x}_0), \boldsymbol{\xi}_3(t; t_0, \mathbf{x}_0)\right\rangle\right|$  (panel c). The individual exponents of  $B_{l_0}^t(\mathbf{x}_0)$  are  $\operatorname{FTLE}_{l_0}^t(\mathbf{x}_0)$  (panel b) and  $\operatorname{VSE}_{l_0}^t(\mathbf{x}_0)$  (panel d).

three-dimensional velocity data [52]. We approximate the leading Lyapunov exponent over a finite time interval as

 $\mu_3^+(\mathbf{x}_0) \sim \text{FTLE}_0^{200}(\mathbf{x}_0).$ 

In the Appendix C, we show that the chosen time interval guarantees statistical convergence for all three Lyapunov exponents and vectors.

Elliptic (or vortical) flow structures are shear dominated areas characterized by high LAVD-values. Material elements stretch algebraically within those regions and we therefore expect a vanishing leading Lyapunov exponent [53]. In contrast, hyperbolic flow areas are characterized by positive dominant Lyapunov exponents because of exponentially stretching material lines.

Fig. 3 reveals that the dominant forward and backward Lyapunov vectors are nearly collinear within elliptic flow regions (see Fig. 3c–d). By approximating  $\mu_3^+(\mathbf{x}_0)$  from a finite time computation, we find that Lyapunov exponents vanish within elliptic flow areas, i.e., generalized KAM tori. Our computations on the steady ABC flow highlight that the vorticity aligns with  $n(\mathbf{x}_0)$  in hyperbolic flow regions characterized by positive leading Lyapunov exponents (see Fig. 3a–b), as predicted by Theorem 1. This is also highlighted by the joint PDFs of the Lyapunov exponent  $\mu_3(\mathbf{x}_0)$  with the alignment diagnostic  $\sin^2 \alpha(\mathbf{x}_0)$  and the rotational diagnostic LAVD<sub>0</sub><sup>200</sup> ( $\mathbf{x}_0$ ) (see Fig. 4).

Fig. 4a confirms that the vorticity statistically aligns with the direction  $n(x_0)$  in regions of high material stretching, i.e., hyperbolic flow regions. The joint PDF in Fig. 4b shows that elliptic flow regions identified by high LAVD-values coincide with minimal stretching regions, i.e., domains with vanishing leading Lyapunov exponent. Therefore,

within elliptic flow regions, the vorticity is no longer guaranteed to align with the direction  $n(x_0)$ , because the assumptions of Theorem 2 cease to be satisfied.

In Fig. 5 we show the alignment between the vorticity and the eigenvectors of the rate–of–strain tensor. We approximate the asymptotic estimates (43)–(47) from a forward and backward computation over a finite time interval  $t \in [0, \pm 200]$  and define

$$L_{0}^{200}(\mathbf{x}_{0}) = 1 - (1 - \left| \int_{0}^{200} e^{-\sigma_{03}(\mathbf{x}_{0}, t_{0})s} \left\langle \mathbf{e}_{03}(\mathbf{x}_{0}, t_{0}), \mathbf{\Gamma}(\mathbf{x}_{0}, s) \frac{\omega_{s}(\mathbf{x}_{0})}{\left|\omega_{t_{0}}(\mathbf{x}_{0})\right|} \right\rangle ds \right| \right)^{2},$$

$$L_{0}^{-200}(\mathbf{x}_{0}) = 1 - (1 - 1)$$
(56)

$$\left| \int_{0}^{-200} e^{-\sigma_{01}(\mathbf{x}_{0},t_{0})s} \left\langle \mathbf{e}_{01}(\mathbf{x}_{0},t_{0}), \mathbf{\Gamma}(\mathbf{x}_{0},s) \frac{\omega_{s}(\mathbf{x}_{0})}{\left|\omega_{t_{0}}(\mathbf{x}_{0})\right|} \right\rangle ds \right| \right|^{2},$$
(57)

$$L_{-200}^{200}(\boldsymbol{x}_0) = L_0^{-200}(\boldsymbol{x}_0) + L_0^{200}(\boldsymbol{x}_0).$$
(58)

According to Corollary 2, the vorticity preferentially aligns with the intermediate eigenvector  $\mathbf{e}_{02}(\mathbf{x}_0)$  of the rate-of-strain tensor  $\mathbf{S}_0(\mathbf{x}_0) = \mathbf{S}(\mathbf{x}_0)$  given that  $\mathbf{L}_{-200}^{200}(\mathbf{x}_0)$  is sufficiently small. Contrary to common observations for homogenous isotropic turbulence, the vorticity shows preferential alignment with  $\mathbf{e}_{02}(\mathbf{x}_0)$  in the ABC flow only for very small regions. These regions coincide with areas where  $\mathbf{L}_{-200}^{200}(\mathbf{x}_0)$  is minimal (see zoomed insets of Fig. 5). The zoomed insets additionally display branches of unstable (white line) and stable (magenta line) manifolds,



Fig. 2. Same as Fig. 1 but over the time interval [0, 10].

which are approximated through ridges of the backward and forward FTLE field. At the intersection between stable and unstable manifolds, the vorticity strongly aligns with the intermediate eigenvectors of the rate-of-strain tensor.

Therefore, the Eulerian alignment observed in homogenous isotropic turbulence generally does not carry over to inviscid flows with bounded vorticity. Instead, the vorticity is exactly orthogonal to the forward and backward leading Lyapunov vectors in unforced inviscid flows along trajectories with positive forward and backward Lyapunov exponents.

## 6.2. Forced homogenous isotropic turbulence from Johns Hopkins Turbulence Database

Next, we investigate vorticity alignment in forced homogenous isotropic turbulence from the Johns Hopkins Turbulence Database (JHTDB) [47]. The isotropic turbulence fields were obtained from a 1024<sup>3</sup>-node direct numerical simulation (DNS) that is publicly available from the JHTDB over a periodic box of size  $[0, 2\pi]^3$ . The Taylor–Reynolds number fluctuates around  $Re_{\lambda} \sim 433$ . In order to keep the velocity field at a statistically stationary state, the flow is externally forced at large scales by keeping the total energy constant in modes such that their wave number magnitude is less than or equal to 2. The main parameters of the simulation are given in Table 1.

The Lagrangian trajectories and the velocity gradients are publicly accessible through a web-based interface. The total simulation time of the stored trajectories is 20 Kolmogorov time scales ( $\tau_{\eta}$ ), with the data provided at a time step of  $0.1\tau_{\eta}$ . Once the trajectories are calculated,

#### Table 1

Parameters for the data from the homogenous isotropic turbulence simulation of the Johns Hopkins Turbulence Database (JHTDB).

| Dataset | Nodes             | ν                   | η                   | $	au_\eta$ | $Re_{\lambda}$ |
|---------|-------------------|---------------------|---------------------|------------|----------------|
| JHTDB   | 1024 <sup>3</sup> | $1.85\cdot 10^{-4}$ | $2.87\cdot 10^{-3}$ | 0.045      | 433            |

the velocity gradients are computed at each point using a 4th-order central finite difference method with 4th-order Lagrange interpolation. We select a 2D slice over a domain-size given by  $50\eta \times 50\eta$ , where  $\eta$  is the Kolmogorov length scale. For this study, we use  $150 \times 150 = 22500$  Lagrangian particle trajectories to compute the relevant statistical quantities. The viscous and non-conservative forces are computed from the vorticity evolution according to Eq. (13). We approximate the asymptotic estimates (43)–(47) over a long but finite time interval  $[t_0, t_0 \pm 20\tau_n]$  through

$$L_{t_0}^{t_0+20\tau_{\eta}}(\mathbf{x}_0) = 1 - (1 - \left\| \int_{t_0}^{t_0+20\tau_{\eta}} e^{\sigma_{03}(\mathbf{x}_0,t_0)(t_0-s)} \langle \mathbf{e}_{03}(\mathbf{x}_0,t_0), \mathbf{h}_s(\mathbf{x}_0;\nu,\mathbf{f}) \rangle \, ds \right\| \right)^2,$$
(59)  
$$L_{t_0}^{t_0-20\tau_{\eta}}(\mathbf{x}_0) = 1 - (1 - 1)^{1/2} \left\| \mathbf{e}_{03}(\mathbf{x}_0,t_0) - \mathbf{e}_{03}(\mathbf{x}_0,t_0) - \mathbf{e}_{03}(\mathbf{x}_0,t_0) \right\|_{t_0}^{1/2} \right\|_{t_0}^{1/2} \left\| \mathbf{e}_{03}(\mathbf{x}_0,t_0) - \mathbf{e}_{03}(\mathbf{x}_0,t_0) - \mathbf{e}_{03}(\mathbf{x}_0,t_0) \right\|_{t_0}^{1/2} \left\| \mathbf{e}_{03}(\mathbf{x}_0,t_0) - \mathbf{e}_{03}(\mathbf{x}_0,t_0) \right\|_{t_0}^{1/2} \right\|_{t_0}^{1/2} \left\| \mathbf{e}_{03}(\mathbf{x}_0,t_0) - \mathbf{e}_{03}(\mathbf{x}_0,t_0) \right\|_{t_0}^{1/2} \right\|_{t_0}^{1/2} \left\| \mathbf{e}_{03}(\mathbf{x}_0,t_0) - \mathbf{e}_{03}(\mathbf{x}_0,t_0) \right\|_{t_0}^{1/2} \left\| \mathbf{e}_{03}(\mathbf{x}_0,t_0$$

$$\left| \int_{t_0}^{t_0 - 20\tau_{\eta}} e^{\sigma_{01}(\mathbf{x}_0, t_0)(t_0 - s)} \langle \mathbf{e}_{01}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \rangle \, ds \right| \right)^2, \tag{60}$$

$$L_{t_0-20\tau_{\eta}}^{t_0+20\tau_{\eta}}(\boldsymbol{x}_0) = L_{t_0}^{t_0+20\tau_{\eta}}(\boldsymbol{x}_0) + L_{t_0}^{t_0-20\tau_{\eta}}(\boldsymbol{x}_0).$$
(61)



Fig. 3. Alignment diagnostics vs. hyperbolic and elliptic regions in the ABC flow. The zoomed insets show an elliptic region with invariant tori surrounded by hyperbolic region with chaotic dynamics.

We have verified in the Appendix D, that the upper bounds (59)–(60) statistically converge over the chosen time interval.

In Fig. 6 we plot the upper bounds defined in Eqs. (59)–(61) (panels (a,d,g)), the alignments between  $\omega_{t_0}$  and the eigenvectors of the rateof-strain tensor  $S_0(x_0, t_0)$  (panels (b,e,h)), and the principal strain rates  $\sigma_{0j}(\mathbf{x}_0, t_0)$  (panels (c,f,i)) over the initial conditions  $\mathbf{x}_0$ . In panel (g), we have used a different colormap for  $L_{t_0-20\tau_\eta}^{t_0+20\tau_\eta}(\mathbf{x}_0) > 1$ . Trenches of the  $L_{t_0}^{t_0-20\tau_\eta}(\mathbf{x}_0)$  and  $L_{t_0}^{t_0-20\tau_\eta}(\mathbf{x}_0)$  fields indicate areas where the vorticity is preferentially orthogonal to  $\mathbf{e}_{03}(\mathbf{x}_0, t_0)$  or  $\mathbf{e}_{01}(\mathbf{x}_0, t_0)$ . By combining the forward and the backward estimates, we obtain that low values of the  $L_{t_0-20\tau}^{t_0+20\tau}(\mathbf{x}_0)$  field guarantee strong alignment between  $\omega_{t_0}$  and  $\mathbf{e}_{02}(\mathbf{x}_0)$ . Furthermore, the scalar fields of the upper bounds (panels (a,d,g)) are topologically similar to the corresponding alignment fields (panels (b,e,h)).

The white continuous inset in Fig. 6 highlights a region where the estimate  $L_{t_0-20\tau_\eta}^{t_0+20\tau_\eta}(\mathbf{x}_0)$  suggests strong alignment between  $\omega_{t_0}$  and  $\mathbf{e}_{02}(\mathbf{x}_0, t_0)$ . Specifically, this area displays high stretching and compression in the  $\mathbf{e}_{03}(\mathbf{x}_0, t_0)$  and  $\mathbf{e}_{01}(\mathbf{x}_0, t_0)$  directions (see panels (c,f)). Therefore, the deformation principally occurs on a 2D-plane spanned by  $\mathbf{e}_{03}(\mathbf{x}_0, t_0)$  and  $\mathbf{e}_{01}(\mathbf{x}_0, t_0)$ . The vorticity is constrained to point out of that plane and align with  $\mathbf{e}_{02}(\mathbf{x}_0)$ . Such high strain areas have frequently been associated with regions where the vorticity preferentially aligns with the intermediate eigenvector of the rate–of–strain tensor [32]. Note, however, that regions of intense alignment between the  $\boldsymbol{\omega}_{t_0}$  and  $\mathbf{e}_{02}(\mathbf{x}_0, t_0)$  can also arise in low strain areas (see dashed white insets in Fig. 6). We emphasize that in those areas, the estimate  $\sum_{t_0-20\tau_{\eta}}^{t_0+20\tau_{\eta}}(\mathbf{x}_0)$  guarantees preferential alignment between  $\boldsymbol{\omega}_{t_0}$  and  $\mathbf{e}_{02}(\mathbf{x}_0, t_0)$ .

Fig. 7 displays the cumulative distributive functions (CDF) for the scalar fields from Fig. 6. For 65% of the initial conditions, the vorticity is preferentially aligned with  $\mathbf{e}_{02}(\mathbf{x}_0)$ . Additionally, the CDFs in panel (b) show that the vorticity is more likely to be orthogonal to  $\mathbf{e}_{01}(\mathbf{x}_0)$  than to  $\mathbf{e}_{03}(\mathbf{x}_0)$ . This is also reflected in the distribution of  $\mathbf{L}_{t_0}^{t_0+20\tau_\eta}(\mathbf{x}_0)$  and  $\mathbf{L}_{t_0}^{t_0-20\tau_\eta}(\mathbf{x}_0)$  (see panel (a)). We attribute this to the fact that the CDF of  $\sigma_{02}(\mathbf{x}_0, t_0)$  is skewed towards positive values and therefore  $\sigma_{03}(\mathbf{x}_0, t_0)$  is on average smaller than  $-\sigma_{01}(\mathbf{x}_0, t_0)$ . This implies that the exponential decay rate of the forward integral is smaller than the one of the backward integral. By taking into account the effect of viscous and non-conservative forces as well as the strain rate variation along a trajectory, the asymptotic estimate  $\mathbf{L}_{t_0}^{t_0+20\tau_\eta}$  guarantees that for 15% of the initial conditions, the vorticity preferentially aligns with the intermediate Eulerian stretching direction (see panel (a)).

Finally, we compare the statistics of the upper bounds (59)–(61) with the corresponding exact vorticity alignments. Each panel in Fig. 8 displays the joint probability density function (PDF) of the estimates



**Fig. 4.** Joint PDF of  $\sin^2 \alpha$  and leading forward Lyapunov exponent  $\mu_3^+$ , where  $\alpha$  denotes the angle between the initial vorticity  $\boldsymbol{\omega}_{t_0}$  and the direction  $\boldsymbol{n}(\mathbf{x}_0, t_0)$  (panel a). Joint PDF of LAVD<sub>0</sub><sup>200</sup> and  $\mu_3^+$  (panel b).

 $L_{t_0}^{t_0+20\tau_\eta}$ ,  $L_{t_0}^{t_0-20\tau_\eta}$  and  $L_{t_0-20\tau_\eta}^{t_0+20\tau_\eta}$  together with their exact vorticity alignments. The colored contour plots in Fig. 8 show the joint PDFs and the histograms on the top and the right of each panel indicate the individual PDFs. The white dots coincide with a ridge in the joint PDF and highlight the average correlation between the exact alignments and the corresponding upper bounds. Panel (a) shows that as the upper bound of the forward estimate  $L_{t_0}^{t_0+20\tau_\eta}$  decreases, the inner product between  $\mathbf{e}_{\omega_{t_0}}$  and  $\mathbf{e}_{03}$  decreases and the vorticity vector becomes preferentially orthogonal to the most stretching direction  $\mathbf{e}_{03}$ . The alignment diagnostic  $\left\langle e_{\omega_{t_0}}, e_{03} \right\rangle^2$ , therefore, varies directly as a function of its upper bound  $L_{t_0}^{t_0+20\tau_\eta}$ . A similar direct relationship holds for the backward alignment estimate in Eq. (60) as well (see panel (b)). Panel (c) displays the joint PDF between

$$\sin^2\beta = 1 - \left\langle \boldsymbol{e}_{\boldsymbol{\omega}_{t_0}}, \boldsymbol{e}_{02} \right\rangle^2$$

and the numerical estimate  $L_{t_0-20\tau_\eta}^{t_+20\tau_\eta}$  from the asymptotic upper bound of Theorem 2. We again obtain that, on average,  $\sin^2 \beta$  varies as a

function of the estimate  $L_{t_0-20\tau_{\eta}}^{t_+20\tau_{\eta}}$ . This shows that the upper bound from Theorem 2 is statistically correlated with the alignment between the vorticity and the intermediate eigenvector of the rate-of-strain tensor.

Our results for homogenous isotropic turbulence demonstrate that Theorem 2 provides a conservative but physically insightful upper bound on the squared sine of the angle between the vorticity and the intermediate eigenvector of the rate-of-strain tensor.

## 7. Conclusions

We have reconsidered here the phenomenon of vorticity alignment (or lack thereof) with the Lyapunov vectors and various distinguished Eulerian stretching directions in flows with pointwise bounded vorticity. As a first observation, we have pointed out that any observed alignment of vorticity with distinguished Eulerian or Lagrangian strain directions is a fundamentally frame-dependent phenomenon. This follows from the frame-dependence of the direction of the vorticity vector and the frame-indifference of the eigenvectors of common Eulerian and Lagrangian strain tensors. As a consequence, any observed preferential alignment reported so far is specific to inertial frames and will not persist in other frames.

Next, we have derived a general Lagrangian estimate for the angle between the vorticity vector and the asymptotic limits of the eigenvectors of the Cauchy-Green strain tensor, whenever those limits exist. This estimate enabled us to conclude that in inviscid flows with pointwise uniformly bounded vorticity and conservative body forces, the vorticity must be exactly orthogonal to the planes spanned by the dominant forward and backward Lyapunov vectors (Theorem 1) along trajectories with positive forward and backward maximal Lyapunov exponents. For steady and time periodic flows defined on compact domains, this conclusion can be strengthened, given that the Lyapunov exponents are known to exist for almost all initial conditions by Oseledets multiplicative ergodic theorem (Corollary 1). If the forwardand backward asymptotic behaviors along a trajectory coincide, then Corollary 1 implies a perfect alignment between the vorticity and the intermediate Lyapunov vector in inviscid flows (Remark 1). To our knowledge, this is the first exact mathematical result that establishes preferential vorticity alignment with intermediate stretching directions, which is the Lagrangian analog of the empirically observed preferential alignment between the vorticity and the intermediate eigenvector of the rate-of-strain tensor in the Eulerian setting.

Our results underline that the previously reported asymptotic alignment between the vorticity and the left singular vector of the deformation gradient cannot happen along trajectories with bounded vorticity and positive Lyapunov exponents (Remark 2). Indeed, such an alignment would imply temporally unbounded vorticity growth and hence establish a contradiction. In homogeneous isotropic turbulence, most initial conditions are expected to have positive Lyapunov exponents, and hence numerical indications of vorticity alignment with the most stretching direction are likely to arise from the unavoidable numerical errors arising in the computation of Lagrangian strain eigenvectors over long time intervals.

Finally, we considered viscous flows with non-conservative body forces and with pointwise bounded vorticity. For such flows, our Theorem 2 gives an exact upper estimate on the deviation between the vorticity and the intermediate eigenvector of the rate-of-strain tensor. The asymptotic estimate provides physical conditions such that the vorticity preferentially aligns with the intermediate eigenvector of the rate-of-strain tensor. This alignment is strong in high strain regions given that over short trajectory segments the variation of the rate-of-strain tensor is small and the curl of viscous and nonconservative forces are small relative to the initial vorticity magnitude. We have demonstrated this in DNS data on homogeneous and isotropic turbulence.



Fig. 5. Vorticity alignment and Eulerian diagnostics for the ABC flow. Numerical estimates (56)–(58) for the asymptotic upper bound from Corollary 2 (panels a, d, g). Exact Eulerian vorticity alignments (panels b, e, h). Principal strains, i.e., eigenvalues of the rate-of-strain tensor (panels c, f, i). The zoomed insets display branches of unstable (white line) and stable (magenta line) manifolds in proximity of a hyperbolic fixed point. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

#### CRediT authorship contribution statement

**A. Encinas-Bartos:** Formal analysis, Investigation, Methodology, Writing – original draft. **G. Haller:** Conceptualization, Funding acquisition, Project administration, Supervision, Writing – review & editing.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

#### Data availability

I have put references to the data in the article.

## Appendix A. Proof of Theorem 2

We recall from Section 5 the integral form of the vorticity transport equation (41),

$$\boldsymbol{\omega}_{t}\left(\mathbf{x}_{0}\right) = e^{\mathbf{S}_{0}(\mathbf{x}_{0},t_{0})\left(t-t_{0}\right)}\boldsymbol{\omega}_{t_{0}}\left(\mathbf{x}_{0}\right) + \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0},t_{0})\left(t-s\right)} \boldsymbol{g}_{s}(\mathbf{x}_{0};v,\boldsymbol{f}) \, ds, \qquad (A.1)$$

where

$$g_t(\mathbf{x}_0; v, \mathbf{f}) := v \Delta \omega_t(\mathbf{x}_0) + \nabla \times f(\mathbf{x}(t; t_0, \mathbf{x}_0), t) + \Gamma(\mathbf{x}_0, t) \omega_t(\mathbf{x}_0)$$
(A.2)

contains the contribution of the viscous and non-potential external forces, and the strain rate variation  $\Gamma$  times the vorticity. Squaring both sides of Eq. (A.1) yields

$$\begin{aligned} \left|\boldsymbol{\omega}_{t}(\mathbf{x}_{0})\right|^{2} &= \left|e^{\mathbf{S}_{0}(\mathbf{x}_{0},t_{0})(t-t_{0})}\boldsymbol{\omega}_{t_{0}}\right|^{2} \\ &+ 2\left\langle e^{\mathbf{S}_{0}(\mathbf{x}_{0},t_{0})(t-t_{0})}\boldsymbol{\omega}_{t_{0}}, \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0},t_{0})(t-s)}\boldsymbol{g}_{s}(\mathbf{x}_{0};\boldsymbol{\nu},\boldsymbol{f})\,ds\right\rangle \\ &+ \left|\int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0},t_{0})(t-s)}\boldsymbol{g}_{s}(\mathbf{x}_{0};\boldsymbol{\nu},\boldsymbol{f})\,ds\right|^{2} \end{aligned} \tag{A.3}$$

The eigenvalue decomposition from formula (39) implies

$$e^{\mathbf{S}_{0}(\mathbf{x}_{0},t_{0})t}e_{0j}(\mathbf{x}_{0},t_{0}) = e^{\sigma_{0j}(\mathbf{x}_{0},t_{0})t}\mathbf{e}_{0j}(\mathbf{x}_{0},t_{0})$$
(A.4)

and we can therefore write

 $\left| e^{\mathbf{S}_0(\mathbf{x}_0,t_0)(t-t_0)} \boldsymbol{\omega}_{\star}(\mathbf{x}_0) \right|^2$ 

$$= \left| e^{\mathbf{S}_{0}(\mathbf{x}_{0},t_{0})t} \sum_{j=1}^{3} \left\langle \boldsymbol{\omega}_{t_{0}}(\mathbf{x}_{0}), \mathbf{e}_{0j}(\mathbf{x}_{0},t_{0}) \right\rangle \mathbf{e}_{0j}(\mathbf{x}_{0},t_{0})} \right|^{2}$$
  
$$= \left| \sum_{j=1}^{3} \left\langle \boldsymbol{\omega}_{t_{0}}(\mathbf{x}_{0}), \mathbf{e}_{0j}(\mathbf{x}_{0},t_{0}) \right\rangle e^{\sigma_{0j}(\mathbf{x}_{0},t_{0})t} \mathbf{e}_{0j}(\mathbf{x}_{0},t_{0}) \right|^{2}$$
  
$$= \sum_{j=1}^{3} \left\langle \boldsymbol{\omega}_{t_{0}}(\mathbf{x}_{0}), \mathbf{e}_{0j}(\mathbf{x}_{0},t_{0}) \right\rangle^{2} e^{2\sigma_{0j}(\mathbf{x}_{0},t_{0})t}$$

European Journal of Mechanics / B Fluids 105 (2024) 259-274



**Fig. 6.** Numerical estimates for the asymptotic upper bounds from Eqs. (59)–(61) in the homogeneous isotropic turbulence dataset (panels a, d, g). Exact vorticity alignment (panels b, e, h). Principal strain rates from  $S_0(x_0, t_0)$  (panels c, f, i). All diagnostics are plotted with respect to the initial conditions  $x_0$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



**Fig. 7.** CDF of the upper bounds defined in Eqs. (59)–(61) (panel a). CDF of  $\langle \mathbf{e}_{\omega_0}, \mathbf{e}_{01}(\mathbf{x}_0, t_0) \rangle^2$ ,  $\langle \mathbf{e}_{\omega_0}, \mathbf{e}_{03}(\mathbf{x}_0, t_0) \rangle^2$  and  $\sin^2 \beta(\mathbf{x}_0, t_0) = 1 - \langle \mathbf{e}_{\omega_0}, \mathbf{e}_{02}(\mathbf{x}_0, t_0) \rangle^2$  (panel b). CDF of the principal strain rates (panel c).

$$\geq \left\langle \boldsymbol{\omega}_{t_{0}}\left(\mathbf{x}_{0}\right), \mathbf{e}_{0j}(\mathbf{x}_{0}, t_{0})\right\rangle^{2} e^{2\sigma_{0j}(\mathbf{x}_{0}, t_{0})t}, \quad j = 1, 2, 3.$$
Combining the last inequality with Eq. (A.3) gives
$$\left\langle \boldsymbol{\omega}_{t_{0}}\left(\mathbf{x}_{0}\right), \mathbf{e}_{03}(\mathbf{x}_{0}, t_{0})\right\rangle^{2} e^{2\sigma_{03}(\mathbf{x}_{0}, t_{0})t}$$

$$\leq \left|\boldsymbol{\omega}_{t}(\mathbf{x}_{0})\right|^{2} - 2 \left\langle e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-t_{0})} \boldsymbol{\omega}_{t_{0}}, \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$

$$= \left| \left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \mathbf{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, f) ds \right|^{2}$$



**Fig. 8.** Joint PDF of  $L_{t_0}^{t_0+20\tau_q}$  and  $\left\langle \mathbf{e}_{\omega_0}, \mathbf{e}_{03} \right\rangle^2$  (panel a),  $L_{t_0}^{t_0-20\tau_q}$  and  $\left\langle \mathbf{e}_{\omega_0}, \mathbf{e}_{01} \right\rangle^2$  (panel b), and  $L_{t_0-20\tau_q}^{t_0+20\tau_q}$  and  $\sin^2 \beta$  restricted to  $L_{t_0-20\tau_q}^{t_0+20\tau_q} \leq 1$  (panel c). The white markers define a ridge in the joint PDFs. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

By dividing both sides of Eq. (A.6) by  $e^{2\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)}$  and introducing the notation

$$\mathbf{e}_{\omega_{t_0}}(\mathbf{x}_0) := \frac{\boldsymbol{\omega}_{t_0}(\mathbf{x}_0)}{\left|\boldsymbol{\omega}_{t_0}(\mathbf{x}_0)\right|},$$

we re-write inequality (A.6) as

$$\begin{split} \left\langle \mathbf{e}_{\omega_{t_{0}}}, \mathbf{e}_{03}(\mathbf{x}_{0}, t_{0}) \right\rangle^{2} \\ &\leq \frac{\left| \boldsymbol{\omega}_{t}(\mathbf{x}_{0}) \right|^{2}}{\left| \boldsymbol{\omega}_{t_{0}}(\mathbf{x}_{0}) \right|^{2}} e^{-2\sigma_{03}(\mathbf{x}_{0}, t_{0})(t-t_{0})} + 2 \frac{\left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(2t-t_{0}-s)} \boldsymbol{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, \boldsymbol{f}) \, ds \right|}{\left| \boldsymbol{\omega}_{t_{0}}(\mathbf{x}_{0}, t_{0}) \right| e^{2\sigma_{03}(\mathbf{x}_{0}, t_{0})(t-t_{0})}} \\ &- \left( \frac{\left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \boldsymbol{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, \boldsymbol{f}) \, ds \right|}{\left| \boldsymbol{\omega}_{t_{0}}(\mathbf{x}_{0}, t_{0}) \right| e^{\sigma_{03}(\mathbf{x}_{0}, t_{0})(t-t_{0})}} \right]^{2} \\ &= e^{2 \left( \text{VSE}_{t_{0}}^{t}(\mathbf{x}_{0}) - \sigma_{03}(\mathbf{x}_{0}, t_{0}) \right)(t-t_{0})} + 2 \frac{\left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(2t-t_{0}-s)} \boldsymbol{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, \boldsymbol{f}) \, ds \right|}{\left| \boldsymbol{\omega}_{t_{0}}(\mathbf{x}_{0}, t_{0}) \right| e^{2\sigma_{03}(\mathbf{x}_{0}, t_{0})(t-t_{0})}} \\ &- \left( \frac{\left| \int_{t_{0}}^{t} e^{\mathbf{S}_{0}(\mathbf{x}_{0}, t_{0})(t-s)} \boldsymbol{g}_{s}(\mathbf{x}_{0}; \mathbf{v}, \boldsymbol{f}) \, ds \right|}{\left| \boldsymbol{\omega}_{t_{0}}(\mathbf{x}_{0}, t_{0}) \right| e^{2\sigma_{03}(\mathbf{x}_{0}, t_{0})(t-t_{0})}} \right\}^{2}, \tag{A.7}$$

where  $VSE_{t_{1}}^{t}(\mathbf{x}_{0})$  is the vorticity stretching exponent defined in formula (14). For flows with bounded vorticity the first term in the inequality (A.7) vanishes in the asymptotic limit  $t \to \infty$ . By using the eigenvalue decomposition (A.4), we write out the individual terms in Eq. (A.7) as

$$\frac{\left|\int_{t_0}^{t} e^{\mathbf{S}_0(\mathbf{x}_0,t_0)(2t-t_0-s)} \mathbf{g}_s(\mathbf{x}_0;\mathbf{v},\mathbf{f}) \, ds\right|}{e^{2\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)}} \tag{A.8}$$
$$= \left| \left( \int_{t_0}^{t} \frac{e^{\sigma_{01}(\mathbf{x}_0,t_0)(2t-t_0-s)}}{e^{2\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)}} \left\langle \mathbf{e}_{01}(\mathbf{x}_0,t_0), \mathbf{g}_s(\mathbf{x}_0;\mathbf{v},\mathbf{f}) \right\rangle \, ds \right| \\\int_{t_0}^{t} \frac{e^{\sigma_{02}(\mathbf{x}_0,t_0)(2t-t_0-s)}}{e^{2\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)}} \left\langle \mathbf{e}_{02}(\mathbf{x}_0,t_0), \mathbf{g}_s(\mathbf{x}_0;\mathbf{v},\mathbf{f}) \right\rangle \, ds \right| \\\int_{t_0}^{t} e^{\sigma_{03}(\mathbf{x}_0,t_0)(t_0-s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0,t_0), \mathbf{g}_s(\mathbf{x}_0;\mathbf{v},\mathbf{f}) \right\rangle \, ds \right|$$
and

$$\frac{\left|\int_{t_0}^{t} e^{\mathbf{S}_0(\mathbf{x}_0,t_0)(t-s)} \mathbf{g}_s(\mathbf{x}_0;\mathbf{v},\mathbf{f}) \, ds\right|}{e^{\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)}} \tag{A.9}$$

$$= \left( \int_{t_0}^{t} \frac{e^{\sigma_{01}(\mathbf{x}_0, t_0)(t-s_0)}}{e^{\sigma_{03}(\mathbf{x}_0, t_0)(t-t_0)}} \left\langle \mathbf{e}_{01}(\mathbf{x}_0, t_0), \mathbf{g}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right) \\ \int_{t_0}^{t} \frac{e^{\sigma_{02}(\mathbf{x}_0, t_0)(t-s_0)}}{e^{\sigma_{03}(\mathbf{x}_0, t_0)(t-t_0)}} \left\langle \mathbf{e}_{02}(\mathbf{x}_0, t_0), \mathbf{g}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \\ \int_{t_0}^{t} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0-s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{g}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right)$$
(A.10)

# With the notation

 $\|\boldsymbol{g}\|_{\infty} = \sup_{t \in \mathbb{R}, \boldsymbol{x}_0 \in U} \left| \boldsymbol{g}_t(\boldsymbol{x}_0, t_0; \boldsymbol{\nu}, \boldsymbol{f}) \right|,$ 

we then have for the individual terms in formulas (A.8)-(A.9)

$$\begin{split} &\int_{t_0}^{t} \frac{e^{\sigma_{01}(\mathbf{x}_{0},t_{0})(2t-t_{0}-s)}}{e^{2\sigma_{03}(\mathbf{x}_{0},t_{0})(t-t_{0})}} \left\langle \mathbf{e}_{01}(\mathbf{x}_{0},t_{0}), \mathbf{g}_{s}(\mathbf{x}_{0};\mathbf{v},\mathbf{f}) \right\rangle ds \tag{A.11} \\ &\leq \frac{e^{\sigma_{01}(\mathbf{x}_{0},t_{0})(2t-t_{0})}}{e^{2\sigma_{03}(\mathbf{x}_{0},t_{0})(t-t_{0})}} \int_{t_0}^{t} e^{-\sigma_{01}(\mathbf{x}_{0},t_{0})s} ds \|\mathbf{g}\|_{\infty} \\ &= \frac{e^{\sigma_{01}(\mathbf{x}_{0},t_{0})(2t-t_{0})}}{e^{2\sigma_{03}(\mathbf{x}_{0},t_{0})(t-t_{0})}} \left[ \frac{e^{-\sigma_{01}(\mathbf{x}_{0},t_{0})s} - e^{-\sigma_{01}(\mathbf{x}_{0},t_{0})t}}{\sigma_{01}(\mathbf{x}_{0},t_{0})} \right] \|\mathbf{g}\|_{\infty} \\ &= \left[ \frac{e^{2\sigma_{01}(\mathbf{x}_{0},t_{0})(t-t_{0})} - e^{\sigma_{01}(\mathbf{x}_{0},t_{0})(t-t_{0})}}{e^{2\sigma_{03}(\mathbf{x}_{0},t_{0})(t-t_{0})}} \right] \frac{\|\mathbf{g}\|_{\infty}}{\sigma_{01}(\mathbf{x}_{0},t_{0})}, \end{split}$$

and similarly

$$\int_{t_0}^{t} \frac{e^{\sigma_{01}(\mathbf{x}_0,t_0)(2t-t_0-s)}}{e^{2\sigma_{01}(\mathbf{x}_0,t_0)(t-t_0)}} \left\langle \mathbf{e}_{02}(\mathbf{x}_0,t_0), \mathbf{g}_s(\mathbf{x}_0;\mathbf{v},\mathbf{f}) \right\rangle ds$$

$$\leq \left[ \frac{e^{2\sigma_{02}(\mathbf{x}_0,t_0)(t-t_0)} - e^{\sigma_{02}(\mathbf{x}_0,t_0)(t-t_0)}}{e^{2\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)}} \right] \frac{\|\mathbf{g}\|_{\infty}}{\sigma_{02}(\mathbf{x}_0,t_0)}, \qquad (A.12)$$

$$\begin{aligned} \int_{t_0} & \frac{1}{e^{\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)}} \left( \mathsf{e}_{01}(\mathbf{x}_0,t_0), \mathbf{g}_s(\mathbf{x}_0,\mathbf{v},\mathbf{f}) \right) ds \\ &\leq e^{-\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)} \left[ 1 - e^{\sigma_{01}(\mathbf{x}_0,t_0)(t-t_0)} \right] \frac{\|\mathbf{g}\|_{\infty}}{\sigma_{01}(\mathbf{x}_0,t_0)}, \\ & \int_{t_0}^t \frac{e^{\sigma_{02}(\mathbf{x}_0,t_0)(t-s)}}{e^{\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)}} \left\langle \mathsf{e}_{02}(\mathbf{x}_0,t_0), \mathbf{g}_s(\mathbf{x}_0;\mathbf{v},\mathbf{f}) \right\rangle ds \\ &\leq e^{-\sigma_{03}(\mathbf{x}_0,t_0)(t-t_0)} \left[ 1 - e^{\sigma_{02}(\mathbf{x}_0,t_0)(t-t_0)} \right] \frac{\|\mathbf{g}\|_{\infty}}{\sigma_{02}(\mathbf{x}_0,t_0)}, \end{aligned} \tag{A.13}$$

Except for trajectories where the eigenvalues of  $S_0(x_0, t_0)$  are repeated, i.e., at singularities of  $S_0(x_0, t_0)$ , the right-hand side of the inequalities (A.11)–(A.14) vanishes in the asymptotic limit  $t \to \infty$ . By combining the latter statements, we obtain the following simplified version of inequality (A.7) in the asymptotic limit  $t \to \infty$ :

$$\left\langle \mathbf{e}_{\omega_{t_0}}, \mathbf{e}_{03}(\mathbf{x}_0, t_0) \right\rangle^2$$

$$\leq 2 \frac{\left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{g}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \rangle \, ds \right|}{\left| \mathbf{\omega}_{t_0}(\mathbf{x}_0, t_0) \right|}$$

$$- \left( \frac{\left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{g}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \rangle \, ds \right|}{\left| \mathbf{\omega}_{t_0}(\mathbf{x}_0, t_0) \right|} \right)^2$$

$$= 1 - \left( 1 - \left| \frac{\left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{g}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \rangle \, ds \right|}{\left| \mathbf{\omega}_{t_0}(\mathbf{x}_0, t_0) \right|} \right)^2$$

$$= 1 - \left( 1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \rangle \, ds \right| \right)^2, \quad (A.15)$$

where we introduced the notation

$$\begin{split} \boldsymbol{h}_{t}(\boldsymbol{x}_{0};\boldsymbol{\nu},\boldsymbol{f}) &:= \frac{\boldsymbol{\nu} \Delta \boldsymbol{\omega}_{t}(\boldsymbol{x}_{0}) + \boldsymbol{\nabla} \times \boldsymbol{f}_{t}(\boldsymbol{x}_{0}) + \boldsymbol{\Gamma}\left(\boldsymbol{x}_{0},t\right) \boldsymbol{\omega}_{t}(\boldsymbol{x}_{0})}{\left|\boldsymbol{\omega}_{t_{0}}(\boldsymbol{x}_{0},t_{0})\right|} \\ &= \frac{\boldsymbol{g}_{t}(\boldsymbol{x}_{0};\boldsymbol{\nu},\boldsymbol{f})}{\left|\boldsymbol{\omega}_{t_{0}}(\boldsymbol{x}_{0},t_{0})\right|}. \end{split}$$

Applying the same reasoning in backward time and taking the asymptotic limit  $t \to -\infty$  yields

$$\left\langle \mathbf{e}_{\omega_{t_0}}, \mathbf{e}_{01}(\mathbf{x}_0, t_0) \right\rangle^2 \le 1 - (1 - \left| \int_{t_0}^{-\infty} e^{\sigma_{01}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{01}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; v, \mathbf{f}) \right\rangle ds \right| \right)^2$$
(A.16)

The eigenvectors  $\mathbf{e}_{0j}(\mathbf{x}_0, t_0)$  are mutually orthogonal and hence

$$\left\langle \mathbf{e}_{01}(\mathbf{x}_{0},t_{0}),\mathbf{e}_{\omega_{t_{0}}}\right\rangle^{2} + \left\langle \mathbf{e}_{02}(\mathbf{x}_{0},t_{0}),\mathbf{e}_{\omega_{t_{0}}}\right\rangle^{2} + \left\langle \mathbf{e}_{03}(\mathbf{x}_{0},t_{0}),\mathbf{e}_{\omega_{t_{0}}}\right\rangle^{2} = 1.$$
 (A.17)

The angle  $\beta(\mathbf{x}_0, t_0)$  between the vorticity  $\boldsymbol{\omega}_0$  and the intermediate eigenvector  $\mathbf{e}_{02}(\mathbf{x}_0, t_0)$  of the rate-of-strain tensor then satisfies

$$\sin^2 \beta(\mathbf{x}_0, t_0) = \left\langle \mathbf{e}_{01}(\mathbf{x}_0, t_0), \mathbf{e}_{\omega_{t_0}} \right\rangle^2 + \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{e}_{\omega_{t_0}} \right\rangle^2.$$
(A.18)

By combining the last equation with the inequalities (A.15)–(A.16) we obtain

$$\sin^2 \beta(\mathbf{x}_0, t_0) \le 2 - \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 - \frac{1}{2} - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 \right)^2 \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 \right)^2 \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 \right)^2 \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 \right)^2 \right)^2 - \frac{1}{2} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \right\rangle ds \right| \right)^2 \right)^2 \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{e}_{03}(\mathbf{x}_0, t_0) \right\rangle ds \right| \right)^2 \right)^2 \right)^2 \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \right| \right)^2 \right)^2 \right)^2 \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0} ds \right)^2 \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \right| \right)^2 \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0} ds \right)^2 \right)^2 \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0, t_0)} ds \right)^2 \left(1 - \left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x$$

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$$\left(1 - \left| \int_{t_0}^{-\infty} e^{\sigma_{01}(\mathbf{x}_0, t_0)(t_0 - s)} \left\langle \mathbf{e}_{01}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \nu, \mathbf{f}) \right\rangle ds \right| \right)^2$$
(A.19)

This concludes the proof of Theorem 2.

# Appendix B. Proof of convergence of the upper bound from Theorem 2

We start by writing out the upper bound from Theorem 2

$$\sin^{2} \beta(\mathbf{x}_{0}, t_{0}) \leq 2 - \left(1 - \left| \int_{t_{0}}^{\infty} e^{\sigma_{03}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \langle \mathbf{e}_{03}(\mathbf{x}_{0}, t_{0}), \mathbf{h}_{s}(\mathbf{x}_{0}; v, \mathbf{f}) \rangle \, ds \right| \right)^{2} - \left(1 - \left| \int_{t_{0}}^{-\infty} e^{\sigma_{01}(\mathbf{x}_{0}, t_{0})(t_{0} - s)} \langle \mathbf{e}_{01}(\mathbf{x}_{0}, t_{0}), \mathbf{h}_{s}(\mathbf{x}_{0}; v, \mathbf{f}) \rangle \, ds \right| \right)^{2}.$$
(B.1)

In order to show that the right-hand side of (B.1) has a well defined asymptotic limit, it suffices to show that both forward and backward improper integrals of inequality (B.1) converge over  $[t_0, \pm \infty]$ . Assume that the combined effect of the viscous and external body forces is uniformly bounded, i.e.,

$$\left\|\boldsymbol{h}_{t}(\boldsymbol{x}_{0};\boldsymbol{\nu},\boldsymbol{f})\right\|_{\infty} < K,\tag{B.2}$$

where  $\|h_t(\mathbf{x}_0; v, f)\|_{\infty} := \sup_{t \in \mathbb{R}, \mathbf{x}_0 \in U} |h_t(\mathbf{x}_0; v, f)|$  denotes the infinity norm of f. The relationship (B.2) then implies that the forward integral of inequality (B.1) is upper bounded by

$$\left| \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} \langle \mathbf{e}_{03}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \nu, \mathbf{f}) \rangle \, ds \right|$$
(B.3)

$$\leq \|\boldsymbol{h}_{t}(\boldsymbol{x}_{0};\nu,\boldsymbol{f})\|_{\infty} \int_{t_{0}}^{\infty} e^{\sigma_{03}(\boldsymbol{x}_{0},t_{0})(t_{0}-s)} ds$$
(B.4)

$$\leq K \int_{t_0}^{\infty} e^{\sigma_{03}(\mathbf{x}_0, t_0)(t_0 - s)} ds$$
(B.5)

$$=\frac{K}{\sigma_{03}(x_0,t_0)}.$$
(B.6)

Similarly the backward integral in formula (B.1) satisfies

$$\left| \int_{t_0}^{-\infty} e^{\sigma_{01}(\mathbf{x}_0, t_0)(t_0 - s)} \langle \mathbf{e}_{01}(\mathbf{x}_0, t_0), \mathbf{h}_s(\mathbf{x}_0; \mathbf{v}, \mathbf{f}) \rangle \, ds \right|$$

$$\leq \frac{K}{|\sigma_{01}(\mathbf{x}_0, t_0)|}.$$
(B.8)

Therefore, both forward and backward improper integrals from inequality (B.1) are bounded and have well defined asymptotic limits. This implies that the upper bound from Theorem 2 converges provided that the viscous and external body forces are uniformly bounded.

# Appendix C. Convergence of Lyapunov exponents and vectors in steady ABC flow

The convergence of the Lyapunov exponents and vectors in the steady ABC flow is guaranteed for almost all initial conditions by Oseledets theorem [41]. We numerically approximate the Lyapunov exponents and vectors through the singular values and vectors of the gradient of the flow  $\nabla \mathbf{F}_{t_0}^t(\mathbf{x}_0)$  over a finite time interval. We check the convergence of the limits (10)–(11) over an ensemble of initial conditions  $\mathbf{x}_0$ . As shown in Fig. C.9, the forward Lyapunov exponents and vectors, approximated from the singular values and vectors of the gradient of the flow map, have converged at t = 200. The forward Lyapunov vectors  $\zeta_i(\mathbf{x}_0)$  are numerically estimated as  $\zeta_i(\mathbf{x}_0) = \xi_i(200; \mathbf{x}_0)$ . A similar statements holds for the backward asymptotics.



**Fig. C.9.** (panel a) The time averaged logarithms of the singular values  $\sqrt{\lambda_i}(t, \mathbf{x}_0)$  of the gradient of the flow map  $\nabla \mathbf{F}'_{t_0}(\mathbf{x}_0)$  statistically converge to the Lyapunov exponents  $\mu_i(\mathbf{x}_0)$ . The overline denotes the average over the initial conditions  $\mathbf{x}_0$ . (panel b) Similarly, the right singular vectors  $\boldsymbol{\xi}_i(t; \mathbf{x}_0)$  of  $\nabla \mathbf{F}'_{t_0}(\mathbf{x}_0)$  converge to the Lyapunov vectors  $\boldsymbol{\zeta}_i(\mathbf{x}_0)$ .



**Fig. D.10.**  $\overline{L_{t_0}^{t_0 \pm t}}(\mathbf{x}_0)$  denotes the average of  $L_{t_0}^{t_0 \pm t}(\mathbf{x}_0)$  over the initial conditions  $\mathbf{x}_0$  as a function of time *t*. The upper bounds  $L_{t_0}^{t_0 \pm t}(\mathbf{x}_0)$  statistically converge.

# Appendix D. Convergence of $L_{t_0}^{t_0\pm t}(x_0)$

The asymptotic estimate of Theorem 2 convergences pointwise provided that the viscous and external body forces are uniformly bounded (see Appendix B). We approximate this estimate over a finite time interval  $[t_0, t_0 \pm t]$  through the forward and backward integrals (59)–(60). Fig. D.10 displays the average over the initial conditions  $\mathbf{x}_0$  of the upper bounds  $L_{t_0}^{t_0\pm t}(\mathbf{x}_0)$  as a function of time *t*. We confirm that both the forward and backward upper bounds  $L_{t_0}^{t_0\pm t}(\mathbf{x}_0)$  have statistically converged at  $t = 20\tau_{\eta}$ . Note that the limits of the forward and backward integrals generally do not coincide. This follows from the fact  $\sigma_{03}$  is on average smaller than  $\sigma_{-01}$  (see Fig. 6).

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#### A. Encinas-Bartos and G. Haller

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