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Blow-up of Koopman eigenfunctions at basin boundaries

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We show that under non-degeneracy conditions on a dynamical system, all of its smooth, non-constant Koopman eigenfunctions blow up along the boundary of the basin of attraction of a fixed point. We give a simple, geometric version of this general result for the case of a basin boundary composed of stable manifolds of edge states. We also give a more technical version of this blow-up result that is applicable to more general basin boundaries. These findings improve available results on the necessary blow-up of at least one Koopman eigenfunction and clarify that examples showing non-trivial bounded Koopman eigenfunctions across basin boundaries are degenerate. Our findings also imply that reduced-order models based on approximating Koopman-invariant subspaces of a typical physical system necessarily break down before reaching any truly nonlinear (i.e. nonlinearizable) behaviour away from a fixed point. Furthermore, Koopman operator-based reduced models, as well as their data-driven numerical approximations, over different domains of attraction cannot be continuously patched together as often envisioned.

1. Introduction

Originally introduced by Koopman [1], the Koopman operator maps values of an observable quantity at initial conditions of an autonomous dynamical system to later values of the same observable along trajectories starting from those initial conditions. The eigenfunctions of this operator, if and where they exist, are special observables whose time evolution obeys a linear system of differential equations.

Passing to subspaces of the observable space spanned by Koopman eigenfunctions has been broadly suggested

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as a way to reduce the study of the original nonlinear dynamical system to that of a linear one, thereby enabling easier system identification, analysis and control (e.g. Mezić [2], Brunton *et al.* [3] and the references cited therein). Based on this expectation, a number of data-driven numerical algorithms have been developed to approximate Koopman eigenfunctions using combinations of other observables (e.g. Schmid [4] for a review).

Amidst these developments, however, a broadly held misunderstanding has also emerged. According to this view, by simply passing to the basis of its Koopman eigenfunctions, one can globally linearize even an intrinsically nonlinear system. An intrinsically nonlinear (or nonlinearizable) dynamical system is one that is not topologically conjugate to any linear system; that is, its trajectories cannot be mapped by a homeomorphism into those of a linear system. For instance, a nonlinear system with multiple isolated limit sets (such as isolated fixed points) cannot be topologically conjugate to a linear system, that is, cannot be mapped into any linear system of equal dimension in a one-to-one and continuous fashion. One might still be able to *embed* such a nonlinear system into a larger, degenerate linear system with a subspace of fixed points, as the nifty examples of Arathoon & Kvalheim [5] show. Such a linearizing embedding, however, is only possible for non-generic systems and will result in degenerate (structurally unstable) linear models. Indeed, as Arathoon & Kvalheim [5] note, no nonlinear system with a distinct domain of attraction or repulsion, or with transitions from one state to another, can be continuously embedded into a linear system. This is confirmed by the general results of Liu *et al.* [6,7], who show the non-existence of continuous linearizing embeddings for systems with multiple limit sets and pre-compact forward trajectories. All these publications seek to eliminate the confusion surrounding the ability of the Koopman operator in rendering a linear reduced model for global nonlinear dynamics.

The above findings in the literature lead to the plausible expectation that Koopman eigenfunctions of typical systems cannot be extended to domains with nonlinearizable dynamics. To our knowledge, the first detailed result confirming this expectation for a specific system is due to Page & Kerswell [8], who showed the blow-up (i.e. lack of uniform boundedness) of Koopman eigenfunctions at basin boundaries in a simple, one-dimensional system with two fixed points. More recently, Haller & Kaszás [9] showed for arbitrary nonlinear systems that at least one Koopman eigenfunction must necessarily blow up at (i.e. take arbitrarily large values near) each point of the basin boundary of a stable hyperbolic fixed point. More general results by Kvalheim & Arathoon [10] imply that linearizing embeddings, as proper maps of the domain of attraction into a linear space, must necessarily blow up along basin boundaries.

All these results, however, do not imply that *all* Koopman eigenfunctions must blow up along basin boundaries. Indeed, it is not hard to construct examples in which only one Koopman eigenfunction blows up but not the rest (see §3 for such examples). This, in principle, raises the possibility that at least some of the Koopman-invariant subspaces may be used to model non-trivial dynamics across boundaries between domains of attraction. Here, we show that under non-degeneracy conditions on a dynamical system, *all* its principal Koopman eigenfunctions blow up along basin boundaries. Specifically, for basin boundaries composed of stable manifolds of normally hyperbolic invariant sets (or edge states), our first blow-up result, theorem 1, only assumes non-resonance conditions on the spectrum of the fixed point contained by the basin and a non-intersection between primary co-spectral submanifolds (co-SSMs) of eigenspaces of the fixed point and the unstable manifolds of saddle-type edge states embedded in the basin boundary. In practice, the latter intersection conditions are only violated if a part of the system is already fully linear or decoupled from the rest, as we shall see in examples. For more general basin boundaries, theorem 2 gives a more technical result that relies on the local, near-equilibrium behaviour of trajectory sequences whose initial conditions approach the basin boundaries.

Kvalheim & Revzen [11] show that all C^∞ Koopman eigenfunctions are products of powers of principal eigenfunctions. Therefore, for typical systems, our results imply that all reduced models within subspaces of non-constant Koopman eigenfunctions (or their sufficiently accurate numerical approximations) necessarily break down as they approach the boundary of linearizable dynamics. These conclusions on the typical blow-up of Koopman eigenfunctions can be used to

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assess the feasibility of numerical procedures that seek to approximate such eigenfunctions. For instance, numerical procedures claiming to construct bounded approximations to Koopman eigenfunctions across basin boundaries of generic systems must be erroneous or must be carrying out something other than what their authors state. As an example, a closer inspection of the extended dynamic mode decomposition (EDMD)-based code of Garcia-Tenorio & Vande Wouwer [12] for Koopman eigenfunction construction reveals a misunderstanding of the EDMD algorithm.

Specifically, the EDMD procedure postulates that a set of observables and their polynomial powers together form a variable whose time evolution obeys a linear system of ordinary differential equations (ODEs). In contrast, the authors of [12] evaluate those polynomial powers of the observables during a fitting procedure, and hence ultimately fit a nonlinear polynomial system of ODEs to the evolution of the original set of observables. Such polynomial ODEs can indeed approximate nonlinear systems with multiple attractors, but have little to do with the eigenfunctions of the Koopman operator. Rather, they are essentially implementations of the sparse identification of nonlinear dynamics procedure of Brunton *et al.* [13] without any sparsification. Buzhardt *et al.* [14] also point out this type of misinterpretation of the EDMD procedure and cite further examples of this approach.

An exact evaluation of the geometric non-degeneracy conditions of our theorem 1 requires the knowledge of the stable manifolds of saddle-type edge states and the primary co-SSMs of eigenspaces at the attracting fixed point. Both of these ingredients can be numerically computed when the governing equations of the systems are known, and primary co-SSMs for a known fixed point can also be located in a purely data-driven setting using the SSMLearn algorithm (see Cenedese *et al.* [15] and Haller [16]). By contrast, non-trivial or higher-dimensional edge states are not readily identifiable for dynamical systems defined by datasets. Our non-degeneracy conditions, however, hold in typical systems, because they involve non-resonance conditions and non-intersection conditions that hold with probability one in generic dynamical systems. Using several simple examples, we illustrate the types of degeneracies that a nonlinear dynamical system would need to have for these non-degeneracy conditions to fail.

2. Main results

(a) Geometric blow-up criterion

We first give an easily interpretable geometric blow-up criterion that is applicable to the most commonly observed basin boundary types. We consider the system of ODEs

$$\dot{x} = Ax + f(x), \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}, \quad f \in C^r, \qquad f(x) = \mathcal{O}\left(|x|^2\right), \tag{2.1}$$

for some integers $n \geq 1$ and $r \geq 2$. We denote solutions of this ODE starting from x_0 at time $t = 0$ by $x(t; x_0)$. The flow map of the system is then defined as $F^t : \mathbb{R}^n \to \mathbb{R}^n$, $F^t(x_0) = x(t; x_0)$.

We assume for simplicity that A is diagonalizable and denote its eigenvalues by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, with corresponding unit right eigenvectors $e_1, \ldots e_n \in \mathbb{C}^n$ and unit left eigenvectors $w_1, \ldots w_n \in \mathbb{C}^n$. We collect these eigenvectors in the matrices

$$T = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix}, \qquad W = T^{-1} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \qquad T, W \in \mathbb{C}^{n\times n}. \tag{2.2}$$

We define the kth real eigenspace E_k as the span of the real and imaginary parts of the eigenvector e_k. Since A is assumed diagonalizable, the dimension of E_k is equal to the multiplicity of the eigenvalue λ_k when λ_k is real. If λ_k is complex, then the dimension of E_k is equal to twice the multiplicity of λ_k. In either case, the subspace E_k is invariant under the dynamics of the linearized system $\dot{x} = Ax$. For later purposes, the direct sum of all but one eigenspace, E_k, will have a special

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significance. We denote such a *co-spectral subspace* for E_k as

$$\mathcal{E}_k = E_1 \oplus \cdots \oplus E_{k-1} \oplus E_{k+1} \oplus \cdots \oplus E_n. \tag{2.3}$$

Note that $\mathcal{E}_k$ is an invariant subspace for the linearized dynamics $\dot{x} = Ax$ and $E_k \oplus \mathcal{E}_k = \mathbb{R}^n$. If $\mathcal{E}_k$ is the empty set (i.e. if $E_k = \mathbb{R}^n$), then we follow the usual convention for the codimension of an empty set:

$$\mathcal{E}_k = \emptyset \quad \Longrightarrow \quad \operatorname{codim} \mathcal{E}_k = \infty. \tag{2.4}$$

We now assume that the origin is a stable hyperbolic fixed point for system (2.1) and the nonlinearities are smooth enough with respect to the spectrum of A, that is,

$$\operatorname{Re}\lambda_n \leq \cdots \leq \operatorname{Re}\lambda_1 < 0, \qquad r > L := \left\lfloor \frac{\max_j \left|\operatorname{Re}\lambda_j\right|}{\min_j \left|\operatorname{Re}\lambda_j\right|} \right\rfloor, \tag{2.5}$$

with $\lfloor a \rfloor$ denoting the integer part of a real number $a > 0$. Under this assumption, the origin has an open domain of attraction $\mathcal{A}$ with a boundary $\partial\mathcal{A}$.

Finally, we assume the following non-resonance conditions for the spectrum of A:

$$\sum_{j=1}^{n} k_j \lambda_j \neq \lambda_\ell, \qquad \ell = 1, \ldots, n, \qquad k_1, \ldots, k_n \in \mathbb{N}, \qquad 2 \leq \sum_{j=1}^{n} k_j \leq L. \tag{2.6}$$

Under assumptions (2.5) and (2.6), the C^r linearization theorem of Sternberg [17] guarantees the existence of a local C^r diffeomorphism,

$$y = h(x), \qquad x \in U \subset \mathbb{R}^n, \tag{2.7}$$

that transforms the nonlinear system (2.1) into its linear part,

$$\dot{y} = Ay, \tag{2.8}$$

in a neighbourhood U of the origin.

For any integer $k \in [1, n]$, the transformation h^{-1} maps the co-spectral subspace $\mathcal{E}_k$ defined in equation (2.3) into a locally defined C^r invariant manifold,

$$W_{\text{loc}}\left(\mathcal{E}_k\right) = h^{-1}\left(\mathcal{E}_k\right),$$

of the nonlinear system (2.1). This invariant manifold can be extended into a globally defined C^r invariant manifold within the basin of attraction $\mathcal{A}$ under the inverse flow map:

$$W\left(\mathcal{E}_k\right) = \cup_{t \leq 0} F^t\left(W_{\text{loc}}\left(\mathcal{E}_k\right)\right) \subset \mathcal{A}, \qquad \dim W\left(\mathcal{E}_k\right) = \dim \mathcal{E}_k, \qquad k \in [1, n]. \tag{2.9}$$

By the results of Cabré *et al.* [18], conditions (2.5) also guarantee that for all $k \in \{1, \ldots, n\}$ $W\left(\mathcal{E}_k\right)$ is the unique C^{L+1} invariant manifold of dimension $\dim \mathcal{E}_k$ that is tangent to $\mathcal{E}_k$ at $x = 0$. For this reason, we refer to $W\left(\mathcal{E}_k\right)$ as the *primary* co-SSM associated with the eigenspace E_k. Secondary (or fractional) co-SSMs also exist with similar invariance, tangency and dimensionality properties, but they are always less smooth than $W\left(\mathcal{E}_k\right)$ (see Haller [16] for more detail). In figure 1, we sketch the geometry of the slowest spectral subspace E_1, its co-spectral subspace $\mathcal{E}_1$, the primary spectral submanifold $W(E_1)$ and the primary co-SSM $W\left(\mathcal{E}_1\right)$.

Another invariant manifold concept we need to state for our main result is the notion of an edge state, which is motivated by the concept of an exact coherent state (ECS) from fluid mechanics (e.g. Waleffe [19]). The stable manifolds of ECSs have generally been observed to act as basin boundaries between stable laminar states and turbulence (e.g. Skufca *et al.* [20], Schneider *et al.*

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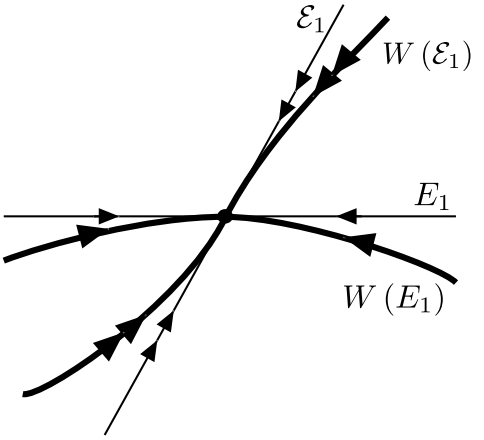


Figure 1. The slowest real spectral subspace E_1, its co-spectral subspace $\mathcal{E}_1$, the primary spectral submanifold $W(E_1)$ and the primary co-SSM $W\left(\mathcal{E}_1\right)$.

[21], and Kaszás & Haller [22]). For some integer $m \in [0, n-1]$, an m-dimensional *edge state*, $\mathcal{S} \subset \partial\mathcal{A}$, of system (2.1) is defined here as an m-dimensional, compact, normally hyperbolic invariant manifold, $\mathcal{S} \subset \partial\mathcal{A}$. Some examples of edge states, labelled as $\mathcal{S}_j$, are shown in figure 2.

Basin boundaries are typically formed by unions of edge states and stable manifolds of edge states. We assume this to be the case for system (2.1). Specifically, we assume that for some integer $q > 0$, the basin boundary $\partial\mathcal{A}$ is the union of edge states $\mathcal{S}_1, \ldots, \mathcal{S}_q$ and their stable manifolds:

$$\partial\mathcal{A} = \bigcup_{i=1}^{q} W^s\left(\mathcal{S}_i\right). \tag{2.10}$$

In this case, since $\partial\mathcal{A}$ is a codimension-one surface, $\mathcal{S}$ must have an $(m+1)$-dimensional unstable manifold $W^u(\mathcal{S})$. Some examples of typical basin boundaries satisfying this assumption are shown in figure 2. A non-generic example violating assumption (2.10) would be a basin boundary composed of fixed points.

By definition, we always have $\mathcal{S}_i \subseteqq W^s\left(\mathcal{S}_i\right)$ because all trajectories in $\mathcal{S}_i$ trivially converge to $\mathcal{S}_i$ in backward time. However, it may specifically happen that $\mathcal{S}_i \equiv W^s\left(\mathcal{S}_i\right)$, as is the case for $\mathcal{S}_2$ in figure 2c. Some examples of basin boundaries illustrate this observation in figure 2.

We also define the *punctured basin boundary*, $\hat{\partial}\mathcal{A}$, which we obtain by removing the edge states that have no open domain of attraction within $\partial\mathcal{A}$:

$$\hat{\partial}\mathcal{A} = \partial\mathcal{A} - \cup_{j\in I}\mathcal{S}_j, \qquad I = \left\{i \in [1,q] \,:\, \mathcal{S}_i = W^s\left(\mathcal{S}_i\right) \subsetneq \partial\mathcal{A}\right\}. \tag{2.11}$$

This definition ensures that $\hat{\partial}\mathcal{A}$ is always non-empty, that is, edge states forming the full basin boundary are not removed from $\partial\mathcal{A}$. Note that $\partial\mathcal{A}$ and $\hat{\partial}\mathcal{A}$ only differ over the set $\cup_{j\in I}\mathcal{S}_j$, whose relative measure within $\partial\mathcal{A}$ is zero. For instance, for the geometry shown in figure 2c, we have $\hat{\partial}\mathcal{A} = \partial\mathcal{A} - \mathcal{S}_2$.

Following the general definition of Kvalheim & Revzen [11], we define a principal Koopman eigenfunction $\varphi : \mathcal{A} \to \mathbb{C}$ with eigenvalue $\lambda \in \mathbb{C}$ on the domain of attraction $\mathcal{A}$ as a C^1 function that satisfies

$$\varphi\left(x\left(t;x_0\right)\right) = e^{\lambda t}\varphi\left(x_0\right), \quad x_0 \in \mathcal{A}, \quad t \in \mathbb{R}, \qquad \varphi(0) = 0, \qquad D_x\varphi(0) \neq 0. \tag{2.12}$$

Under appropriate smoothness and non-resonance conditions, Kvalheim & Revzen [11] showed the existence and uniqueness of n independent eigenfunctions for systems of the form equation (2.1) with diagonalizable, non-resonant linear parts.

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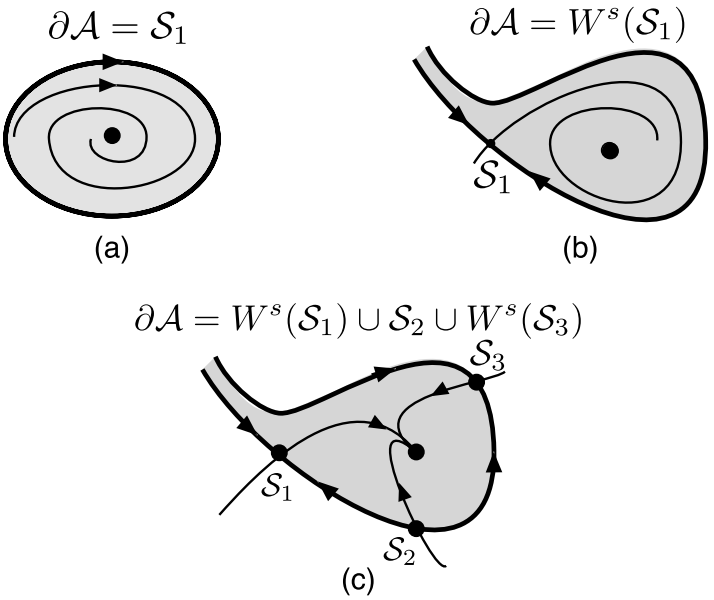


Figure 2. Examples of typical basin boundaries satisfying assumption (2.10). (a) limit cycle. (b) stable manifold of a fixed point. (c) chain of stable manifolds of fixed points with $W^s\left(\mathcal{S}_2\right) = \mathcal{S}_2$.

Using the C^2 linearization theorem of Hartman [23], Lan & Mezić [24] explicitly constructed n principal Koopman eigenfunctions $\varphi_1(x), \ldots, \varphi_n(x) : \mathcal{A} \mapsto \mathbb{C}$ on $\mathcal{A}$ with their corresponding eigenvalues coinciding with those of A:

$$\varphi_k\left(x\left(t; x_0\right)\right) = e^{\lambda_k t}\varphi_k\left(x_0\right), \quad x_0 \in \mathcal{A}, \quad t \in \mathbb{R}. \tag{2.13}$$

The following theorem states that under non-degeneracy and smoothness conditions on system (2.1), all eigenfunctions in equation (2.13) blow up along the punctured basin boundary $\hat{\partial}\mathcal{A}$.

Theorem 1. ***[Blow-up of Koopman eigenfunctions along basin boundaries]*** *Assume that conditions (2.5) and (2.6) are satisfied. Assume further that for some $k \in \{1, \ldots, n\}$, the primary co-SSM, $W\left(\mathcal{E}_k\right)$, does not intersect the unstable manifold of any edge state within $\hat{\partial}\mathcal{A}$, that is,*

$$W\left(\mathcal{E}_k\right) \cap \bigcup_{i \notin I} W^u\left(\mathcal{S}_i\right) = \emptyset. \tag{2.14}$$

*Then, the norm $\left|\varphi_k(x)\right|$ of the k*th *principal Koopman eigenfunction $\varphi_k(x)$ takes arbitrarily large values in arbitrarily small neighbourhoods of the punctured basin boundary $\hat{\partial}\mathcal{A}$.*

Proof. See appendix A of the supplementary material. ■

Several remarks are in order related to theorem 1.

Remark 1. **[Implication for general Koopman eigenfunctions]** Kvalheim & Revzen [11] show that any C^∞ Koopman eigenfunction is a finite linear combination of products of integer powers of principal eigenfunctions and their conjugates. Therefore, principal eigenfunctions are the important ones to find, as all other smooth eigenfunctions can be generated from them. Specifically, if all principal eigenfunctions blow up at domain boundaries, then all non-constant C^∞ eigenfunctions also blow up at basin boundaries under the non-resonance condition (2.6). Indeed, under the latter condition, individual members in the linear combinations of integer powers of principal eigenfunctions and their conjugates cannot cancel each other out. This follows from eqn (67) of appendix A of the supplementary material, which shows all principal eigenfunctions are products

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of an exponential (with its exponent in the spectrum of A) and a uniformly bounded factor.

Remark 2. **[The blow-up of φ_k does not follow trivially by definition]** It might be tempting to argue that assumption (2.5) and definition (2.12) of the Koopman eigenfunctions imply theorem 1 without any further assumption if one simply takes the $t \to -\infty$ limit in equation (2.12). Note, however, that in that limit, a typical trajectory $x\left(t; x_0\right)$ will *not* converge to a point of the basin boundary but will tend to grow unbounded (e.g. our examples 1 and 2 in §3). Another notable example is the damped Duffing equation $\dot{x} = y$, $\dot{y} = -y + x - x^3$, in which only a single orbit in either domain of attraction converges to a basin boundary point, the origin. Along all other orbits in these examples, the Koopman eigenfunctions only grow unbounded over unbounded domains. This by itself would not be an obstruction to using them for model reduction of nonlinearizable phenomena arising over a compact domain of the phase space.

Remark 3. **[Extension to basins of repulsion for unstable fixed points]** A result similar to theorem 1 holds for the typical blow-up of Koopman eigenfunctions along basins of repulsion of unstable hyperbolic fixed points. Specifically, one has to replace assumption (2.5) with

$$\operatorname{Re}\lambda_n \geq \cdots \geq \operatorname{Re}\lambda_1 > 0, \tag{2.15}$$

and replace assumption (2.14) with

$$W\left(\mathcal{E}_k\right) \cap \bigcup_{i \notin I}^{q} W^s\left(\mathcal{S}_i\right) = \emptyset, \qquad I = \left\{i \in [1,q] \,:\, \mathcal{S}_i = W^u\left(\mathcal{S}_i\right) \subsetneq \partial\mathcal{A}\right\}, \tag{2.16}$$

for the stable manifolds $W^s\left(\mathcal{S}_i\right)$ of edge states $\mathcal{S}_i \subset \hat{\partial}\mathcal{A}$. In this case, the punctured basin of repulsion boundary, $\hat{\partial}\mathcal{A}$, is defined as the union of unstable manifolds of edge states with an open domain of repulsion within $\mathcal{A}$. In that case, the proof of theorem 1 can be repeated in backward time.

Remark 4. **[Extension to iterated maps]** A result similar to theorem 1 can be concluded for Koopman eigenfunctions defined in the basin of attraction or repulsion of hyperbolic limit cycles. This follows by applying the versions of the Hartman [23] and Sternberg's [17] linearization results formulated for maps with stable hyperbolic fixed points. The necessary existence results on principal Koopman eigenfunctions for fixed points of maps are available from Lan & Mezić [24] and Kvalheim & Revzen [10].

Remark 5. **[Extension to time-periodic systems]** The extension noted in remark 4 also enables an extension to basin boundaries of stable or unstable periodic orbits that are fixed points of a period-T Poincaré map of a T-periodic non-autonomous system. A similar extension follows for fixed points of a first-return map defined on a codimension-one section transverse to a periodic orbit of an autonomous dynamical system.

Remark 6. **[Extension to basins of attractions or repulsions of general invariant sets]** Using the extended linearization results of Eldering *et al.* [25], similar results could, in principle, be obtained for basin boundaries of more general invariant sets. These linearization results, however, rely on the existence of an appropriate local linearization near the invariant set, which only exists under very restrictive conditions. Even for invariant quasiperiodic tori, a Floquet-type local linearization result is generally unattainable (see Sell [26]).

Remark 7. **[Significance of the non-resonance conditions]** The non-resonance conditions (2.6) are satisfied with probability one within the family of general dynamical systems. (The probability measure in this statement is the Lebesgue measure normalized to 1 over any finite volume of

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interest in $\mathbb{R}^{n\times n}$, the space of potential entries for the matrix A.) Under conditions (2.6), the primary co-SSM, $W\left(\mathcal{E}_k\right)$, can be uniquely defined and used in the main condition (2.14) of theorem 1. Even in the case of a resonance, however, one can obtain a result similar to theorem 1 by using a C^2 version of the linearizing transformation h in equation (2.7) that is due to Hartman [23]. This C^2 linearization theorem requires no non-resonance conditions. The co-SSM, $W\left(\mathcal{E}_k\right)$, as the preimage of $\mathcal{E}_k$ under Hartman's linearization, can then be defined as

$$W\left(\mathcal{E}_k\right) = \cup_{t\leq 0} F^t\left(h^{-1}\left(\mathcal{E}_k\right)\right),$$

and used in condition (2.14) of theorem 1. While $W\left(\mathcal{E}_k\right)$ is not known to be the unique smoothest co-SSM in this case, its intersection with $\bigcup_{i\notin I} W^u\left(\mathcal{S}_i\right)$ is generally still empty, and hence theorem 1 still implies the blow-up of φ_k along the punctured basin boundary.

Remark 8. **[Robustness of the intersection** condition (2.14)**]** If an edge state $\mathcal{S}_i$ is m-dimensional and the real eigenspace $E_k \subsetneq \mathbb{R}^n$ is d-dimensional, then

$$\operatorname{codim} W\left(\mathcal{E}_k\right) + \operatorname{codim} W^u\left(\mathcal{S}_i\right) = d + (n - (m+1)) = n + d - m - 1. \tag{2.17}$$

A robust (i.e. transverse) intersection between $W\left(\mathcal{E}_k\right)$ and $W^u\left(\mathcal{S}_i\right)$ is only possible if their intersection set is at least one-dimensional, that is, the codimension of their intersection set is less than n. Since the codimension of a transverse intersection is the sum of the codimensions of the intersection sets (e.g. Guillemin & Pollack [27]), equation (2.17) gives the necessary condition for a transverse intersection along at least one (one-dimensional) trajectory in the form $n + d - m - 1 \leq n - 1$, or, equivalently,

$$d < m + 1. \tag{2.18}$$

This inequality gives a condition under which the main assumption (2.14) of theorem 1 can be violated in a robust fashion, that is, in a way that is persistent under generic small perturbations to the nonlinear system (2.1). We note that, by equation (2.4), we have

$$W\left(\mathcal{E}_k\right) = \emptyset \quad \Longrightarrow \quad d = \infty, \tag{2.19}$$

for the purposes of evaluating the inequality (2.18). In that case, the inequality (2.18) always fails and hence no structurally stable violation of the main assumption of theorem 1 is possible (e.g. example 3 of §3).

As an example to illustrate inequality (2.18), assume that the eigenspace E_k is one-dimensional ($d = 1$) and the edge state $\mathcal{S}_i$ is a fixed point ($m = 0$), then inequality (2.18) is violated and hence any potential violation of assumption (2.14) is non-generic, that is, can be removed by arbitrarily small perturbations to the original system (2.1).

As another example, assume that the real eigenspace E_k is two-dimensional ($d = 2$), corresponding to a complex conjugate pair of eigenvalues. Then, the edge state $\mathcal{S}_i$ would have to be at least two-dimensional ($m = 2$) to satisfy the requirement of inequality (2.18). Even then, there is no *a priori* reason for inequality (2.18) to be violated, that is, there is no reason why $W\left(\mathcal{E}_k\right)$ and $\bigcup_{i\notin I} W^u\left(\mathcal{S}_i\right)$ should intersect. However, should they intersect transversely, their intersection is not removable by small perturbations to the nonlinear system (2.1). Even if such a robust intersection occurs, the codimension of the intersection set equals $n + 2 - m - 1$. Consequently, the maximal codimension of the intersection set is 2, which is attained when $m = n - 1$. Excluding a compact neighbourhood of such an intersection set within the set $C_\delta\left(\tilde{\mathcal{S}}_i\right)$ used in the proof of theorem 1, the same proof would still apply. As a result, the blow-up of $\varphi_k(x)$ can still be concluded with the exception of a measure zero subset of the punctured basin boundary $\hat{\partial}\mathcal{A}$. In theorem 2, we give a more general (but less intuitive) technical condition under which the blow-up of $\varphi_k(x)$ can still be concluded in such cases, even along the full basin boundary.

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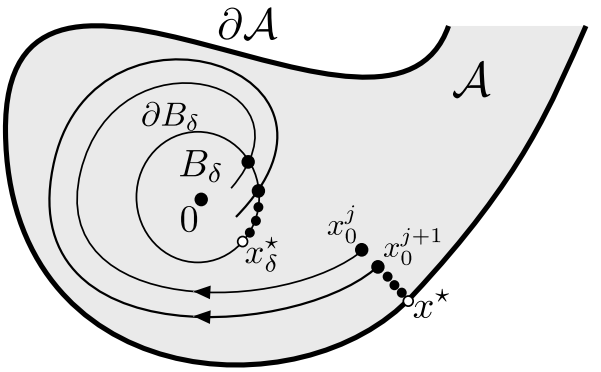


Figure 3. The basin of attraction $\mathcal{A}$ of the origin, its boundary $\partial\mathcal{A}$ and the ball B_δ. Also shown is a sequence $\left\{x_0^j\right\}_{j=1}^{\infty}$ converging to a basin boundary point $x^\star \in \partial\mathcal{A}$. The forward trajectories starting from elements of this sequence will have a sequence of first intersection points with ∂B_δ with at least one point of accumulation, $x_\delta^\star$.

(b) Analytic blow-up criterion

We now give a more technical version of the results in theorem 1, that is, applicable to more general basin boundaries and where we do not necessarily assume non-resonance conditions on the underlying stable fixed point. The conditions obtained here will be more technical, but still applicable to all of our examples without explicit calculations. In some examples, they will be seen to yield blow-up of Koopman eigenfunctions for the full basin boundary $\mathcal{A}$, not just for the punctured basin boundary $\hat{\partial}\mathcal{A}$ covered by theorem 1. In other examples, they will yield eigenfunction blow-up without the non-resonance assumptions required by theorem 1.

Let B_δ be a small closed ball of radius $\delta > 0$ centred at the origin. For a basin boundary point $x^\star \in \partial\mathcal{A}$, we consider a sequence of initial conditions, $\left\{x_0^j\right\}_{j=1}^{\infty} \subset \mathcal{A}$ that converges to $x^\star$, as shown in figure 3. As seen in the figure, trajectories starting from the initial condition sequence $\left\{x_0^j\right\}_{j=1}^{\infty}$ intersect the boundary ∂B_δ of B_δ in an infinite sequence of points. Since this intersection sequence is bounded, it has at least one point of accumulation, marked as $x_\delta^\star$ in figure 3. In general, there may be more points of accumulation for this intersection sequence. We denote by $C_\delta(x^*) \subset \partial B_\delta$ the set of *all* such accumulation points for *all* possible intersection sequences whose initial condition sequence satisfies $\left\{x_0^j\right\}_{j=1}^{\infty} \subset \mathcal{A}$ and $\lim_{j\to\infty} x_0^j = x^\star$.

The following theorem states that under non-degeneracy and smoothness conditions on system (2.1), all principal Koopman eigenfunctions blow up at all basin boundary points $x^\star \in \partial\mathcal{A}$.

Theorem 2. ***[Blow-up of Koopman eigenfunctions along basin boundaries]*** *Consider a basin boundary point $x^\star \in \partial\mathcal{A}$ and an integer $k \in \{1, \ldots, n\}$. Assume that one of the following two non-degeneracy conditions holds:*

(i)

$$\liminf_{\delta\to 0} \sup_{x_\delta^\star \in C_\delta(x^*)} \frac{\left|w_k \cdot x_\delta^\star\right|}{\delta} > 0. \tag{2.20}$$

(ii) *$f \in C^r$ with $r \geq 2$ satisfying*

$$r > \frac{\max_j \left|\mathrm{Re}\,\lambda_j\right|}{\min_j \left|\mathrm{Re}\,\lambda_j\right|}, \qquad \sum_{2\leq|\mathbf{k}|\leq r} k_j\lambda_j \neq \lambda_\ell, \qquad \ell = 1, \ldots, n, \tag{2.21}$$

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$$\lim_{\delta\to 0}\sup_{x_\delta^\star\in C_\delta(x^*)}\frac{\left|w_k\cdot x_\delta^\star\right|}{\delta}=0,\qquad \liminf_{\delta\to 0}\sup_{x_\delta^\star\in C_\delta(x^*)}\frac{1}{\delta^2}\left|\sum_{|\mathbf{k}|=2}\frac{w_k\cdot\partial_{z^\mathbf{k}}\left[f(Tz)\right]\big|_{z=0}\left(Wx_\delta^\star\right)^\mathbf{k}}{\mathbf{k}!\left(\lambda_k-\sum_{\ell=1}^n k_\ell\lambda_\ell\right)}\right|>0, \tag{2.22}$$

with the notation $\mathbf{k}=(k_1,\dots,k_n)\in\mathbb{N}^n$ *and* $\mathbf{k}!=k_1!\cdots k_n!$, *and with* $z^\mathbf{k}:=z_1^{k_1}\cdots z_n^{k_n}$ *denoting a scalar monomial built of the coordinates of the vector* $z=Wx$.

Then the norm $|\varphi_k(x)|$ *of the kth principal Koopman eigenfunction* $\varphi_k(x)$ *takes arbitrarily large values arbitrarily close to the basin boundary point* $x^\star$.

Proof. See appendix B of the supplementary material. ∎

The conditions of theorem 2 are less intuitive than those of theorem 1 but may also be more effective. Specifically, in some of our examples, theorem 2 will yield blow-up of Koopman eigenfunctions for the full basin boundary $\mathcal{A}$, not just for the punctured basin boundary $\hat{\partial}\mathcal{A}$. In other examples, theorem 2 will imply eigenfunction blow-up without the non-resonance assumptions required by theorem 1.

3. Examples

The following examples illustrate how theorems 1 and 2 allow conclusions on the blow-up of Koopman eigenfunctions without their explicit knowledge. These examples also illustrate the degenerate features that lead to uniformly bounded eigenfunctions across domain boundaries. Those features include linearity in certain dimensions or the full decoupling of an equation from the rest.

Our first three examples are not only simple for didactic reasons. In order to illustrate theorems 1 and 2 in practice, we have to consider nonlinear examples in which the principal Koopman eigenfunctions as ground truth are exactly known. This necessarily restricts the discussion to simple systems, as there is no general procedure for the explicit construction of eigenfunctions for the Koopman operator. In our fourth example, however, we illustrate the use of theorems 1 and 2 in a case where an explicit form of smooth principal Koopman eigenfunctions is not available.

Example 1. Consider the two-dimensional system

$$\begin{aligned}\dot{x}&=x(x-1),\\ \dot{y}&=-ay,\qquad a>0.\end{aligned} \tag{3.1}$$

This system has a stable hyperbolic fixed point at $x=0$; its right-hand side is analytic, and hence equation (3.1) is of the general form of equation (2.1). The phase portrait of the system is shown in figure 4 for three different values of $a>0$. The basin of attraction of the asymptotically stable origin and its boundary, respectively, are given by

$$\mathcal{A}=\left\{(x,y)\in\mathbb{R}^2:\ x<1\right\},\qquad \partial\mathcal{A}=\left\{(x,y)\in\mathbb{R}^2:\ x=1\right\}. \tag{3.2}$$

The first of equation (3.1) decouples from the full system and has a principal Koopman eigenfunction–eigenvalue pair:

$$\varphi_1(x,y)=\frac{x}{x-1},\qquad \lambda_1=-1,\qquad (x,y)\in\mathcal{A}, \tag{3.3}$$

as we show in appendix C.1 of the supplementary material. Therefore, φ_1 blows up along $\partial\mathcal{A}$. The second equation in system (3.1) also decouples and is linear. This enables us to conclude immediately that the second principal Koopman eigenfunction of system (3.1) is

$$\varphi_2(x,y)=y,\qquad (x,y)\in\mathcal{A}, \tag{3.4}$$

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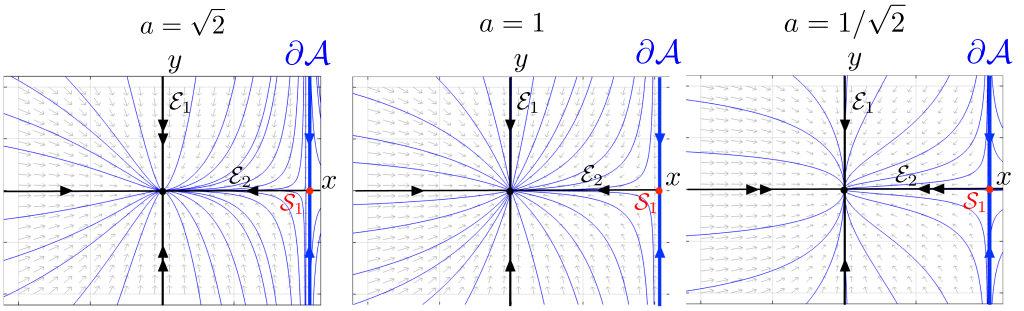


Figure 4. The phase portrait of system (3.1) for three different values of the parameter a.

which does *not* blow up at any point of $\partial\mathcal{A}$.

Application of theorem 1: We now apply theorem 1 to this example to reach conclusions about the blow-up of Koopman eigenfunctions without using their specific form. We start by noting that the punctured basin boundary $\hat{\partial}\mathcal{A}$ is formed by the stable manifold of a fixed point at the location $(x, y) = (1, 0)$; thus, we have

$$\hat{\partial}\mathcal{A} \equiv \partial\mathcal{A} = W^s\left(\mathcal{S}_1\right), \qquad \mathcal{S}_1 = (1, 0),$$

where the fixed point $\mathcal{S}_1$ is an $m = 0$-dimensional edge state. The unstable manifold of this edge state is given by

$$W^u\left(\mathcal{S}_1\right) = \left\{(x, y) : \quad x \in (0, \infty), \quad y = 0\right\}. \tag{3.5}$$

The coefficient matrix of the linear part of the system is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -a \end{pmatrix}, \tag{3.6}$$

giving rise to the eigenspaces

$$E_1 = \text{span } (1, 0)^{\mathrm{T}}, \qquad E_2 = \text{span } (0, 1)^{\mathrm{T}},$$

which in turn yield the co-spectral subspaces

$$\mathcal{E}_1 = \text{span } (0, 1)^{\mathrm{T}}, \qquad E_2 = \text{span } (1, 0)^{\mathrm{T}}.$$

The non-resonance conditions (2.6) take the specific form

$$\begin{aligned} k_1 + k_2 a &\neq a, \qquad k_1, k_2 \in \mathbb{N}, \qquad 2 \leq k_1 + k_2 \leq r, \\ k_1 + k_2 a &\neq 1, \qquad k_1, k_2 \in \mathbb{N}, \qquad 2 \leq k_1 + k_2 \leq r, \end{aligned} \tag{3.7}$$

or equivalently,

$$a \notin \left\{ \frac{1}{r}, \ldots, \frac{1}{3}, \frac{1}{2}, 2, 3, \ldots, r \right\}. \tag{3.8}$$

Choosing

$$r > L = \max\left(\lfloor a \rfloor, \left\lfloor \frac{1}{a} \right\rfloor \right),$$

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we obtain from the results of Cabré *et al.* [18] that the primary co-SSM submanifolds tangent to $\mathcal{E}_i$ at the origin are unique among C^{L+1} spectral submanifolds. As the subspaces E_1 and E_2 are arbitrarily smooth invariant manifolds for the nonlinear system (3.1), they must necessarily have

$$W(\mathcal{E}_1) \equiv \mathcal{E}_1 = E_2, \qquad W(\mathcal{E}_2) \equiv \mathcal{E}_2 = E_1. \tag{3.9}$$

Therefore, by equations (3.5) and (3.9), we have

$$W(\mathcal{E}_1) \cap W^u(S_1) = \emptyset, \qquad W(\mathcal{E}_2) \cap W^u(S_1) \neq \emptyset. \tag{3.10}$$

Hence, under the non-resonance conditions (3.8), theorem 1 implies the blow-up of the principal eigenfunction φ_1 along $\partial\mathcal{A}$, confirming our conclusion from the explicit form equation (3.3) of φ_1. At the same time, by the second relation in equation (3.10), the blow-up of the principal Koopman eigenfunction φ_2 cannot be concluded from theorem 1. Indeed, as already concluded from equation (3.4), φ_2 does not blow up along the basin boundary $\partial\mathcal{A}$.

Application of theorem 2: Next, we illustrate how the sufficient conditions in theorem 2 perform on this example without relying on an explicit form of the Koopman eigenfunctions, equations (3.3) and (3.4) of system (3.1). The coefficient matrix equation (3.6) of the linear part of the system is diagonal, and hence its left eigenvectors w_j are simply transposes of its right eigenvectors e_j:

$$w_1 = e_1^{\mathrm{T}} = (1,0), \qquad w_2 = e_2^{\mathrm{T}} = (0,1).$$

Furthermore, we have

$$W = T = I, \tag{3.11}$$

for the matrices defined in equation (2.2).

For any sequence of the form

$$\left\{\left(x_0^j, y_0^j\right)\right\}_{j=1}^{\infty} \subset \mathcal{A}, \quad \lim_{j\to\infty}\left(x_0^j, y_0^j\right) = (1, y^\star) \in \partial\mathcal{A}, \tag{3.12}$$

the flow map F^t of the system satisfies

$$\lim_{\left(x_0^j, y_0^j\right)\to(1,y^\star)} F^{\tau\left(x_0^j, y_0^j\right)}\left(x_0^j, y_0^j\right) = \begin{pmatrix} \delta \\ 0 \end{pmatrix}. \tag{3.13}$$

Therefore, the set $C_\delta(x^*, y^*)$ is composed of a single point for any choice of the basin boundary point:

$$C_\delta(x^*, y^*) = \{(\delta, 0)\}.$$

This implies

$$\left|w_1 \cdot \left(x_\delta^*, y_\delta^*\right)\right| = \delta, \qquad \left|w_2 \cdot \left(x_\delta^*, y_\delta^*\right)\right| = 0, \tag{3.14}$$

for all $\delta > 0$. Using these two equalities in evaluating condition (2.20) gives

$$\lim_{\delta\to 0} \frac{\left|w_1 \cdot \left(x_\delta^*, y_\delta^*\right)\right|}{\delta} = 1 > 0, \qquad \lim_{\delta\to 0} \frac{\left|w_2 \cdot \left(x_\delta^*, y_\delta^*\right)\right|}{\delta} = 0, \tag{3.15}$$

for any choice of $(x^*, y^\star)$. Therefore, by the first of equation (3.15), the principal Koopman eigenfunction $\varphi_1(x, y)$ must blow up at any boundary point $(1, y^\star) \in \partial\mathcal{A}$ by condition (2.20) of theorem 2. This conclusion extends the findings obtained from theorem 1 for this example to arbitrary values of the constant $a > 0$.

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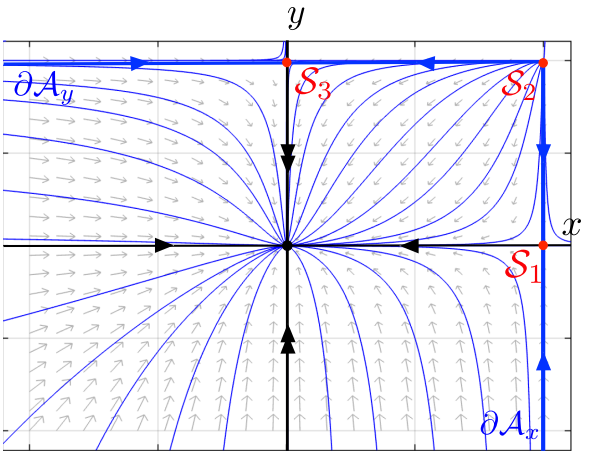


Figure 5. The the phase portrait of system (3.16) for $a = 3/2$.

At the same time, the blow-up of the principal Koopman eigenfunction $\varphi_2(x, y)$ cannot be concluded from theorem 2 because of the degeneracy (lack of coupling and full linearity in one direction) of system (3.1). Indeed, condition (2.20) fails for $k = 2$ by the second equality in equation (3.15) and the second inequality in equation (2.22) fails for $k = 2$ owing to the lack of nonlinear coupling between the two equations in system (3.1). Indeed, as already noted, φ_2 does not blow up along the basin boundary $\partial\mathcal{A}$.

Example 2. Consider the two-dimensional system of ODEs

$$\begin{aligned} \dot{x} &= x\left(x-1\right), \\ \dot{y} &= ay\left(y-1\right), \qquad a > 1. \end{aligned} \tag{3.16}$$

The origin is again a fixed point with the basin of attraction,

$$\mathcal{A} = \left\{(x, y) \,:\, x < 1, y < 1\right\}.$$

The basin boundary is now $\partial\mathcal{A} = \partial\mathcal{A}_x \cup \partial\mathcal{A}_y$, where

$$\partial\mathcal{A}_x = \left\{(x, y) \,:\, x, y \le 1,\ x = 1\right\}, \qquad \partial\mathcal{A}_y = \left\{(x, y) \,:\, x, y \le 1,\ y = 1\right\},$$

as shown in figure 5.

Using the form of the first principal Koopman eigenfunction from equation (3.3), we find that the two principal Koopman eigenfunctions and their corresponding eigenvalues on the domain $\mathcal{A}$ are

$$\varphi_1(x, y) = \frac{x}{x-1}, \quad \lambda_1 = -1, \qquad \varphi_2(x, y) = \frac{y}{y-1}, \quad \lambda_2 = -a. \tag{3.17}$$

Therefore, we have

$$\lim_{x\uparrow 1} \left|\varphi_1(x, y)\right| = \infty, \qquad \lim_{y\uparrow 1} \left|\varphi_2(x, y)\right| = \infty, \tag{3.18}$$

that is, φ_1 blows up along the basin boundary segment $\partial\mathcal{A}_x$ and φ_2 along the basin boundary segment $\partial\mathcal{A}_y$.

Application of theorem 1: First, we again illustrate how the necessary conditions in theorem 1 perform on this example without relying on an explicit construction of the Koopman eigenfunctions of system (3.1). We first note that the basin boundary $\partial\mathcal{A}$ is formed by the stable manifolds

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of the three fixed points

$$S_1 = (1,0)\,, \qquad S_2 = W^s\left(S_1\right) = (1,1)\,, \qquad S_3 = (0,1)\,,$$

which act as zero-dimensional edge states. Therefore, the punctured basin boundary defined in equation (2.11) takes the specific form

$$\hat{\partial}\mathcal{A} = W^s\left(S_1\right) \cup W^s\left(S_3\right).$$

The unstable manifold of the edge states contained in $\hat{\partial}\mathcal{A}$ is

$$W^u\left(S_1\right) = \left\{(x,y):\ \ x \in (0,\infty)\,, \quad y = 0\right\}, \qquad W^u\left(S_3\right) = \left\{(x,y):\ \ x = 0, \quad y \in (0,\infty)\right\}. \tag{3.19}$$

As in example 1, the coefficient matrix of the linearized system at the origin is again $A = \mathrm{diag}\,(-1,-\mathrm{a})$, giving rise to the eigenspaces

$$E_1 = \mathrm{span}\ (1,0)^{\mathrm{T}}\,, \qquad E_2 = \mathrm{span}\ (0,1)^{\mathrm{T}}\,,$$

and their co-spectral subspaces

$$\mathcal{E}_1 = \mathrm{span}\ (0,1)^{\mathrm{T}}\,, \qquad E_2 = \mathrm{span}\ (1,0)^{\mathrm{T}}\,.$$

The non-resonance conditions (2.6) are identical to those in example 1, and hence choosing

$$r > L = \max\left(\lfloor a \rfloor\,, \left\lfloor \frac{1}{a} \right\rfloor\right),$$

we find that the primary co-SSMs associated with E_1 and E_2 are

$$W\left(\mathcal{E}_1\right) \equiv \mathcal{E}_1, \qquad W\left(\mathcal{E}_2\right) \equiv \mathcal{E}_2. \tag{3.20}$$

Therefore, by equations (3.5) and (3.20), we have

$$W\left(\mathcal{E}_1\right) \cap W^u\left(S_1\right) = \emptyset, \qquad W\left(\mathcal{E}_2\right) \cap W^u\left(S_1\right) \neq \emptyset, \qquad W\left(\mathcal{E}_1\right) \cap W^u\left(S_3\right) \neq \emptyset, \qquad W\left(\mathcal{E}_2\right) \cap W^u\left(S_3\right) = \emptyset. \tag{3.21}$$

Therefore, under the non-resonance conditions (3.8), theorem 1 implies the blow-up of the principal eigenfunction φ_1 along $\partial\mathcal{A}_x - S_2$, confirming our conclusion from the explicit form equation (3.17) of φ_1. At the same time, by the second relation in equation (3.21), the blow-up of the principal Koopman eigenfunction φ_2 along $\partial\mathcal{A}_x - S_2$ cannot be concluded from theorem 1. Indeed, as already concluded from equation (3.17), φ_2 does not blow up along the basin boundary $\partial\mathcal{A}_x$.

Similarly, by the last of equation (3.21), theorem 1 implies the blow-up of the principal eigenfunction φ_2 along $\partial\mathcal{A}_y - S_2$, confirming our conclusion from the explicit form equation (3.17) of φ_2. At the same time, by the third of equation (3.21), the blow-up of the principal Koopman eigenfunction φ_1 along $\partial\mathcal{A}_y - S_2$ cannot be concluded from theorem 1. Indeed, as already concluded from equation (3.17), φ_1 does not blow up along the basin boundary $\partial\mathcal{A}_y$.

Application of theorem 2: The basin boundary is $\partial\mathcal{A} = \partial\mathcal{A}_x \cup \partial\mathcal{A}_y$, where

$$\partial\mathcal{A}_x = \left\{(x,y):\ \ x,y \leq 1,\ x = 1\right\}, \qquad \partial\mathcal{A}_y = \left\{(x,y):\ \ x,y \leq 1,\ y = 1\right\},$$

as shown in figure 5. The matrix A, its eigenvalues and the matrices T and W are the same as in example 1. Again, for any sequence of the form

$$\left\{\left(x_0^j, y_0^j\right)\right\}_{j=1}^{\infty} \subset \mathcal{A}, \quad \lim_{j\to\infty}\left(x_0^j, y_0^j\right) = \left(1, y^\star\right) \in \partial\mathcal{A}_x, \tag{3.22}$$

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the intersection sequence $\left\{F^{\tau(x_0^j,y_0^j)}\left(x_0^j,y_0^j\right)\right\}_{j=1}^{\infty}$ has the limit

$$\lim_{j\to\infty}\left\{F^{\tau(x_0^j,y_0^j)}\left(x_0^j,y_0^j\right)\right\}_{j=1}^{\infty}=(\delta,0),$$

and hence

$$C_\delta(x^*,y^\star)=\{(\delta,0)\},\qquad \left(x^*,y^\star\right)\in\partial\mathcal{A}_x.$$

From an identical argument carried out on sequences converging to a point in the basin boundary component $\partial\mathcal{A}_y$, we obtain

$$C_\delta(x^*,y^*)=\{(0,\delta)\},\qquad \left(x^*,y^*\right)\in\partial\mathcal{A}_y.$$

This implies

$$\begin{aligned}&\left|w_1\cdot\left(x_\delta^*,y_\delta^*\right)\right|=\delta,\qquad \left|w_2\cdot\left(x_\delta^*,y_\delta^*\right)\right|=0,\qquad \left(x^*,y^*\right)\in\partial\mathcal{A}_x,\\&\left|w_1\cdot\left(x_\delta^*,y_\delta^*\right)\right|=0,\qquad \left|w_2\cdot\left(x_\delta^*,y_\delta^*\right)\right|=\delta,\qquad \left(x^*,y^*\right)\in\partial\mathcal{A}_y,\end{aligned}\tag{3.23}$$

for all $\delta>0$. Using these two equalities in evaluating condition (2.20) gives

$$\begin{aligned}&\lim_{\delta\to0}\frac{\left|w_1\cdot\left(x_\delta^*,y_\delta^*\right)\right|}{\delta}=1>0,\qquad \lim_{\delta\to0}\frac{\left|w_2\cdot\left(x_\delta^*,y_\delta^*\right)\right|}{\delta}=0,\qquad \left(x^*,y^*\right)\in\partial\mathcal{A}_x,\\&\lim_{\delta\to0}\frac{\left|w_1\cdot\left(x_\delta^*,y_\delta^*\right)\right|}{\delta}=0,\qquad \lim_{\delta\to0}\frac{\left|w_2\cdot\left(x_\delta^*,y_\delta^*\right)\right|}{\delta}=1>0,\qquad \left(x^*,y^*\right)\in\partial\mathcal{A}_y.\end{aligned}$$

Applying condition (2.20) of theorem 2 to these last findings confirms the known blow-ups listed in equation (3.18), again without the actual construction of the eigenfunctions involved. Note that this result is stronger than what we have obtained from theorem 1, because it does not require the exclusion of $\mathcal{S}_2$.

At the same time, both assumptions (i) and (ii) of theorem 2 fail for the blow-up of $\varphi_1\left(x,y\right)$ along $\partial\mathcal{A}_y$ and for the blow-up of $\varphi_2\left(x,y\right)$ along $\partial\mathcal{A}_x$. This is consistent with the fact that $\varphi_1\left(x,y\right)$ is independent of y and hence is indeed uniformly bounded along $\partial\mathcal{A}_y$. Similarly, $\varphi_2\left(x,y\right)$ is independent of x and hence is indeed uniformly bounded along $\partial\mathcal{A}_x$. Therefore, both eigenfunctions are degenerate and remain uniformly bounded along certain portions of the domain boundary owing to the degeneracy of system (3.16).

Example 3. Consider the two-dimensional system

$$\begin{aligned}\dot{x}&=-x-y+x\left(x^2+y^2\right),\\\dot{y}&=x-y+y\left(x^2+y^2\right),\end{aligned}\tag{3.24}$$

which has a stable fixed point at the origin and an unstable limit cycle at $\left|\left(x,y\right)\right|=1$, as shown in figure 6. Therefore, the basin of attraction and its boundary are given by

$$\mathcal{A}=\left\{x\in\mathbb{R}:\ \left|\left(x,y\right)\right|<1\right\},\qquad \partial\mathcal{A}=\left\{x\in\mathbb{R}:\ \left|\left(x,y\right)\right|=1\right\}.$$

Writing system (3.24) in polar coordinates and using the results of appendix C.2 of the supplementary material, we obtain a complex principal Koopman eigenfunction of the form

$$\varphi_1(x,y)=\frac{x+iy}{\sqrt{1-x^2-y^2}}.\tag{3.25}$$

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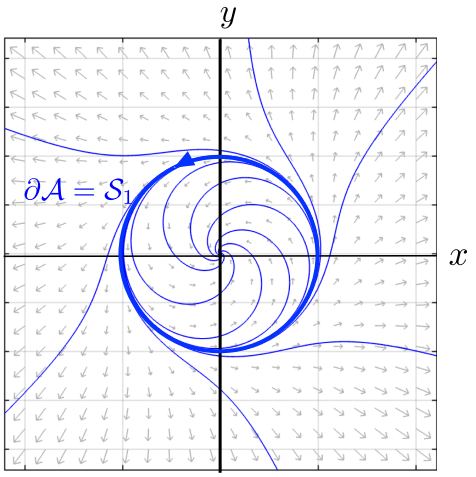


Figure 6. The the phase portrait of system (3.24).

This formula shows that φ_1 blows up at all points of the unstable limit cycle that bounds the basin of attraction $\mathcal{A}$ at $\left|(x,y)\right| = 1$:

$$\lim_{(x,y)\to(x^*,y^\star)} \left|\varphi_1(x,y)\right| = \infty, \qquad (x^*,y^\star) \in \partial\mathcal{A}. \tag{3.26}$$

Application of theorem 1: To conclude this blow-up without the explicit knowledge of the principal eigenfunctions, we now apply theorem 1. The linearized system at the origin has eigenvalues

$$\lambda_{1,2} = -1 \pm i,$$

and hence the non-resonance conditions (2.6) always hold for any choice of the smoothness degree $r \in \mathbb{N}^+$. For system (3.24), we have a single real eigenspace E_1 and corresponding primary spectral submanifold $W\left(E_1\right)$,

$$E_1 = \mathbb{R}^2, \qquad W\left(E_1\right) = \mathcal{A}.$$

Consequently,

$$W\left(\mathcal{E}_1\right) = \emptyset. \tag{3.27}$$

The punctured basin boundary in this example is just the limit cycle

$$\hat{\partial}\mathcal{A} = \partial\mathcal{A} = \mathcal{S}_1,$$

whose unstable manifold is

$$W^u\left(\mathcal{S}_1\right) = \mathbb{R}^2. \tag{3.28}$$

Therefore, by equations (3.27) and (3.28), we have

$$W\left(\mathcal{E}_1\right) \cap W^u\left(\mathcal{S}_1\right) = \emptyset.$$

Therefore, theorem 1 implies the blow-up of the Koopman eigenfunction $\varphi_1(x,y)$ defined in equation (3.25) without any reliance on its explicit form. This confirms our assessment equation (3.26) based on the explicit form of the eigenfunction.

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Application of theorem 2: For system (3.24), we have

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \lambda_{1,2} = -1 \pm i, \quad f(x) = \begin{pmatrix} x\left(x^2+y^2\right) \\ y\left(x^2+y^2\right) \end{pmatrix}.$$

The unit left eigenvectors of A are

$$w_1 = \frac{1}{\sqrt{2}}(1, i), \qquad w_1 = \frac{1}{\sqrt{2}}(1, -i).$$

The system has an unstable limit cycle satisfying $\left|(x,y)\right| = 1$ that acts as a boundary for the basin of attraction of the origin:

$$\mathcal{A} = \left\{x \in \mathbb{R} : \ \left|(x,y)\right| < 1\right\}, \qquad \partial\mathcal{A} = \left\{x \in \mathbb{R} : \ \left|(x,y)\right| = 1\right\}.$$

For any point $(x^*, y^\star) \in \partial\mathcal{A}$, we have $C_\delta(x^*, y^\star) \equiv \partial B_\delta$. Therefore, for an arbitrary

$$\left(x^*_\delta, y^*_\delta\right) = \delta \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \in \partial B_\delta, \qquad \psi \in S^1,$$

we have

$$\frac{\left|w_{1,2} \cdot \left(x^*_\delta, y^*_\delta\right)\right|}{\delta} = \frac{1}{\sqrt{2}} \left|(\cos\psi + i \sin\psi)\right| = \frac{1}{\sqrt{2}} > 0.$$

Hence, assumption (i) of theorem 2 always holds. Therefore, we deduce from the theorem that the complex principal eigenfunctions φ_1 blow up at all points of the basin boundary $\partial\mathcal{A}$. This confirms our assessment equation (3.26) based on the known principal eigenfunctions.

Example 4. We consider the dynamical system:

$$\begin{aligned} \dot{x_1} &= x_1(x_2 - 1), \\ \dot{x_2} &= \frac{5}{2} x_2(x_1 - 1), \end{aligned} \tag{3.29}$$

with its phase portrait shown in figure 7. This system was considered by Haller *et al.* [28] in the study of fractional SSMs, and it also serves as a conceptual model for the edge of chaos in shear flows (e.g. Skufca *et al.* [20], Schneider *et al.* [21] and Kaszás & Haller [22]).

We note that for system (3.29), the conserved quantity

$$H(x) = x_2 - \log x_2 - \frac{5}{2}\left(x_1 - \log x_1\right) \tag{3.30}$$

for $x_1, x_2 > 0$. The quantity H defined in equation (3.30) is a Koopman eigenfunction for system (3.29) with eigenvalue zero. However, H is not a principal eigenfunction, in that case as it does not satisfy the requirement $\varphi(0) = 0$ in definition (2.12).

Application of theorem 1: The basin boundary is formed by the stable manifold of the fixed point

$$S_1 = (1, 1),$$

which acts as a zero-dimensional edge state. Therefore, the punctured basin boundary defined in equation (2.11) is

$$\hat{\partial}\mathcal{A} \equiv \partial\mathcal{A} = W^s\left(S_1\right).$$

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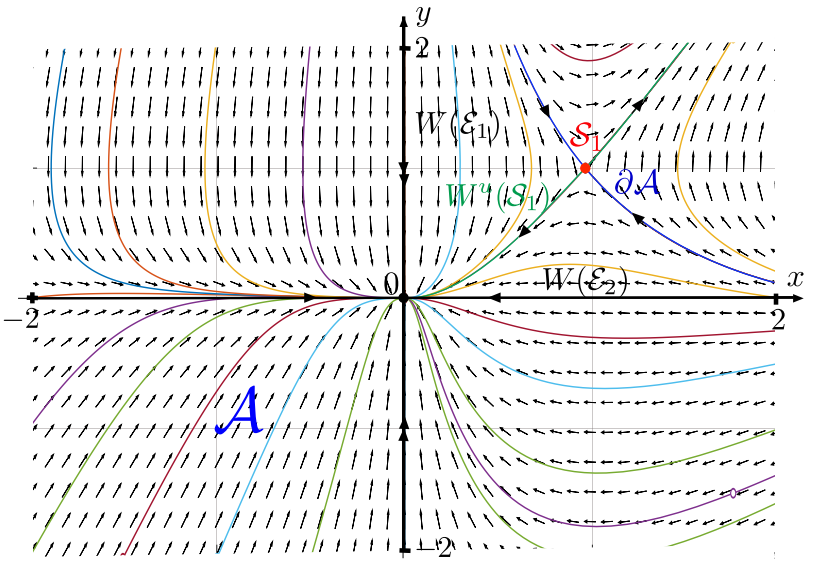


Figure 7. The phase portrait of system (3.29), with an edge state S_1 and primary SSMs $W\left(\mathcal{E}_1\right) \equiv E_2$ and $W\left(\mathcal{E}_2\right) \equiv E_1$.

The coefficient matrix of the linearized system at the origin is $A = \text{diag}\left(-1, -\frac{5}{2}\right)$, giving rise to the eigenspaces

$$E_1 = \text{span}\ (1,0)^{\mathrm{T}}, \qquad E_2 = \text{span}\ (0,1)^{\mathrm{T}},$$

and their co-spectral subspaces

$$\mathcal{E}_1 = \text{span}\ (0,1)^{\mathrm{T}}, \qquad E_2 = \text{span}\ (1,0)^{\mathrm{T}}.$$

The non-resonance conditions (2.6) are satisfied for the eigenvalues $\lambda = -1$ and $\lambda_2 = -\frac{5}{2}$ of A and hence choosing $r = \infty$, we find that the (unique) primary co-SSMs relevant for the main condition of theorem 1 are

$$W\left(\mathcal{E}_1\right) \equiv \mathcal{E}_1 = E_2, \qquad W\left(\mathcal{E}_2\right) \equiv \mathcal{E}_2 = E_1. \tag{3.31}$$

Therefore, by figure 7 and relations (3.31), we have

$$W\left(\mathcal{E}_1\right) \cap W^u\left(S_1\right) = \emptyset, \qquad W\left(\mathcal{E}_2\right) \cap W^u\left(S_1\right) = \emptyset. \tag{3.32}$$

Therefore, theorem 1 implies the blow-up of the principal eigenfunctions φ_1 and φ_2 along the basin boundary $\partial\mathcal{A}$.

Application of theorem 2: We modify system (3.33) and rewrite it here for convenience as

$$\begin{aligned} \dot{x} &= x(y-1), \\ \dot{y} &= \frac{5}{2}y(x-1) + \epsilon x^2, \qquad \epsilon > 0. \end{aligned} \tag{3.33}$$

Here, the phase portrait is only an $\mathcal{O}(\epsilon)$ perturbation of that of system (3.33). Under this perturbation, function (3.30) is no longer a first integral and hence not a Koopman eigenfunction.

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For any basin boundary point, the geometry of the phase space shown in figure 7 implies

$$\lim_{\delta\to 0}\sup_{x_\delta^\star\in C_\delta(x^*)}\frac{\left|w_2\cdot x_\delta^\star\right|}{\delta}=\left|w_2\cdot e_1\right|=\left|e_2\cdot e_1\right|=0.$$

Therefore, assumption (i) of theorem 2 fails and hence the first equation in assumption (ii) of that theorem is satisfied. Because the right-hand side of equation (3.33) is C^r for any $r>0$, we can specifically select

$$r=3>\frac{\max_j\left|\operatorname{Re}\lambda_j\right|}{\min_j\left|\operatorname{Re}\lambda_j\right|}=\frac{5}{2},\tag{3.34}$$

for which

$$\sum_{2\le k_1+k_2\le 3,\quad k_1,k_2\ge 0}k_1\cdot(-1)+k_2\cdot\left(-\frac{5}{2}\right)\notin\left\{-1,-\frac{5}{2}\right\},$$

and hence the non-resonance condition (3.34) also holds.

It remains to verify whether the second inequality in assumption (ii) of theorem 2 holds. Noting that $T=W=I$, we obtain

$$f(Tz)=\begin{pmatrix}z_1z_2\\ \frac{5}{2}z_1z_2\end{pmatrix},$$

and

$$\sum_{|\mathbf{k}|=2}\frac{w_2\cdot\partial_{z^{\mathbf{k}}}\left[f(Tz)\right]\big|_{z=0}\left(Wx_\delta^\star\right)^{\mathbf{k}}}{\mathbf{k}!\left(\lambda_k-\sum_{\ell=1}^n k_\ell\lambda_\ell\right)}=\frac{w_2\cdot\begin{pmatrix}x_\delta^\star y_\delta^\star\\ \frac{5}{2}x_\delta^\star y_\delta^\star+\frac{\epsilon}{2}\left(x_\delta^\star\right)^2\end{pmatrix}}{\frac{5}{2}-1\cdot\frac{1}{2}-1\cdot\frac{5}{2}}$$
$$=-5x_\delta^\star y_\delta^\star-\epsilon\left(x_\delta^\star\right)^2.$$

Consequently, we have

$$\lim_{\delta\to 0}\sup_{x_\delta^\star\in C_\delta(x^*)}\frac{1}{\delta^2}\left|\sum_{|\mathbf{k}|=2}\frac{w_k\cdot\partial_{z^{\mathbf{k}}}\left[f(Tz)\right]\big|_{z=0}\left(Wx_\delta^\star\right)^{\mathbf{k}}}{\mathbf{k}!\left(\lambda_k-\sum_{\ell=1}^n k_\ell\lambda_\ell\right)}\right|=\epsilon>0,\tag{3.35}$$

showing that the second inequality in condition (2.22) is satisfied. Therefore, theorem 2 implies that φ_2 must also blow up along the basin boundary $\partial\mathcal{A}$ for any $\epsilon>0$. Note that in this example, theorem 1 performs better, as it also establishes the blow-up of both eigenfunctions for $\epsilon=0$.

4. Conclusions

We have shown that in non-degenerate nonlinear dynamical systems, all principal Koopman eigenfunctions defined over the basin of attraction of a fixed point blow up along basin boundary segments formed by stable manifolds of saddle-type edge states. We have also given a more general set of blow-up conditions that can be applied under weaker assumptions. These blow-up results imply, by a result of Kvalheim & Revzen [11], that all non-constant C^∞ Koopman eigenfunctions in systems without degeneracies blow up along basin boundaries. The present results extend the findings of Haller & Kaszás [9] and Kvalheim & Arathoon [10] on the necessary blow-up of at least one principal Koopman eigenfunction.

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Specifically, we have shown that the Koopman eigenfunctions associated with non-resonant, attracting fixed points blow up along basin boundaries owing to a lack of intersections between the co-SSMs of the eigenspaces of the linearized system and the unstable manifolds of edge states whose stable manifolds form the basin boundary. This is the statement of our theorem 1, which we have evaluated on simple examples without any calculation, just based on the assessment of the geometry of the invariant manifolds involved. Since the non-intersection between the invariant manifolds involved is to be expected, in general, non-degenerate dynamical systems, the blow-up of Koopman eigenfunctions is also to be generally expected in such systems. Our theorem 2 gives more technical sufficient conditions for the blow-up of eigenfunctions. These conditions do not necessarily involve non-resonance assumptions, allow for more general basin boundary segments and also allow for intersections between co-spectral manifolds of the fixed point and unstable manifolds of edge states.

We have discussed similarities and differences in the applicability of theorems 1 and 2 in all our examples. These examples also highlight that eigenfunctions that do not blow up are insensitive to the nonlinearities creating a basin boundary to begin with. In other words, the Koopman eigenfunctions that remain bounded over all compact subsets of the basin boundaries are degenerate observables with respect to the nonlinearizable dynamics of the system and hence cannot be used in reduced-order modelling across basin boundaries.

To circumvent such a blow-up in reduced modelling, using different Koopman eigenfunctions in different parts of the phase space is sometimes suggested. This is, however, not a realistic programme to carry out for a typical nonlinear system. First, the eigenfunctions grow without bound near basin boundaries and hence one cannot accurately compute them even close to the boundary. Second, to know when to switch from one set of eigenfunctions to the next would require exact knowledge of the location of the basin boundaries, which are generally convoluted (potentially fractal type) and, more importantly, unknown. In other words, one would *a priori* need to know the full phase space geometry of the original nonlinear system in such detail that it would render model reduction moot.

As a ground principle, nonlinear systems with multiple attractors cannot admit linear embeddings beyond individual domains of attraction. This has already been pointed out by multiple authors in settings that are substantially more general than ours (e.g. Arathoon & Kvalheim [5] and Kvalheim & Arathoon [10]). The advance here is that any such embedding does not simply remain confined to domains of attraction. Rather, it generally also blows up in all coordinate directions and hence becomes numerically unmanageable near basin boundaries.

As a consequence, numerically constructed, Koopman–eigenfunction-based approaches are expected to exhibit large inaccuracies even on trajectories that are technically outside the domain of classic local linearization results but are still well separated from the boundary. This has indeed been documented by Page & Kerswell [8], who show that expansions of observables with respect to eigenfunctions can already blow up well inside the basin boundary. Similarly, large inaccuracies have been noted by several studies in the Koopman-based model–predictive control of soft robots even at large distances from the basin boundary of their stable equilibrium state (see Haggerty *et al.* [29] for a brief review of those studies).

We hope that the results obtained here contribute to the correct assessment of the power and limitations of linearized approximations to nonlinear phenomena.

Data accessibility. This article has no additional data.

Supplementary material is available online [30].

Declaration of AI use. I have not used AI-assisted technologies in creating this article.

Authors' contributions. G.H.: conceptualization, formal analysis, writing—original draft.

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