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Predicting subharmonic and superharmonic backbone curves in nonlinear mechanical systems

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We show that conservative (or primary) backbone curves, defined as amplitude–frequency curves of conservative nonlinear normal modes, have a tight mathematical connection with all subharmonic and superharmonic (or secondary) backbone curves, defined as curves connecting peaks of forced response at higher-order resonances. Specifically, under weak periodic forcing and damping, a single primary or secondary member of a backbone curve family determines all other members of that family via a simple linear scaling relationship that we illustrate in examples. This observation provides a quick theoretical prediction for forced response peaks arising from higher-order external resonances without the need for numerical simulations or experiments.

1. Introduction

Higher or lower harmonics in the response of a periodically forced nonlinear system can activate secondary (i.e. subharmonic, ultra-subharmonic, superharmonic and ultra-superharmonic) resonances. The resulting backbone curves (i.e. curves connecting locally maximal amplitudes of forced response diagrams) have been treated as entities separate from the primary backbone curve that emerges near forcing frequencies close to a natural frequency of the system.

While analytical and numerical case studies of subharmonic resonances have been carried out in the literature (e.g. [1,2]), the reliable identification of secondary backbone curves in general mechanical systems has only been pursued more recently. Notably, Volvert and

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Kerschen [3] introduced the idea of phase-resonance nonlinear modes to locate secondary backbone curves from numerical simulations. This new approach infers backbone curve points from distinguished phase differences between the forcing and the response rather than from peaks in forced response diagrams. Specifically, at a $k : \nu$ resonance, the phase lag between the forcing and the response is $\pi/2$ when k and ν are both odd, whereas the phase lag equals $3\pi/4\nu$ otherwise. This phase resonance criterion enables a reliable identification of secondary backbone curves without reliance on response amplitude peaks. This is advantageous because secondary resonance peaks might be overlooked in a sliding frequency sweep as they occur over a narrower frequency domain and take longer to develop. Abeloos *et al.* [4] and Zhou and Kerschen [5] used the same approach to construct a phase-locked feedback control loop to find secondary backbone curves in beam experiment.

Volvert [6] presented detailed averaging calculations for backbone curves in harmonically forced one-degree-of-freedom (1-DOF) oscillators with linear damping and polynomial nonlinearities. From these averaging calculations, the shapes of the backbone curves as phase resonance curves are explicitly deducible. However, for general higher-DOF systems with additional nonlinearities, nonlinear damping and general periodic forcing, such explicit calculations are not feasible. Similarly, in experimental situations, the information necessary for locating phase-resonance backbone curves analytically is generally unavailable.

An exploration of connections among primary and subharmonic resonances was initiated by Raze *et al.* [7], who constructed explicitly a subharmonic forcing amplitude and frequency that would maintain a periodic response already observed near a primary resonance. Using the Duffing oscillator, they illustrated their results by transitioning a primarily resonant periodic response into a $1 : 3$ subharmonically resonant periodic response while maintaining the frequency and the amplitude of the response. In more recent publications, Raze and Kerschen [8,9] have developed an algorithm to achieve such a transition for more general systems and demonstrated its effectiveness even in experimental settings.

Here, we deduce a general theoretical relation between primary and secondary backbone curves bifurcating from the same periodic orbits family of the conservative limit under small dissipation. The resulting simple scaling relation follows from a Poincaré map argument applied to the Lyapunov subcentre manifolds associated with the conservative limit of the system. The secondary backbone curves obtained in this fashion are mathematically exact in the limit of vanishing damping and forcing. Therefore, in lightly damped mechanical systems subject to moderate forcing, the actual secondary backbone curves must be in a small neighbourhood of our theoretically predicted ones, which we will illustrate with the examples of Volvert and Kerschen [3], Volvert [6], and Zhou and Kerschen [5].

For larger damping and forcing, numerical computations or experimental detection of individual backbone curves via phase resonance criteria should ultimately be more accurate than the quick and simple theoretical predictions we obtain here for small damping and forcing. Nevertheless, our predictions are accurate enough to serve as approximate benchmarks for the backbones expected from more precise calculations or experiments. In particular, scaled backbone curves can quickly reveal insufficient numerical resolution or experimental inconsistency in the extraction of higher-order backbone curves.

2. Setup

Consider an n degree-of-freedom, periodically forced and damped, nonlinear mechanical system of the form

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} + \mathbf{G})\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{f}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{f}_{\text{ext}}(\mathbf{q}, \dot{\mathbf{q}}, \Omega t), \quad \mathbf{M}, \mathbf{C}, \mathbf{G}, \mathbf{K} \in \mathbb{R}^{n \times n}, \quad \mathbf{f}_{\text{nl}}, \mathbf{f}_{\text{ext}} \in C^1. \quad (2.1)$$

Here, $\mathbf{q} \in \mathbb{R}^n$ is the vector of generalized coordinates; $\mathbf{M} = \mathbf{M}^T$ is the positive definite mass matrix; $\mathbf{K} = \mathbf{K}^T$ is the stiffness matrix; $\mathbf{C} = \mathbf{C}^T$ is the positive semidefinite damping matrix, and $\mathbf{G} = -\mathbf{G}^T$ is

the gyroscopic force matrix. The external forcing function $\mathbf{f}_{\text{ext}}(\mathbf{q}, \dot{\mathbf{q}}, \Omega t)$ is T -periodic in time with

$$T = \frac{2\pi}{\Omega},$$

where $\Omega > 0$ denotes the forcing frequency.

The nonlinear part of the unforced system is contained in the function

$$\mathbf{f}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{O}(|\mathbf{q}|^2, |\mathbf{q}| |\dot{\mathbf{q}}|, |\dot{\mathbf{q}}|^2) = \nabla U_{\text{nl}}(\mathbf{q}) + \mathbf{G}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}, \quad \mathbf{G}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}}) = -\mathbf{G}_{\text{nl}}^T(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n \times n},$$

where U_{nl} denotes a potential function for geometric nonlinearities and $\mathbf{G}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}})$ is the coefficient matrix of nonlinear gyroscopic terms. Any additional nonconservative forces or nonlinearities in the system are assumed to be subsumed in the forcing function $\mathbf{f}_{\text{ext}}(\mathbf{q}, \dot{\mathbf{q}}, \Omega t)$. In practice, system (2.1) arises when a nonlinear mechanical system is analysed near its $\mathbf{q} = 0$ equilibrium, with all conservative nonlinear terms with respect to this equilibrium included in $\mathbf{f}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}})$.

We are interested in nonlinear oscillations of system (2.1) under small damping and external forcing, which is frequently the case in structural dynamics settings. To this end, we rescale the damping and forcing terms as

$$\mathbf{C} \rightarrow \epsilon \mathbf{C}, \quad \mathbf{f}_{\text{ext}} \rightarrow \epsilon \mathbf{f}_{\text{ext}}, \quad 0 \leq \epsilon \ll 1,$$

and rewrite equation (2.1) accordingly as

$$\mathbf{M}\ddot{\mathbf{q}} + (\epsilon \mathbf{C} + \mathbf{G})\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{f}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}}) = \epsilon \mathbf{f}_{\text{ext}}(\mathbf{q}, \dot{\mathbf{q}}, \Omega t). \quad (2.2)$$

Note that $\epsilon \mathbf{f}_{\text{ext}}(\mathbf{q}, \dot{\mathbf{q}}, \Omega t)$ may also contain small nonlinear damping terms that have no explicit dependence on time.

In order to study system (2.2), we will need to work in its associated phase space, i.e. the space of the variable

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} \in \mathbb{R}^{2n}.$$

Solutions starting from the initial condition $\mathbf{x}_0 = (\mathbf{q}_0, \dot{\mathbf{q}}_0)^T$ in the phase space take the form

$$\mathbf{x}(t; t_0, \mathbf{x}_0, \epsilon) = \begin{pmatrix} \mathbf{q}(t; t_0, \mathbf{q}_0, \dot{\mathbf{q}}_0, \epsilon) \\ \dot{\mathbf{q}}(t; t_0, \mathbf{q}_0, \dot{\mathbf{q}}_0, \epsilon) \end{pmatrix}, \quad (2.3)$$

satisfying the first-order ODE

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{x}, \Omega t, \epsilon), \quad (2.4)$$

with

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{G} \end{pmatrix}, \quad \mathbf{F}(\mathbf{x}, \Omega t, \epsilon) = \begin{pmatrix} \mathbf{0} \\ -\epsilon \mathbf{C}\mathbf{x}_2 - \mathbf{f}_{\text{nl}}(\mathbf{x}_1, \mathbf{x}_2) - \epsilon \mathbf{f}_{\text{ext}}(\mathbf{x}_1, \mathbf{x}_2, \Omega t) \end{pmatrix}. \quad (2.5)$$

We fix the initial time $t_0 = 0$ in the solution (2.3) and define the period- T map (or Poincaré map), $\mathbf{P}_\epsilon : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$, as

$$\mathbf{P}_\epsilon(\mathbf{x}_0) = \mathbf{x}(T; 0, \mathbf{x}_0, \epsilon). \quad (2.6)$$

This mapping, as a finite-time sampling of the flow map of system (2.4), is C^∞ smooth with respect to ϵ . This follows by the smooth dependence of the trajectories of equation (2.4) on parameters with respect to which the right-hand side of the system is smooth (e.g. [10]).

A stroboscopic sampling of the solutions of system (2.4) at times that are integer multiples, νT , of T can be obtained by iterating the discrete dynamical system defined by the mapping $\mathbf{P}_\epsilon$:

$$\mathbf{x}(\nu T; 0, \mathbf{x}_0, \epsilon) = \mathbf{P}_\epsilon^\nu(\mathbf{x}_0) = \underbrace{\mathbf{P}_\epsilon \circ \dots \circ \mathbf{P}_\epsilon}_{\nu}(\mathbf{x}_0).$$

3. The conservative limit ($\epsilon = 0$)

The $\epsilon = 0$ limit of system (2.2),

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{f}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0}, \quad (3.1)$$

conserves the total mechanical energy

$$E(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \langle \dot{\mathbf{q}}, \mathbf{M}\dot{\mathbf{q}} \rangle + \frac{1}{2} \langle \mathbf{q}, \mathbf{K}\mathbf{q} \rangle + U_{\text{nl}}(\mathbf{q}),$$

given that

$$\frac{D}{Dt} E(\mathbf{q}, \dot{\mathbf{q}}) = \langle \dot{\mathbf{q}}, \mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \nabla U_{\text{nl}}(\mathbf{q}) \rangle = - \langle \dot{\mathbf{q}}, \mathbf{G}_{\text{nl}}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \rangle \equiv 0.$$

Furthermore, the unforced system (3.1) has an equilibrium at $\mathbf{q} = \mathbf{0}$, which we assume to be Lyapunov stable under linear approximation. This means that the characteristic equation,

$$\det(\lambda^2 \mathbf{M} + \lambda \mathbf{G} + \mathbf{K}) = 0,$$

of the linearized system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

is assumed to have n pairs of nonzero, purely imaginary eigenvalues,

$$\pm i\omega_1, \dots, \pm i\omega_n, \quad (3.2)$$

whose geometric and algebraic multiplicities are equal. This is always the case if $\mathbf{G} = \mathbf{0}$ (no gyroscopic forces) and $\mathbf{K}$ is positive definite (potential energy minimum at $\mathbf{q} = \mathbf{0}$). The same eigenvalue configuration, however, may also arise in the classic case of gyroscopic stabilization wherein the skew-symmetric matrix $\mathbf{G}$ is nonzero and $\mathbf{K}$ is indefinite.

We finally assume that there is no resonance among the eigenvalues in (3.2), i.e.

$$m\omega_j \neq \omega_k, \quad m \in \mathbb{N}^+, \quad 1 \leq j, k \leq n, \quad j \neq k. \quad (3.3)$$

Under these assumptions, by an extension of the classic results of Lyapunov [11] by Kelley [12,13], the unforced system (3.1) has n unique, class- C^1 , two-dimensional invariant manifolds, $W_1, \dots, W_n$, generally called *Lyapunov Subcentre Manifolds* (LSMs). Each W_j is tangent to the two-dimensional spectral subspace (eigenspace) E_j spanned by the real and imaginary parts of the complex eigenvectors corresponding to the pair $\pm i\omega_j$ of eigenvalues in the list (3.2), as sketched in the top left plot of figure 1. If system (3.1) is analytic, then all LSMs are also known to be analytic (e.g. [12,13]).

Each LSM W_j is filled with a nested, one-parameter family of periodic orbits,

$$\gamma_j^s(t) = \gamma_j^s(t + T_j^s), \quad \left\{ \bigcup_{s>0} \gamma_j^s \right\} \cup \{\mathbf{0}\} = W_j, \quad (3.4)$$

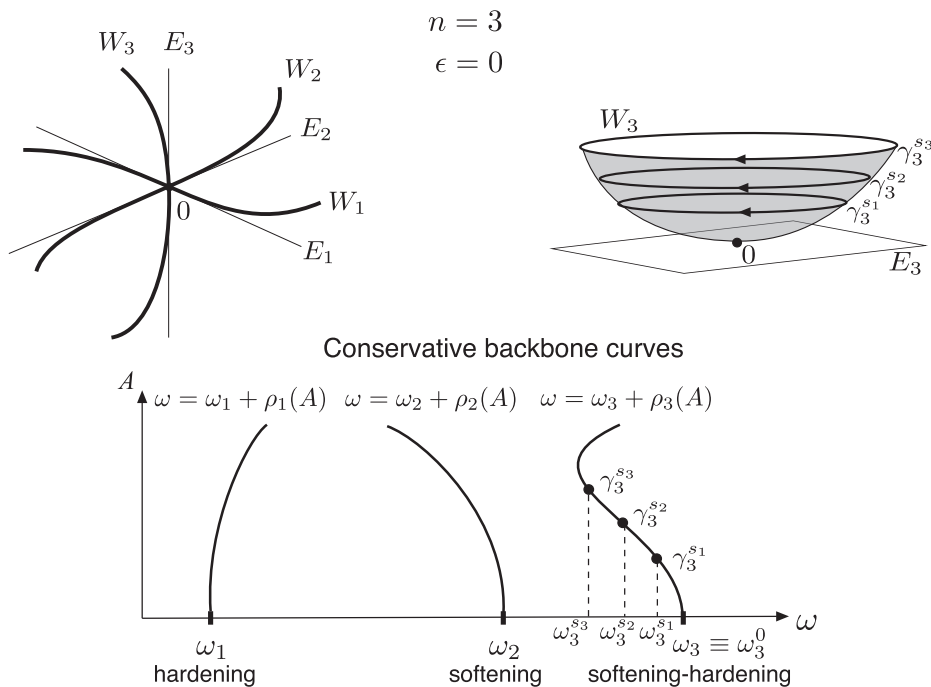


Figure 1. Top left: Geometry of the LSMs, W_1 , W_2 , and W_3 , in the phase space of the unforced system (3.1) for $n = 3$. Top right: The LSM, W_3 , filled with a one-parameter family of periodic orbits, $\{\gamma_3^s\}_{s>0}$. Bottom: Examples of conservative backbone curves, graphed in the plane of their maximal amplitudes A and their angular frequency ω . Also marked are some members of the periodic (Lyapunov) orbit family on the LSM W_3 .

parametrized by a scalar variable $s > 0$, which can be taken to be, for instance, the maximal amplitude of the periodic orbit γ_j^s , as illustrated in the top right plot of figure 1. For that choice of s , the period T_j^s converges to $T_j^0 = 2\pi/\omega_j$ as $s \rightarrow 0$ (see the bottom plot of figure 1 for an illustration). The results we obtain below will be independent of the choice of the parametrization of LSM orbits.

The amplitude-frequency plots of these *Lyapunov orbit families* are often referred to as *conservative backbone curves*, examples of which are sketched in the bottom plot of figure 1. Once we fix a unique and smooth way of defining the amplitude A of a periodic orbit of angular frequency ω , each conservative backbone curve can be characterized as a graph

$$\omega = \omega_j + \rho_j(A), \quad \rho_j(0) = 0, \quad j = 1, \dots, n, \quad (3.5)$$

as indicated in the bottom plot of figure 1. For reasons discussed later, we will also refer to conservative backbone curves as *primary backbone curves*.

The conservative limit, $\mathbf{P}_0$, of the Poincaré map (2.6) admits the same invariant manifolds $W_1, \dots, W_n$ in its dynamics, given that this map is just a stroboscopic sampling of the full flow map. A periodic orbit $\gamma_j^s \subset W_j$ of the continuous dynamical system (3.1) serves as a closed invariant curve for $\mathbf{P}_0$.

4. The weakly forced and damped system ($0 < \epsilon \ll 1$)

For the special case of autonomous perturbations ($\Omega = 0$) in system equation (2.2), de la Llave and Kogelbauer [14] obtained conditions for the smooth persistence of an LSM, W_j , as a primary spectral submanifold (or SSM), $W(E_j)$. The SSM $W(E_j)$ is defined as the unique smoothest two-dimensional invariant manifold that is tangent to the perturbed eigenspace E_j at the origin

Table 1. Terminology for different types of $k : \nu$ resonances in forced nonlinear systems as functions of the positive integers k and ν introduced in formula (4.4).

k	ν	name of $k : \nu$ resonance
$k = 1$	$\nu = 1$	primary
$k > 1$	$\nu = 1$	superharmonic
$k = 1$	$\nu > 1$	subharmonic
$k > \nu > 1$		ultra-superharmonic
$\nu > k > 1$		ultra-subharmonic

(see Haller and Ponsioen [15] and Haller [16]). This smooth persistence result, however, is inapplicable under periodic forcing, which is our main focus here. Therefore, as an alternative approach, we will discuss here the fate of individual periodic orbits under time-periodic forcing, as opposed to the fate of LSMs.

We generally seek a νT -periodic forced-damped response for $\epsilon > 0$ in system (2.2) as a fixed point for some iterate, $\mathbf{P}_\epsilon^\nu$, of the perturbed Poincaré map $\mathbf{P}_\epsilon$ defined in (2.6). Such a fixed point solves the equation

$$\mathbf{R}(\mathbf{x}, \epsilon, \nu) := \mathbf{P}_\epsilon^\nu(\mathbf{x}) - \mathbf{x} = \mathbf{0}. \quad (4.1)$$

Assume that for $\mathbf{x}$ taken from a compact domain C and for a given positive integer ν , we have $\mathbf{R}(\mathbf{x}, 0, \nu) \neq \mathbf{0}$. Then, by the continuity of $\mathbf{P}_\epsilon^\nu(\mathbf{x})$ in ϵ and $\mathbf{x}$, the equation (4.1) will have no solution for $\epsilon > 0$ small enough either.

Therefore, to obtain a νT -periodic forced-damped response for *all* arbitrarily small ϵ , we must have a limiting fixed point $\mathbf{x}_{0,\nu} \in C$ of $\mathbf{P}_0^\nu$ satisfying

$$\mathbf{R}(\mathbf{x}_{0,\nu}, 0, \nu) = \mathbf{0}. \quad (4.2)$$

Such a point $\mathbf{x}_{0,\nu}$ must be, by definition, a period- ν point of the conservative Poincaré map $\mathbf{P}_0(\mathbf{x})$. All points on a periodic orbit $\gamma_j^s \subset W_j$ of the unperturbed system equation (3.1) satisfy this requirement if the period T_j^s of the orbit γ_j^s satisfies the resonance condition

$$\nu T = k T_j^s \quad (4.3)$$

for some positive integer k . In that case, γ_j^s is a closed curve of fixed points for the mapping $\mathbf{P}_0^\nu$. The existence of a resonance of the form (4.3) is, therefore, necessary for a νT -periodic forced response to be born out of a Lyapunov family of conservative periodic orbits.

Expressed in terms of frequencies, the necessary resonance condition (4.3) takes the form

$$\omega_j^s = \frac{k}{\nu} \Omega, \quad k, \nu \in \mathbb{N}^+, \quad (4.4)$$

which we will refer to as a $k : \nu$ resonance following the notation of Volvert and Kerschen [17]. In table 1, we recall commonly used terminology for various sub-classes of $k : \nu$ resonances (e.g. [17,18]).

For any fixed forcing frequency $\Omega > 0$, condition (4.4) holds at points where the infinite set of vertical lines

$$V = \left\{ (\omega, A) : \omega = \frac{k}{\nu} \Omega, \quad k, \nu \in \mathbb{N}^+ \right\}$$

intersects the conservative backbone curves, as shown in figure 2.

Nonlinear $k : \nu$ resonance between the forcing and the third mode in the conservative limit

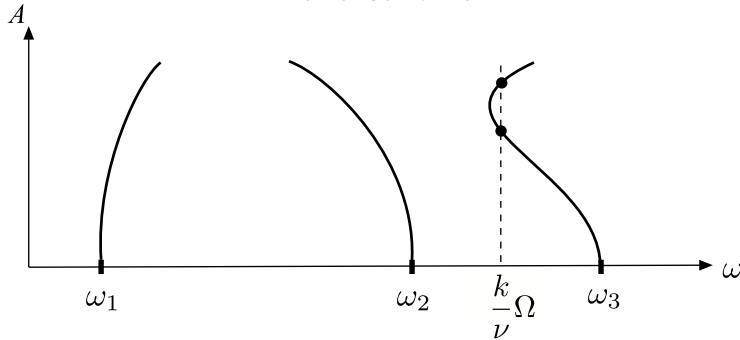


Figure 2. Geometry of the necessary resonance condition (4.4) for the case of $n = 3$ and for a fixed pair of positive integers (k, ν) .

5. Weakly damped-forced response: scaled backbone curves

(a) Backbones for forced response arising from LSMs

We now return to the general necessary condition (4.4), which must hold for any forced response that exists in the range of arbitrary small $\epsilon > 0$.

Recall that these periodic orbits may exist owing to a $k : \nu$ resonance between a member γ_j^s of the continuous, conservative periodic orbit family $\{\gamma_j^s\}_{s>0}$ that forms the LSM, W_j , and the external forcing. Therefore, any given Lyapunov periodic orbit in W_j with frequency ω_j^s will be in resonance with infinitely many external forcing frequencies satisfying the resonance relationship (equation (4.4)) for some choice of the positive integers k and ν . Using the primary backbone curves defined in equation (3.5), we find that the $k : \nu$ resonance relationships in equation (4.4) lead to backbone curves satisfying the following criterion.

Theorem 1. Assume that the nonresonance condition (3.3) holds among the eigenvalues (3.2) in the $\epsilon = 0$ limit. Then, for $\epsilon > 0$ small enough, the backbone curve associated with the $k : \nu$ resonant forced response is given by

$$\Omega = b_{j,k,\nu}(A) := \frac{\nu}{k} b_{j,1,1}(A), \quad j = 1, \dots, n, \quad (5.1)$$

where

$$b_{j,1,1}(A) = \omega_j + \rho_j(A), \quad \rho_j(0) = 0, \quad (5.2)$$

is the graph of the primary backbone curve corresponding to a $1 : 1$ resonance between the forcing frequency Ω and the j^{th} natural frequency ω_j of the linearized system at the origin for $\epsilon = 0$.

This theorem implies the general scaling relationship

$$b_{j,k,\nu}(A) = \frac{\nu \ell}{k \mu} b_{j,\ell,\mu}(A), \quad k, \ell, \nu, \mu \in \mathbb{N}^+, \quad (5.3)$$

between any two members of the j^{th} generalized backbone curve family

$$\mathcal{B}_j = \left\{ (\Omega, A) : \Omega = b_{j,k,\nu}(A) = \frac{\nu}{k} b_{j,1,1}(A), \quad k, \nu \in \mathbb{N}^+ \right\}, \quad j = 1, \dots, n. \quad (5.4)$$

As an illustration, we take a hypothetical primary backbone curve $b_{j,1,1}(A)$ in figure 3 and sketch some of the scaled members of the $\mathcal{B}_j$ family consistent with the chosen $b_{j,1,1}(A)$.

Examples of primary and secondary (scaled) backbone curves for the j^{th} mode

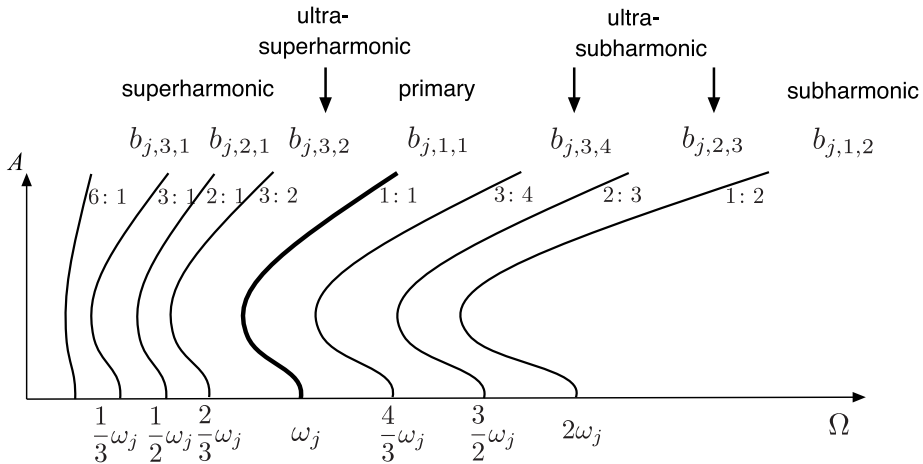


Figure 3. Geometry of the necessary resonance condition (4.4) for the case of $n = 0$ and for a fixed pair of positive integers (k, ν) . The scaled (i.e. superharmonic, ultra-superharmonic, subharmonic and ultra-subharmonic) backbone curves $b_{j,k,m}(\omega)$ were obtained from the assumed primary backbone curve $b_{j,1,1}(A)$ via the linear scaling law (equation (5.1)).

The scaled backbone curves give the undamped-unforced limits of the peaks of subharmonic, superharmonic, ultra-subharmonic and ultra-superharmonic forced-damped response in the weakly forced-damped system (2.2). Whether or not a point on these curves will indeed give rise to an observed forced response depends on the interplay between damping, forcing and the nonlinearities of the system.

To quantify that interplay, Cenedese and Haller [19,20] found that a formal energy-balance argument (i.e. the positive work done by forcing and the negative work done by damping along a Lyapunov orbit must add up to zero) can be rigorously justified to identify sections of backbone curves that give rise to actual nearby forced response. Such points must be transverse zeros of a Melnikov function that approximates the leading order change in energy for $\epsilon > 0$ along a periodic orbit of the $\epsilon = 0$ limit of the system. The forced response amplitudes will then be $\mathcal{O}(\epsilon)$ -close to the scaled backbone curves, with ϵ characterizing the magnitude of the damping and forcing acting on the system.

There are already some specific results in the literature on the actual existence of resonant forced response near backbone curves. Specifically, for the one-degree-of-freedom example of a forced-damped pendulum, Liu and Gu [21] used a second-order Melnikov function to show that the damping must be an order of magnitude weaker than the forcing for ultra-subharmonic orbits to arise. As Jing-Wu and Qing-Shi [22] discussed further, the non-existence of ultra-subharmonics had been generally suspected in the case of damping and zero-mean time-periodic forcing of the same order of magnitude. The physical argument for this is that the damping term is constantly decreasing energy, whereas the forcing term does both positive and negative work, and hence the energy-decreasing property of the damping cannot be balanced by the forcing over more than one oscillation period. Jing-Wu and Qing-Shi [22] also showed that ultra-superharmonic response under time-periodic forcing cannot exist for a general class of dynamical systems containing system (2.1), as long as the forcing is separable, i.e. $\mathbf{f}_{\text{ext}}(\mathbf{q}, \dot{\mathbf{q}}, \Omega t) = \mathbf{\Psi}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{h}(\Omega t)$, where $\mathbf{\Psi}(\mathbf{q}, \dot{\mathbf{q}})$ is an invertible matrix-valued function and $\mathbf{h}(\Omega t)$ is a $T = 2\pi/\Omega$ -periodic, vector-valued function.

All these analytic methods can be used to compute forced response curves from which the full families of active backbone curves can subsequently be inferred. These methods, however, require precise knowledge of a mechanical model, the analytic or at least numerical identification

of its Lyapunov orbit families, and the numerical evaluation of appropriate Melnikov functions. As a result, the analytic results reviewed above cannot be applied to data to predict or verify the location and shape of the backbone curve family from one known member of the family. By contrast, Theorem 1 enables such data-based prediction or verification for numerically and experimentally generated backbone curves, as we will show through examples in §6.

(b) Backbones for forced response outside LSMs

In the conservative limiting system (equation (3.1)), periodic orbits outside the LSM W_j may also exist, but their exact location and shape is specific to the given mechanical system. All non-degenerate periodic orbits of conservative systems, however, exist in smooth families because they can be smoothly continued as fixed points of a first return map from one energy surface to nearby energy surfaces (see, e.g. [23]). An explicit construction of such a subharmonic orbit family in a near-integrable, conservative example system has been given recently by Axâs *et al.* [24].

As a consequence, all non-degenerate periodic orbit families in the conservative system (3.1) fill smooth two-dimensional surfaces to which the argument we have used to deduce Theorem 1 still applies. These orbit families, however, do not generally limit on the $\mathbf{x} = \mathbf{0}$ fixed point of system (3.1) but on a nontrivial periodic orbit of the conservative system that corresponds to a zero $\mathbf{x}_{0,v}$ of equation (4.2) of the conservative system.

For this reason, the conservative backbone curves associated with non-Lyapunov periodic orbit families do not generally reach down to the zero amplitude level and their frequencies do not converge to an *a priori* known natural frequency ω_j of the linearized conservative limit at the fixed point. Consequently, while the formula

$$b_{j,k,v}(A) = \frac{v\ell}{k\mu} b_{j,\ell,\mu}(A), \quad k, \ell, v, \mu \in \mathbb{N}^+, \quad (5.5)$$

is still applicable to such non-Lyapunov periodic orbit families, the relationship (5.2) will generally not hold as these orbit families have no general relationship to the primary backbone curves.

6. Examples

As the most detailed studies of secondary backbone curves to date have been given by Volvert and Kerschen [3,17], Abeloos *et al.* [4] and Volvert [6], all examples we consider here will be taken from these publications. As we recalled in the Introduction, these authors developed a phase resonance criterion between the forcing and a given modal response to infer the location of secondary backbone curves.

Our examples will involve no additional numerical simulation. Rather, we will simply fit a smooth curve to one of the backbone curves reported in the above references, then use the scaling relationship (5.1) to predict the approximate location of other members of the backbone curve family that were independently and individually computed in the literature cited.

(a) Forced-damped Duffing oscillator

As our first example, we consider the harmonically forced and damped Duffing oscillator,

$$m\ddot{x} + c\dot{x} + a_1x + \alpha_3x^3 = \gamma \sin \omega t \quad (6.1)$$

with $m = 1$ [kg], $c = 0.1$ [kgs⁻¹], $a_1 = 1$ [Nm⁻¹] and $\alpha_3 = 1$ [Nm⁻³]. For this parameter setting, Volvert and Kerschen [3] calculated forced response curves using the harmonic balance (HB) method with 8 harmonics. They also calculate backbone curves for primary and secondary resonances using their proposed nonlinear phase-resonance criterion.

- Phase resonance backbones, obtained from HB with 8 harmonics
- Scaled backbones obtained from the 3:1 backbone curve
- 7:1 phase resonance backbone curve from HB with 31 harmonics

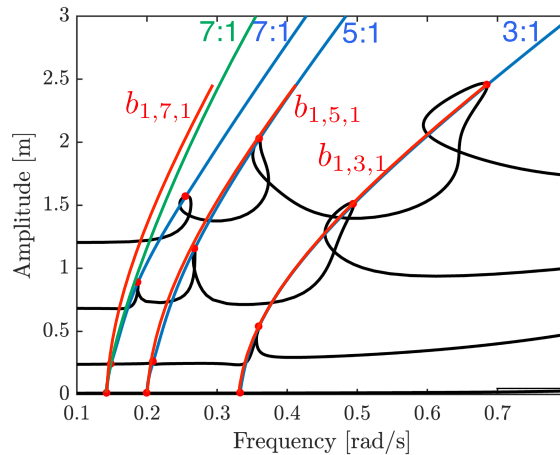


Figure 4. Graphical construction of secondary backbone curves for odd-order superharmonic resonances from the simulations of Volvert & Kerschen [3] for system (6.1). Black curves: Forced response curves for the Duffing oscillator, obtained numerically from the HB method by Volvert & Kerschen [3]. Blue curves: Primary and secondary backbone curves for the Duffing oscillator identified from the phase resonance criterion of Volvert & Kerschen [17]. Red curves: Secondary backbones constructed from the secondary 3 : 1 backbone curve scaling using the scaling relationship (5.1). Green curve: Revised computation of the 7 : 1 phase resonance backbone curve by Gaëtan Kerschen, using 31 harmonics in the HB method.

Figure 4 shows in black the forced response curves obtained by Volvert and Kerschen [3] for three different values of γ . Red dots in these curves mark local maxima of the forced response amplitude, indicating three superharmonic resonances: 3 : 1, 5 : 1 and 7 : 1. These peaks align closely with the blue backbone curves obtained from the phase-resonance criterion put forward by Volvert and Kerschen [3].

Shown in red in figure 4 is our $b_{1,3,1}$ superharmonic backbone curve, obtained as a fit to the blue 3 : 1 curve in the figure. By scaling this fitted curve in the horizontal direction by the factors 3/5 and 3/7, we obtain the scaled backbone curves $b_{1,5,1}$ and $b_{1,7,1}$, obtained from the scaling relationship (5.1).

Note that while our prediction $b_{1,5,1}$ is very close to the numerically computed blue 5 : 1 curve, our prediction $b_{1,7,1}$ noticeably departs for the blue curve marked 7 : 1. Following our inquiry to the second author of Volvert and Kerschen [3], he kindly repeated their analysis with increased precision, using 31 modes (instead of 8) in the HB calculation. Shown in green, this revised 7 : 1 curve now lies in close proximity to our predicted $b_{1,7,1}$ superharmonic backbone curve.

This is, therefore, an example of the envisioned use of our scaled backbone curve result: it provides an approximate but mathematically firm estimate that the backbone curve family of any mode must satisfy. Such estimates are quick to perform and require no actual data beyond a visual identification of a single reference backbone. They can nevertheless reveal inconsistencies owing to experimental imperfections or numerical inaccuracy in a reported backbone curve family.

As fig. 18 of Volvert and Kerschen [3] shows, even-order superharmonic resonances are not observed in the forced Duffing oscillator for arbitrarily small $\gamma > 0$ values. Indeed, as confirmed by Kerschen [25], the corresponding periodic orbits arise from a different, symmetry-breaking bifurcation in this example. This suggests that a non-Lyapunov family of periodic orbits of the unforced, conservative limit gives rise to the periodic response near these even-order resonances.

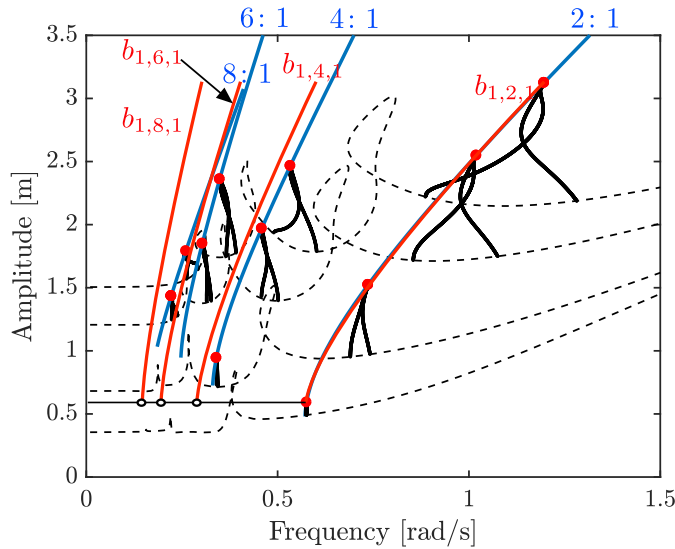


Figure 5. Same as figure 4 but for even-order superharmonic resonances of the forced Duffing oscillator.

Applying the corresponding scaling relation (5.5) with $\ell = 2$ and $\mu = 1$ to the 2 : 1 backbone curve computed by Volvert and Kerschen [3], we obtain the scaled backbone curve predictions shown in figure 5.

The theoretical base points of the predicted backbone curves in figure 5 are given by the appropriately scaled base point position of the 2 : 1 curve. As these resonances become active only for larger forcing amplitudes, the $\mathcal{O}(\epsilon)$ error in these predictions is noticeably larger than what we have seen for odd-order resonances in figure 4. The predicted shapes, however, are accurate in the present case as well. A larger difference between the predicted and computed backbone shapes is again noticeable for the 8 : 1 resonance, as for the 1 : 7 resonance in figure 4. According to Kerschen [25], this is again owing to the use of only 8 harmonics in the harmonics balance computations of Volvert and Kerschen [3].

(b) Helmholtz–Duffing oscillator

Volvert [6] considered the Helmholtz–Duffing oscillator

$$\ddot{x} + c\dot{x} + \omega_0^2 x + \alpha_2 x^2 + \alpha_3 x^3 = \gamma \sin \omega t \quad (6.2)$$

with $c = 0.1$ [1/s], $\omega_0 = 1$ [rad/s], $\alpha_2 = 1.2$ [N(kgm²)⁻¹] and $\alpha_3 = 1$ [N(kgm³)⁻¹]. He calculated the forced response using the harmonic balance method with $8 \times k$ harmonics, with k denoting the positive integer appearing in the $m : k$ relationship with backbone curve $b_{1,k,m}$. This example represents a softening-hardening oscillator: its primary backbone curve shows a sign change in its curvature under increasing forcing amplitudes.

Figure 6 shows the simulation results of Volvert [6] with our scaled backbone curve prediction superimposed in red. Apart from secondary bifurcations not emanating from the zero amplitude level, the scaled backbone curves provide a very accurate prediction for backbone locations even for larger amplitudes for the subharmonic resonances. For superharmonic resonances confined to low forcing levels, the accuracy remains high. The 2 : 1 is the only superharmonic resonance that is also observed for slightly higher amplitudes at which the scaled backbone curves start deviating from the numerical results.

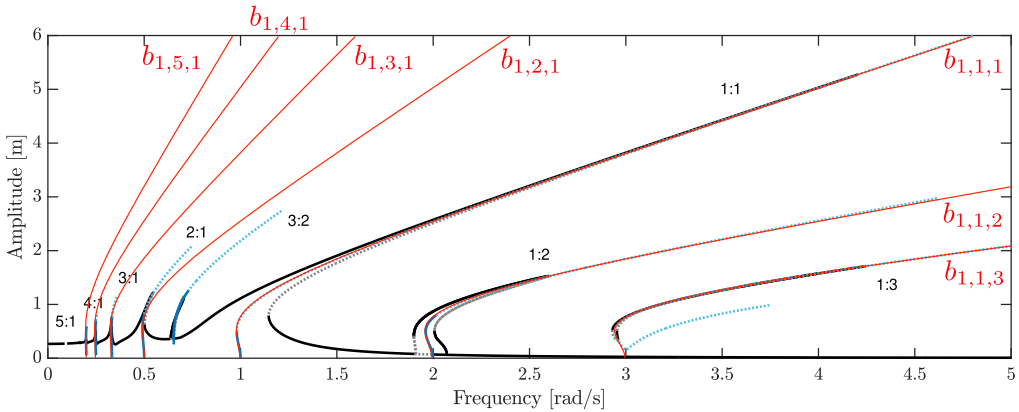


Figure 6. Graphical construction of secondary backbone curves from the simulations of Volvert [6] for system (6.2). Black curve: Forced response curve for the Helmholtz–Duffing oscillator, obtained numerically from the HB method by Volvert [6]. Dotted portions of this curve denote unstable forced response obtained from numerical continuation. Blue curves: Primary and secondary backbone curves for the Helmholtz–Duffing oscillator identified from the phase resonance criterion of Volvert and Kerschen [17]. Dotted portions of the blue curves are predicted unstable parts of the backbone curves. Red curves: Secondary backbones constructed from the primary 1 : 1 backbone curve scaling using the scaling relationship (5.1), using only the 1 : 1 backbone curve of Volvert [6] and superimposing the scaled curves in the figure.

(c) Forced-damped and coupled Duffing oscillator system

Nex, we consider a 2DOF system, the forced, coupled system of Duffing equations

$$\begin{aligned}\ddot{x}_1 + 0.02\dot{x}_1 - 0.01\dot{x}_2 + 2x_1 - x_2 + x_1^3 &= f \sin \omega t, \\ \ddot{x}_2 + 0.11\dot{x}_2 - 0.01\dot{x}_1 + 2x_2 - x_1 &= 0,\end{aligned}\quad (6.3)$$

studied by Volvert and Kerschen [17]. Figure 7 shows peaks (red) of the forced response curves of the two modes, which are connected by primary and secondary (superharmonic) backbone curves. Volvert and Kerschen [17] obtain these superharmonic backbone curves by simulating the forced response of the system for a large number of different forcing frequencies and amplitudes, then connecting the peaks of these curves. Using formula (5.1), one can immediately draw these superharmonic backbone curves without any simulation for any $m : k$ superharmonic resonance as

$$\Omega = \frac{k}{m} b_{j,1,1}(A), \quad j = 1, 2, \quad (6.4)$$

once the primary backbone curves (defined as $\Omega = b_{j,1,1}(A)$) are known, as shown in figure 7. The peaks of the forced responses generally do not lie on these secondary backbone curves but converge to the limit of zero damping and forcing (e.g. [26]). Therefore, the scaled backbone curves are only meant to be leading-order approximations to the blue and green curves in figure 7b. The red dots were computed from the phase-resonance concept of Volvert and Kerschen [17] and hence will generally also differ from the location of locally maximal amplitudes, as seen in figure 7b for the dashed line connecting the local maxima of forced response curves. The same difference is not so clearly visible in figure 7a owing to the different horizontal scaling of that plot.

(d) Forced-damped beam experiments

Our last example is taken from the experimental study of Zhou and Kerschen [5]. We choose the first of their two reported experiments (which they refer to as Beam 1) because a clearly

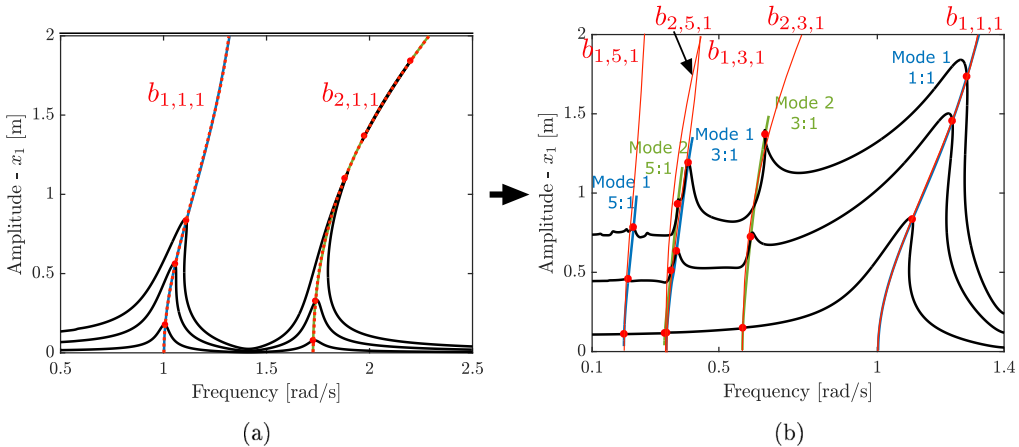


Figure 7. Graphical construction of secondary backbone curves from the simulations of Volvert & Kerschen [17] for system (6.3). (a) Primary backbone curves for the two modes, extracted as the black dotted lines laid over the blue and green curves in fig. 21a of Volvert and Kerschen [17]. (b) The 3 : 1 and 5 : 1 superharmonic backbone curves (red lines) constructed by applying the scaling relationship (5.1) to the dotted lines on the left and superimposing the results on fig. 21b of Volvert and Kerschen [17]. (An additional horizontal scaling by a factor of 1.5385 was also applied to account for the difference in the horizontal ranges of the two plots.)

extracted primary backbone curve is available for this case. The experiment is performed on a clamped–clamped, thin aluminum beam of length 0.678 m and cross section of 2 mm × 20 mm. The response of the beam is measured via an accelerometer placed near the centre of the beam. The experimental setup is shown in the photo within figure 8.

The study employs phase-locked loop testing (PLLT), which involves feedback-controlled excitation via an electromagnetic shaker to impose a user-specified phase lag between the harmonic forcing and the measured response. The controller adjusts the frequency automatically until the desired phase lags in the phase-resonance criterion are achieved. Points of the backbone curve corresponding to the primary 1 : 1 resonance and the 1 : 2 subharmonic resonance are extracted over a range of different forcing magnitudes. For more detail on the setup, we refer the reader to Zhou and Kerschen [5].

We use fig. 18a of Zhou and Kerschen [5] to extract the primary backbone curve reported in the experiments, as shown in the left subplot of figure 8. To obtain the $b_{1,1,2}$ curve, we scale the $b_{1,1,1}$ backbone horizontally by a factor of 2 and shift its base point horizontally to twice the frequency of that base point. The $s_{b_{1,1,2}}$ curve obtained in this fashion is overlaid on the appropriately shifted and scaled version of fig. 19a of Zhou and Kerschen [5]. These shifts and scalings are necessary because the ranges and the scales of the two figures are different both horizontally and vertically. As seen in the right subplot of figure 8, our predicted subharmonic backbone curve runs almost parallel to the 1 : 2 backbone curve reported by Zhou and Kerschen [5].

The difference between the two curves may be partly owing to the large zoom factor used along the frequency axis. Another factor might be experimental imperfections, because the reported 1 : 2 backbone curve does not appear to tend to the double of the inferred natural frequency (approx. 26.2 Hz) at zero forcing amplitude. This is only possible theoretically if the 1 : 2 periodic response does not bifurcate from the Lyapunov orbit family, but arises from another periodic orbit family of the conservative limit. Such a nontrivial subharmonic resonance phenomenon was uncovered recently by Axås *et al.* [24] in the nonlinear dynamics of fluid sloshing under harmonic excitation.

In summary, our simple scaled backbone curve analysis of this experiment signals either potential experimental imperfections and/or the presence of a non-Lyapunov orbit family behind the 1 : 2 subharmonic backbone reported by Zhou and Kerschen [5]. Either way, the observed

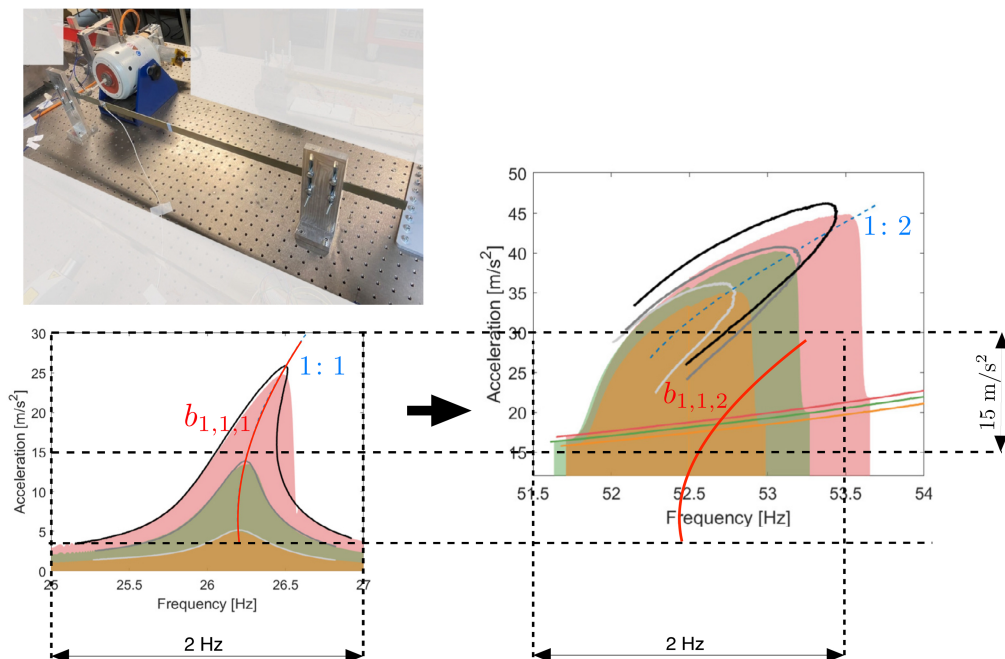


Figure 8. Experimental damped-clamped beam setup and the graphical construction of the 1 : 2 subharmonic backbone curve $b_{1,1,2}$ (red) from the experimentally reported $b_{1,1,2}$ backbone curve of Zhou and Kerschen [5]. Also shown on the right in blue the 1 : 2 backbone curve identified by the authors from the phase-resonance criterion. The background plot on the right, adopted from fig. 19a Zhou and Kerschen [5], has been scaled to match the axis scales of their fig. 18a. See the text for more detail.

discrepancy between an otherwise already validated theoretical approximation and the experimental results indicates that further analysis of this experimental setting might be in order.

7. Conclusions

Using the existence of Lyapunov subcentre manifolds in the conservative limit of forced-damped mechanical systems, we have obtained a simple graphical scaling result that relates sub- and super-harmonic backbone curves to the primary backbone curve emanating from a natural frequency of the system. As a consequence, knowing just a single (primary or secondary) member of the family of backbone curves is enough to obtain leading-order approximations for all other members of the same family that extend to the zero amplitude response limit. Our arguments leading to this result assume small damping and forcing, as is often the case in structural dynamics.

We have illustrated the accuracy of our predictions for secondary backbone curves on examples taken from Volvert and Kerschen [17] and Volvert [6]. Specifically, using only the numerically generated primary backbone curves from these studies, we have predicted their numerically generated secondary backbone curves with high accuracy without using any numerical simulation. In one case, the noticeable discrepancy between our prediction and the available numerical computation has led to the identification of insufficient resolution in the original, published results.

Our scaled backbone curve results also extend to families of periodic responses that arise from periodic orbits families of the conservative limit that are not contained in Lyapunov subcentre manifolds. We have illustrated this with the even-order subharmonic resonance computations of Volvert and Kerschen [17] for the forced Duffing equation.

The simple scaling results derived here for secondary backbone curves should be useful when their detailed experimental identification would be time-consuming or expensive (see [5] for a related experimental technique). In that case, available resources can be focused on finding the primary backbone curve accurately, from which our results lead to the immediate graphical construction of the secondary backbones. These approximate theoretical predictions can then be used as approximate benchmarks for numerical and experimental algorithms that seek to extract such backbone curves with higher accuracy.

Data accessibility. This article has no additional data.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. G.H.: conceptualization, formal analysis, funding acquisition, investigation, methodology, project administration.

Conflict of interest declaration. We declare no competing interests.

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