Uncovering how conservative backbone curves survive the advent of forcing and damping

ETH zürich

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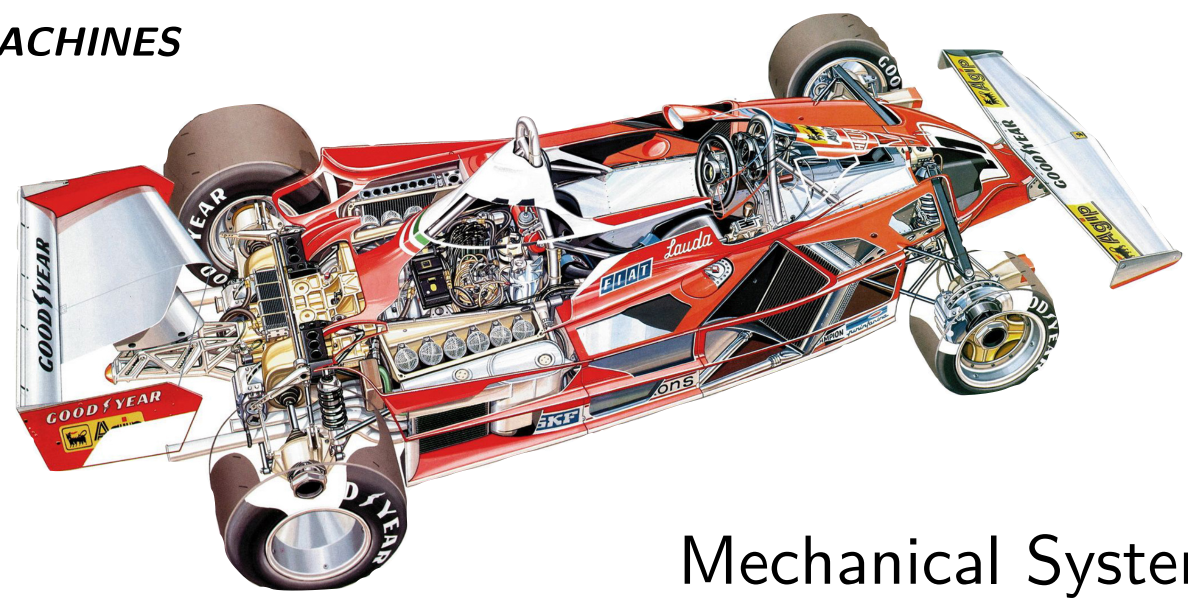
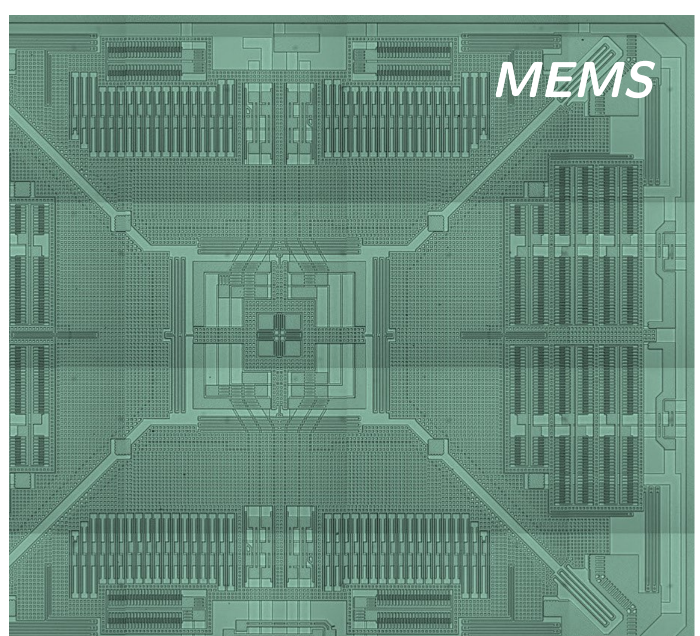
The Role of Periodic Orbits in Dynamics

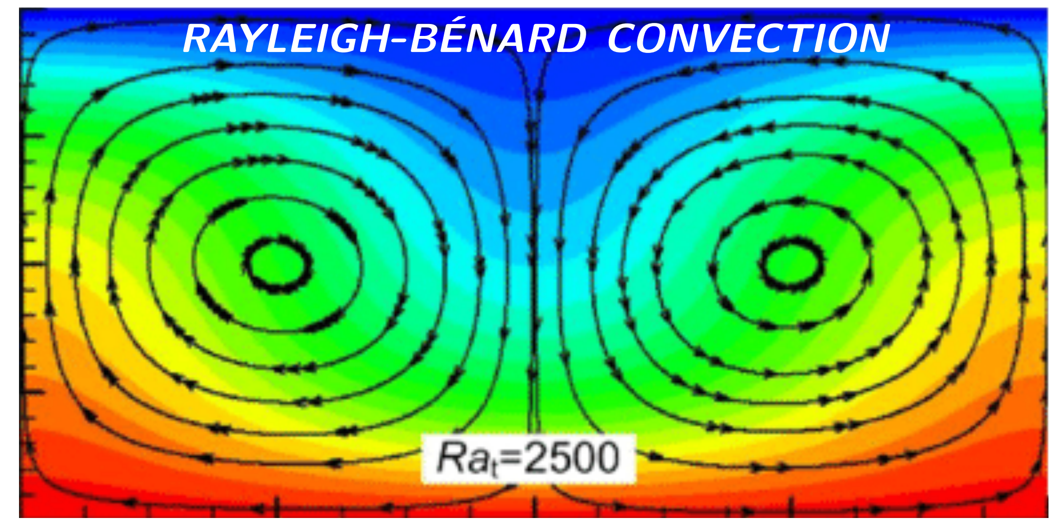
Periodic orbit: *motion that repeats identically after a finite period of time*

MACHINES

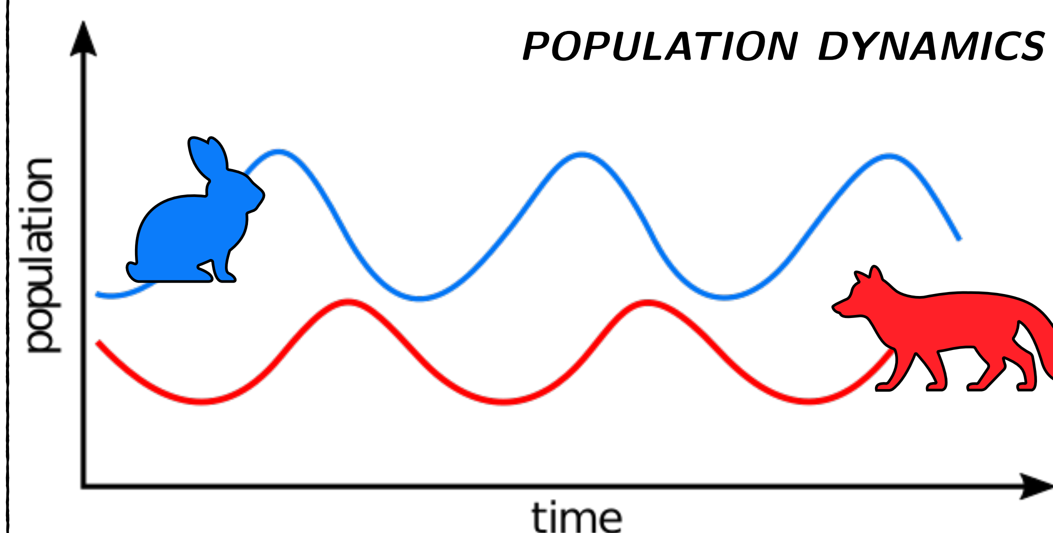


Mechanical Systems





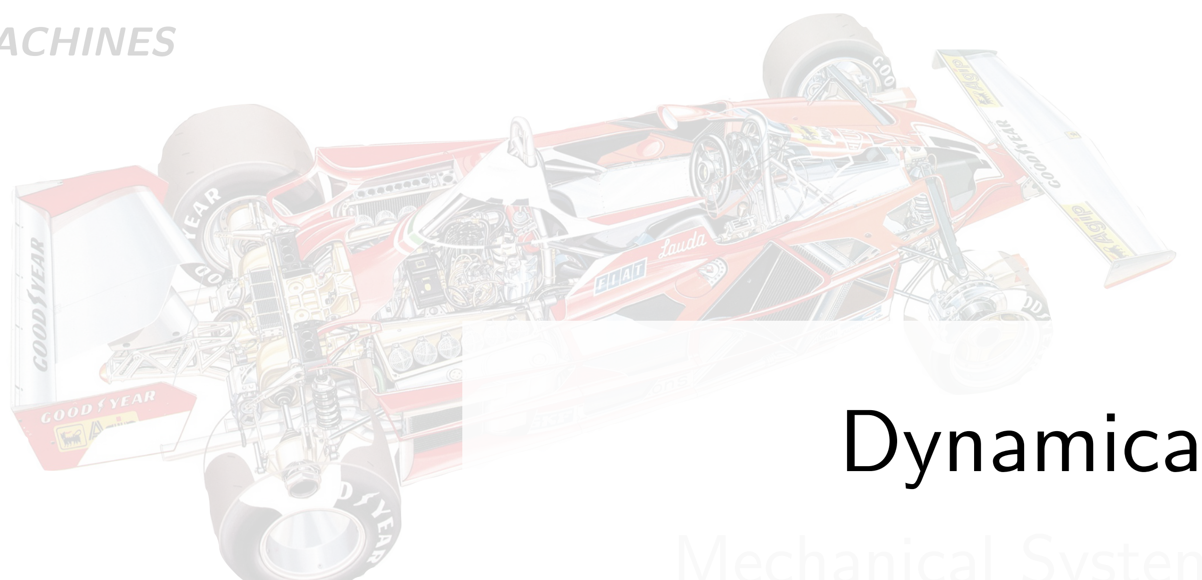
Fluid Dynamics

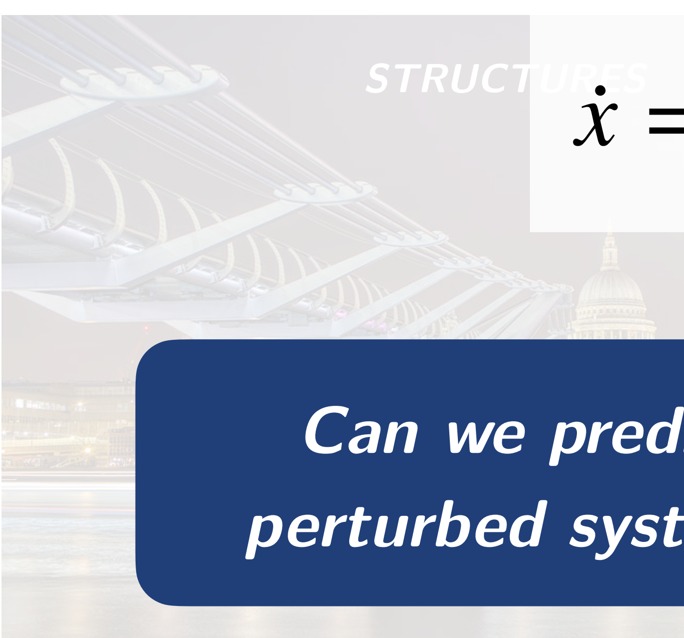


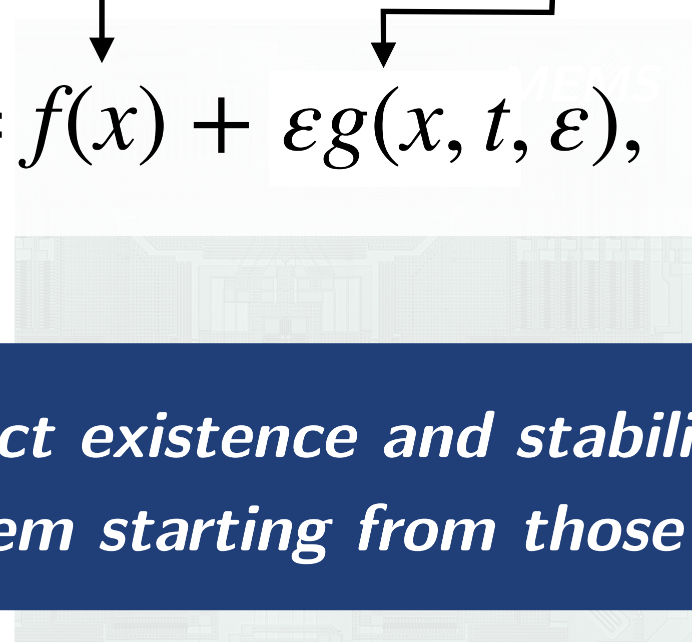
Socio-biological Dynamics

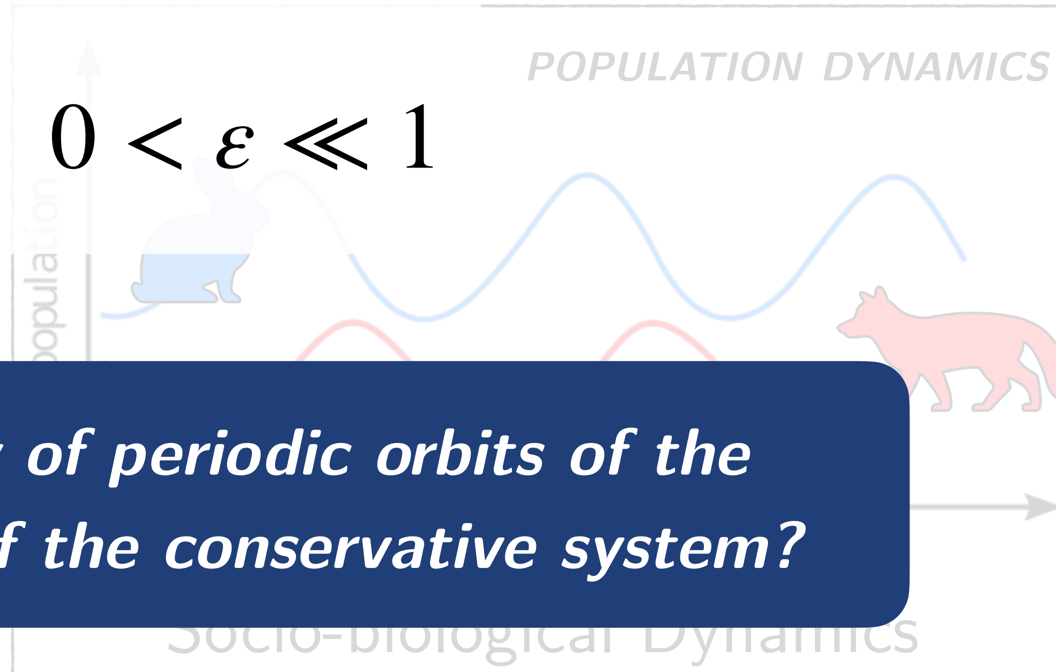
The Role of Periodic Orbits in Dynamics

Periodic orbit: *motion that repeats identically after a finite period of time*









Dynamical models

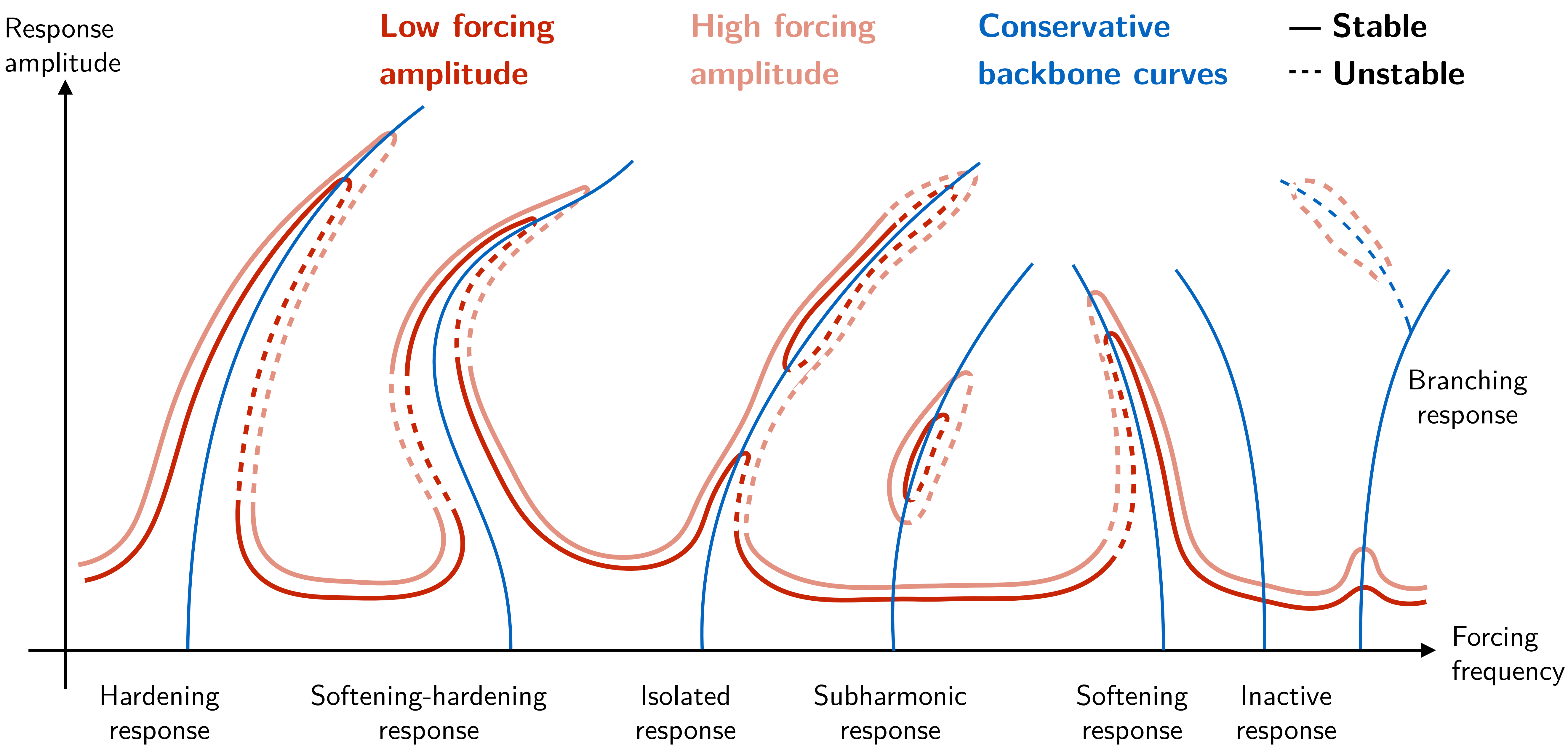
conservative → $f(x)$, *time-periodic* → $g(x,t,\varepsilon)$

$$\dot{x} = f(x) + \varepsilon g(x, t, \varepsilon), \qquad 0 < \varepsilon \ll 1$$

Can we predict existence and stability of periodic orbits of the perturbed system starting from those of the conservative system?

Motivations: the case of Mechanical Vibrations

Consider **N** coupled, periodically forced and damped oscillators for **arbitrary motion amplitude**. Some nonlinear phenomena



Motivations: the case of Mechanical Vibrations

Why would practitioners capitalize on analytical tools?

Computational speed-up for studies of the effect forcing & damping terms

Find isolas: identification is challenging from numerical continuation

Validate and extend experimental routines using the phase-lag quadrature

Available methods:

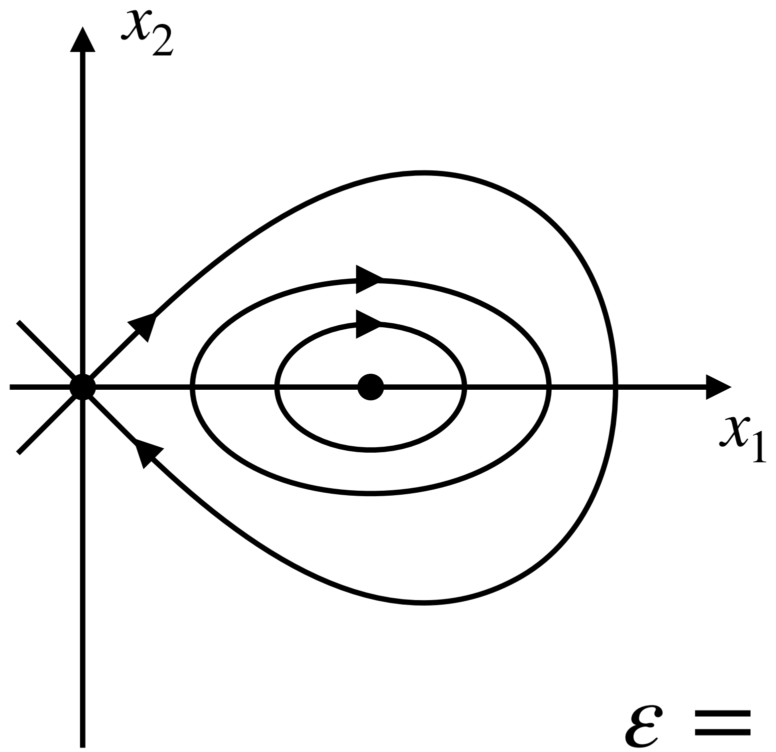
- *Asymptotic expansions from an equilibrium*[1]
- *LSM & SSM*[2]
- *Energy-type arguments*[3]
- *Melnikov methods*

1: Nayfeh & Mook (2007); 2: De la Llave & Kogelbauer (2020); 3: Peeters, G. Kerschen, & Golinval (2011), Hill, Cammarano, Neild & Barton (2017)

Overview of the Classic Melnikov Method

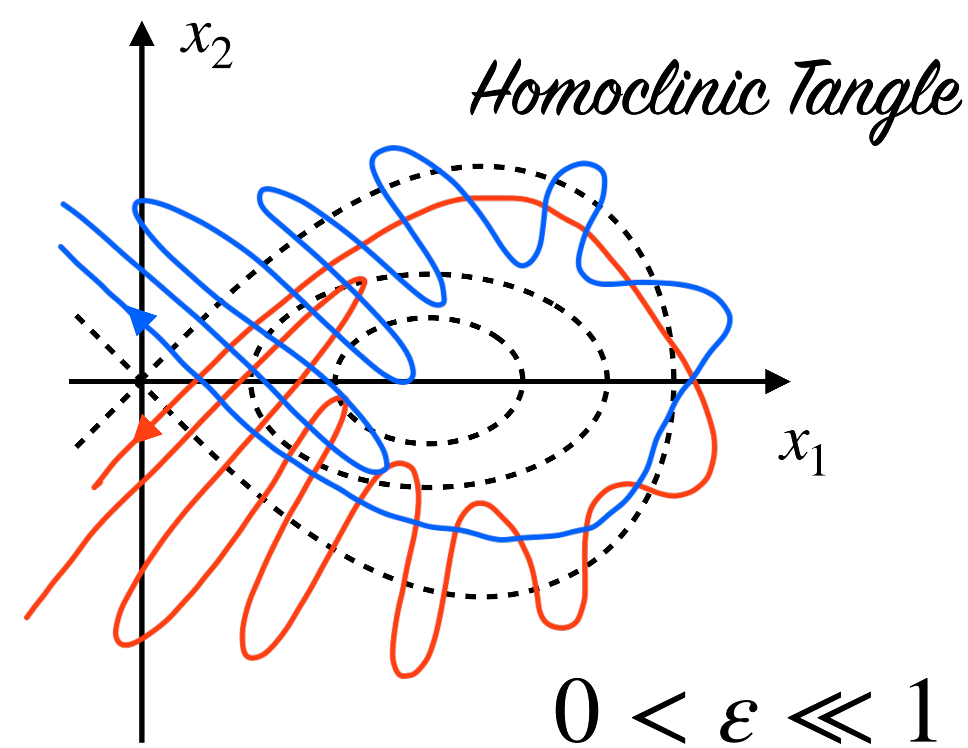
- Or better, the Poincaré-Arnold-Melnikov method (1963)

- Originally: $\dot{x} = JDH(x) + \varepsilon g(x, t), \quad g(x, t + T) = g(x, t), \quad x \in \mathbb{R}^2$



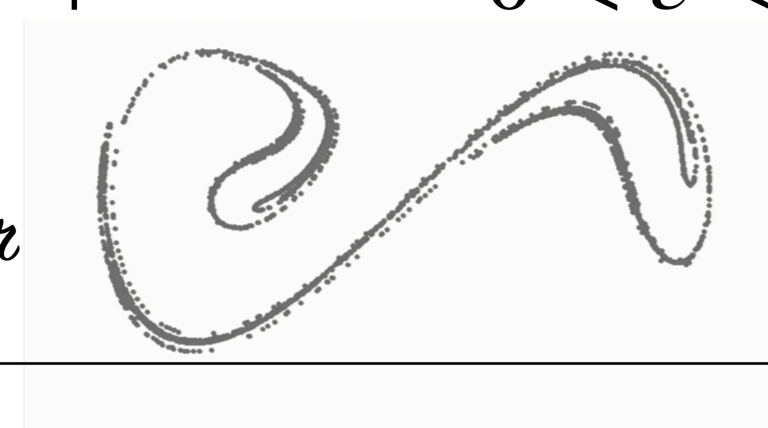
$\varepsilon = 0$

If $\mathscr{M}(s)$ has a transverse zero



$0 < \varepsilon \ll 1$

Chaotic attractor



Melnikov (1963), Guckenheimer & Holmes (1983), Yagasaki (1996)

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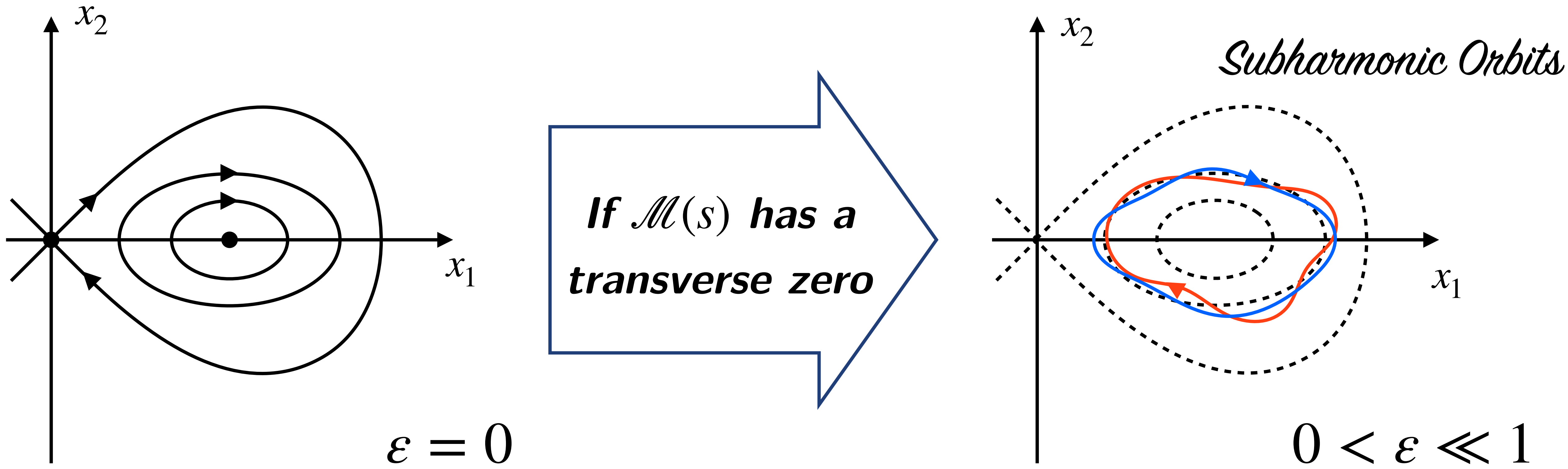


- Extended to integrable, low-dimensional hamiltonian systems

... not the case for structural problems in practical applications.

Melnikov (1963), Guckenheimer & Holmes (1983), Yagasaki (1996)

Setup: Weakly Forced and Damped Systems

- Mechanical system with n degrees of freedom, whose conservative limit is defined by the Lagrangian $q \in \mathbb{R}^n$

$$L(q,\dot{q}) = \frac{1}{2}\langle\, \dot{q}\,, M(q)\dot{q}\rangle + \langle\, \dot{q}\,, G_1(q)\rangle + G_0(q) - V(q)$$

and its energy reads: $H(q,\dot{q}) = \frac{1}{2}\langle\, \dot{q}\,, M(q)\dot{q}\rangle - G_0(q) + V(q)$

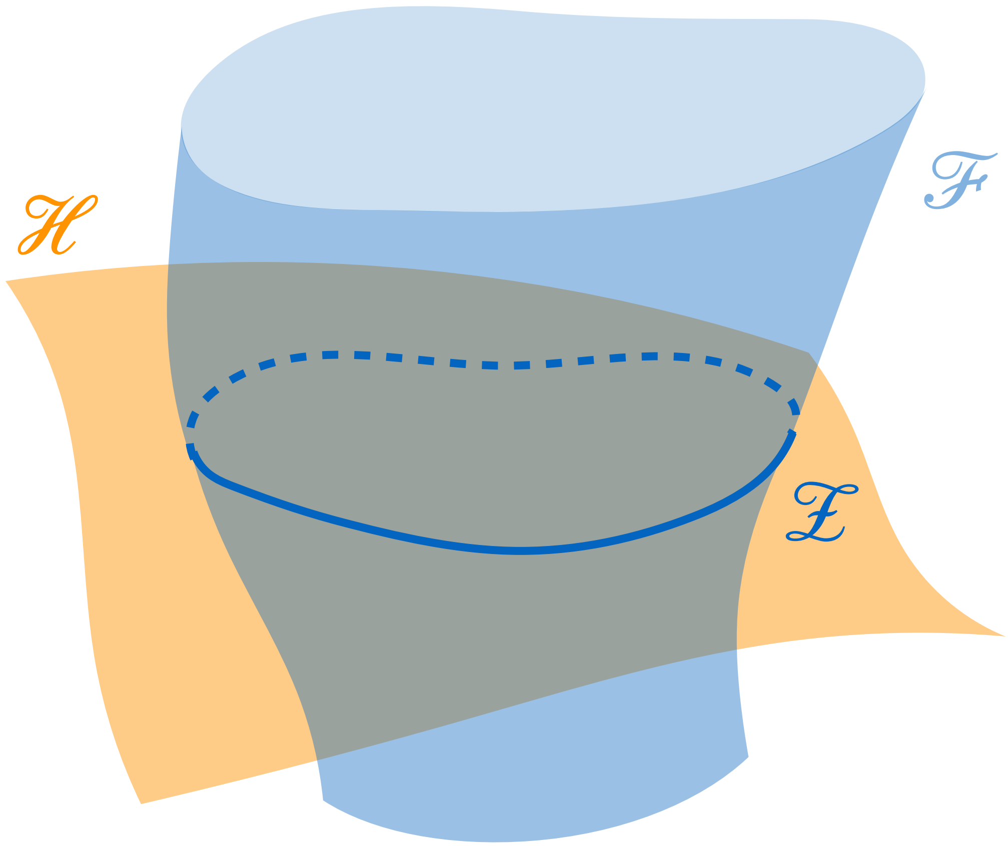
- Collecting any dissipative or active force in the small, time-periodic Lagrangian component Q with frequency Ω, the equations of motion are

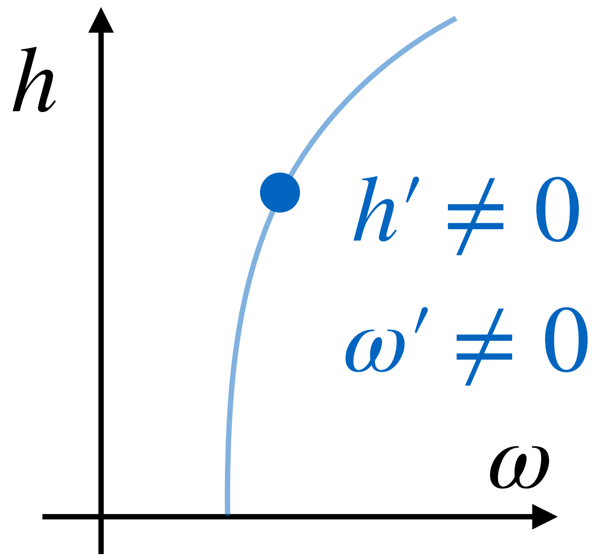
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \varepsilon Q(q,\dot{q},t;\Omega,\varepsilon), \qquad 0 \le \varepsilon \ll 1$$

Periodic Orbits of Conservative Systems

- Present in almost all energy levels
- Generically, they exist in families

  NNMs
- Not structurally stable
- Types of orbits in 1 parameter families:
  - Regular periodic orbits





Muñoz-Almaraz, Freire, Galán, Doedel & Vanderbauwhede (2003)

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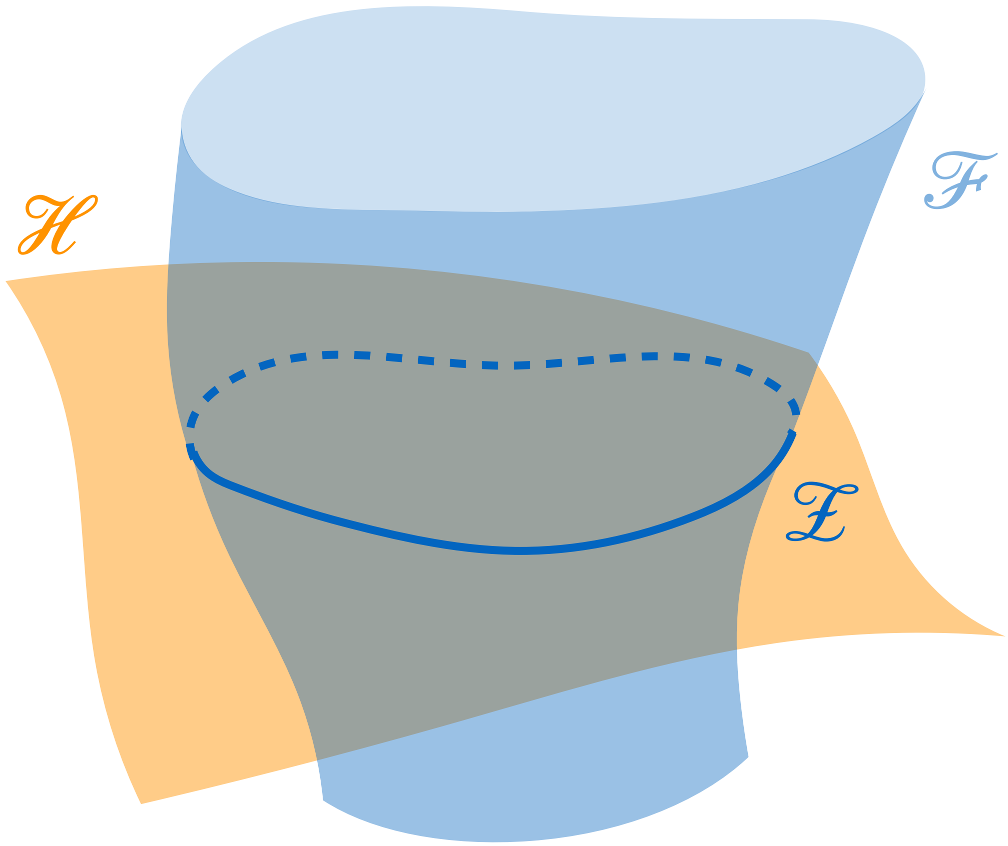
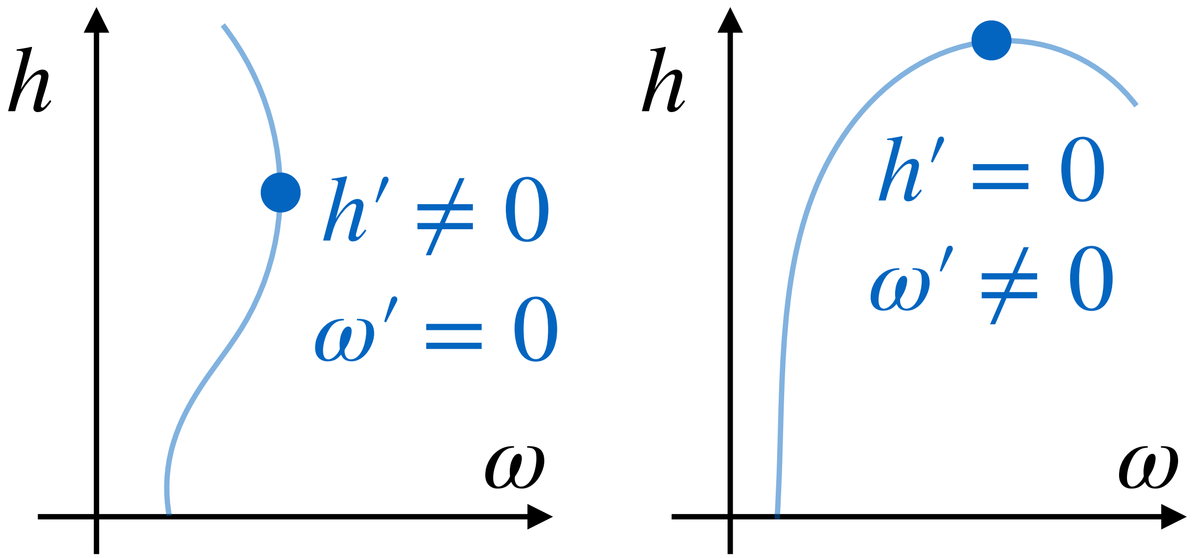


- Types of orbits in 1 parameter families:
  - Regular periodic orbits
  - Folding periodic orbits



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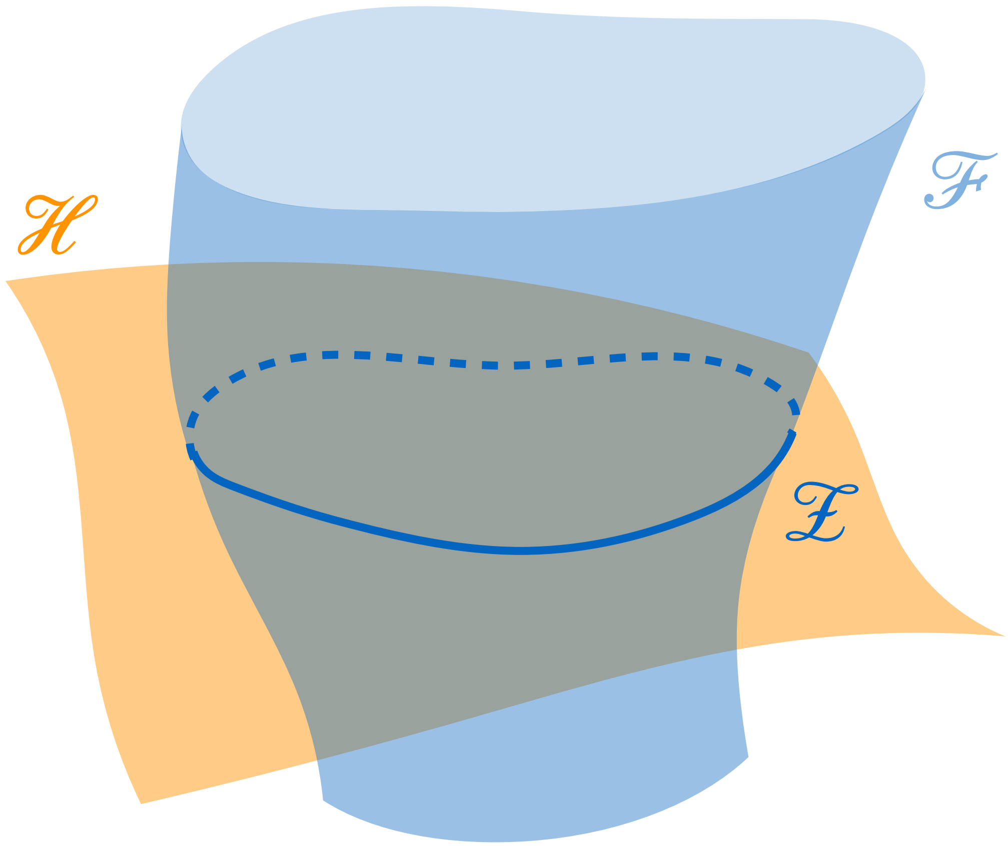
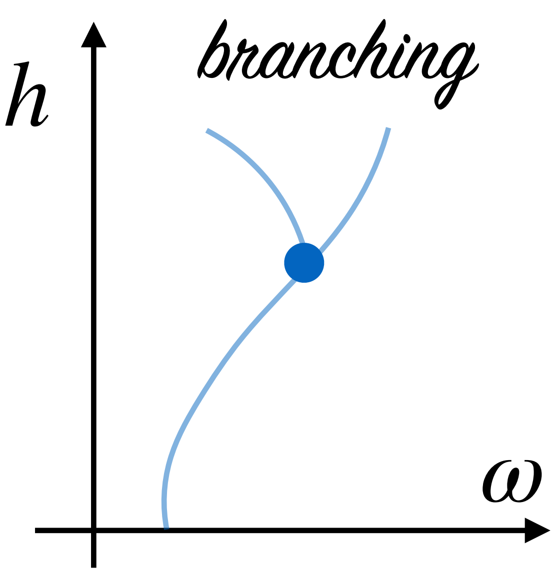
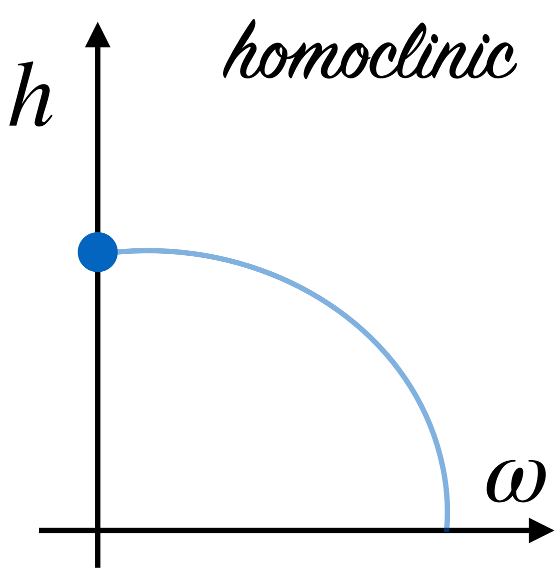


- Types of orbits in 1 parameter families:
  - Regular periodic orbits
  - Folding periodic orbits
  - ~~Critical cases~~

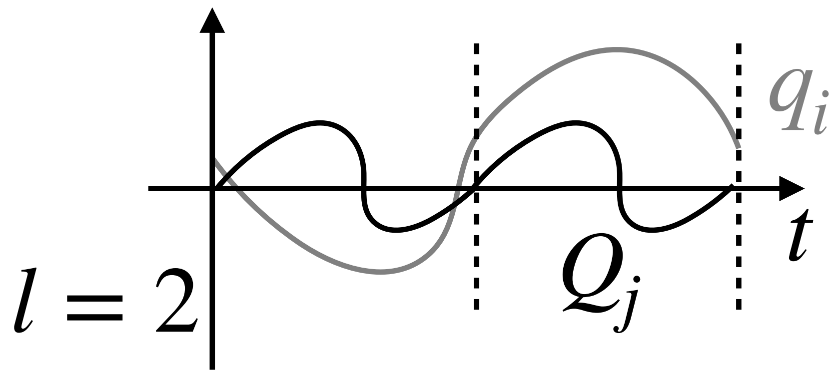




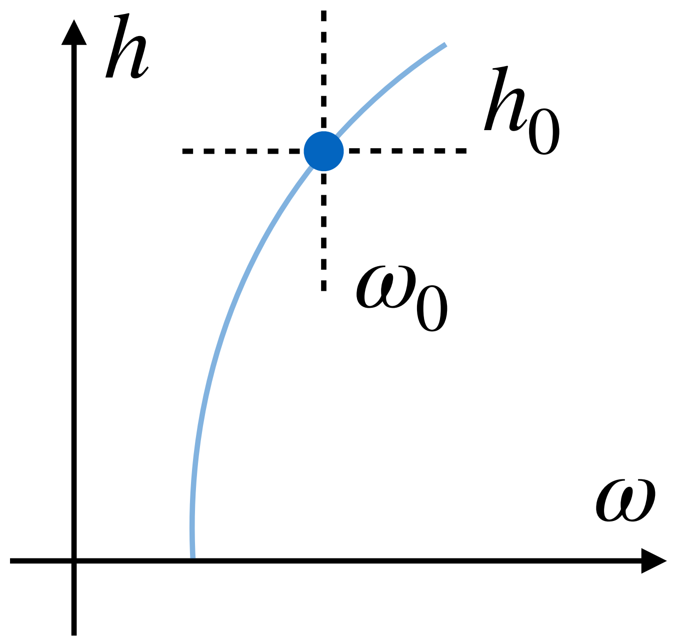
Muñoz-Almaraz, Freire, Galán, Doedel & Vanderbauwhede (2003)

Perturbation from the Conservative limit

- We look for subharmonic orbits of order $l \in \mathbb{N}$ in the forced-damped system



- Pick a regular orbit $q_0(t)$ with period τ_0 of the conservative backbone curve at (ω_0, h_0)



- Set $q(t) = q_0(t+s) + O(\varepsilon)$ as well as a resonance constraint to fix Ω, either

  (a) **Exact resonance**: $m\Omega = l\omega_0$ with m, l being relatively prime integers, or

  (b) **Near resonance**: $m\Omega = l\omega_0 + O(\varepsilon)$ and $H(q(0), \dot{q}(0)) = h_0$

Main Result: Existence

Work done by non-conservative forces evaluated at the conservative limit!

- Define the Melnikov function

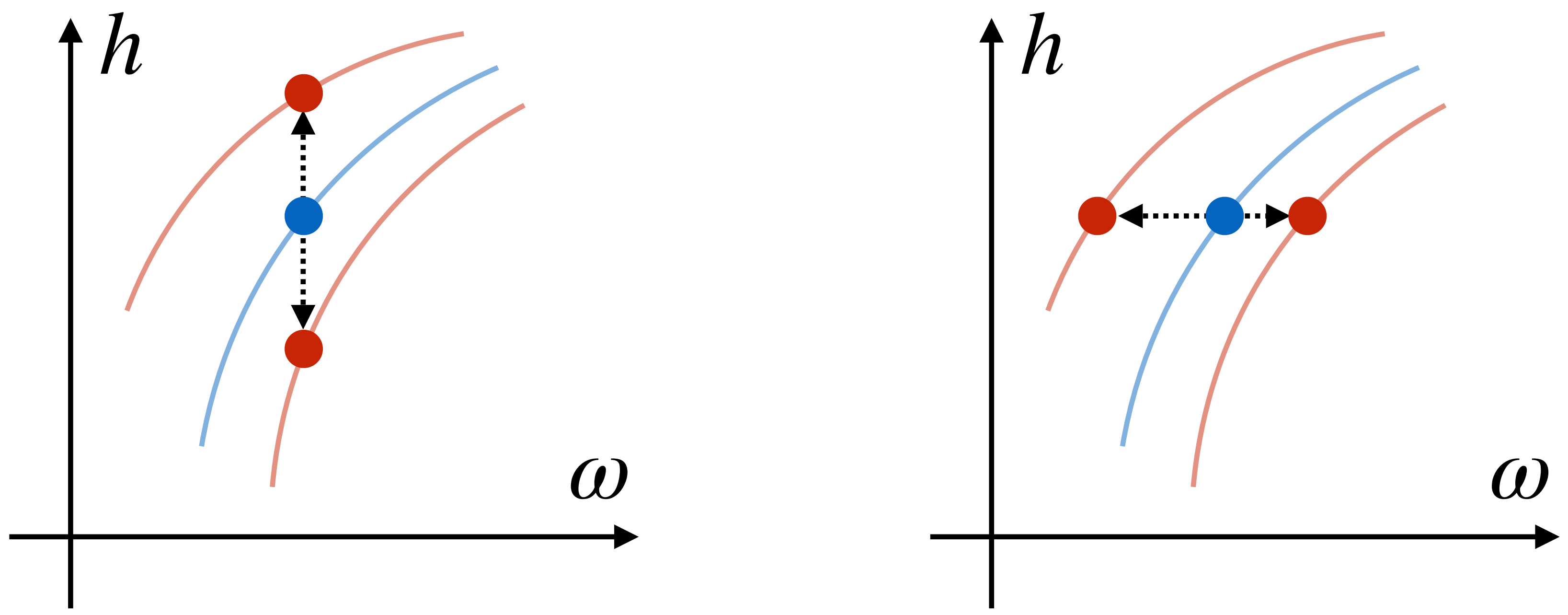
$$\mathscr{M}_{m:l}(s) = \int_0^{m\tau_0} \langle \dot{q}_0(t+s), Q(q_0(t+s), \dot{q}_0(t+s), t; l\omega_0/m, 0)\rangle dt$$

- If $\mathscr{M}_{m:l}(s_0) = 0$ & $\mathscr{M}'_{m:l}(s_0) \neq 0$, the conservative limit $q_0(s_0 + t)$ persists for the weakly damped, periodically forced system

Exact resonance *Near resonance*

$\varepsilon = 0$

$0 < \varepsilon \ll 1$



Cenedese & Haller, *How do conservative backbone curves perturb into frequency responses? A Melnikov function analysis*, P.R.S.A (2020)

Main Result: Existence

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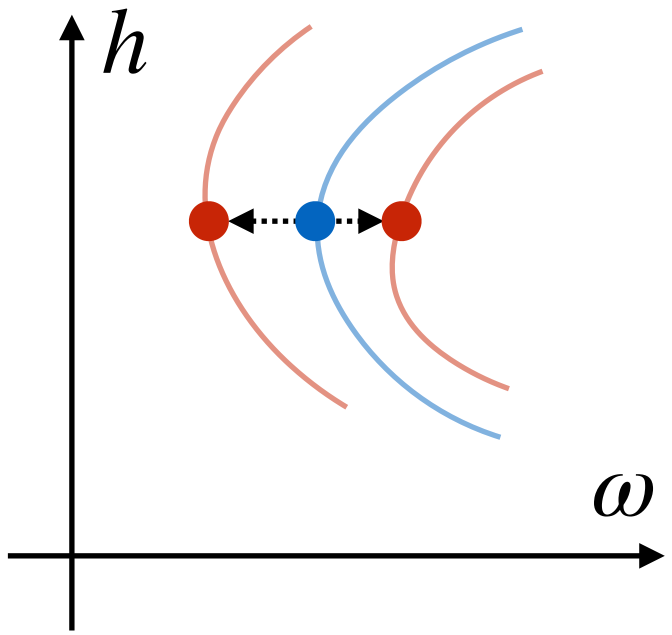
- If $\mathscr{M}_{m:l}(s_0) = 0$ & $\mathscr{M}'_{m:l}(s_0) \neq 0$, but the backbone curve has a fold at (ω_0, h_0), then $q_0(s_0 + t)$ persists in any direction transverse to the folding direction

Fold in ω

$\varepsilon = 0$

$0 < \varepsilon \ll 1$



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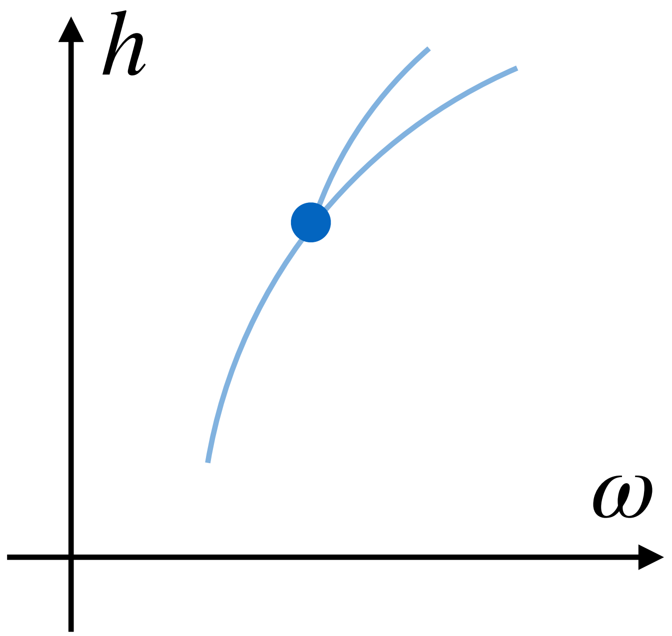
Main Result: Existence

- Define the Melnikov function

$$\mathscr{M}_{m:l}(s) = \int_0^{m\tau_0} \langle \dot{q}_0(t+s), Q(q_0(t+s), \dot{q}_0(t+s), t; l\omega_0/m, 0) \rangle dt$$

- If $|\mathscr{M}_{m:l}(s)| > 0$, the conservative limit does not persist for the weakly damped, periodically forced system

- If the conservative periodic orbit $q_0(t)$ is a critical orbit, the Melnikov function alone is not sufficient to predict the fate of the fate of $q_0(t)$



Cenedese & Haller, *How do conservative backbone curves perturb into frequency responses? A Melnikov function analysis*, P.R.S.A (2020)

Towards Stability

- Write the system in Hamiltonian form

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \qquad g = \begin{pmatrix} 0 \\ Q \end{pmatrix}$$

$$p = \frac{\partial L}{\partial \dot{q}} = M(q)\dot{q} + G_1(q)$$

$$x = (q, p)$$

$$\Rightarrow \quad \dot{x} = JDH(x) + \varepsilon g(x, t; \Omega, \varepsilon)$$

- For the stability of a periodic orbit with period $l\Omega$ we need to study the eigenvalues of the monodromy matrix $X(l\Omega) \in \mathbb{R}^{n \times n}$

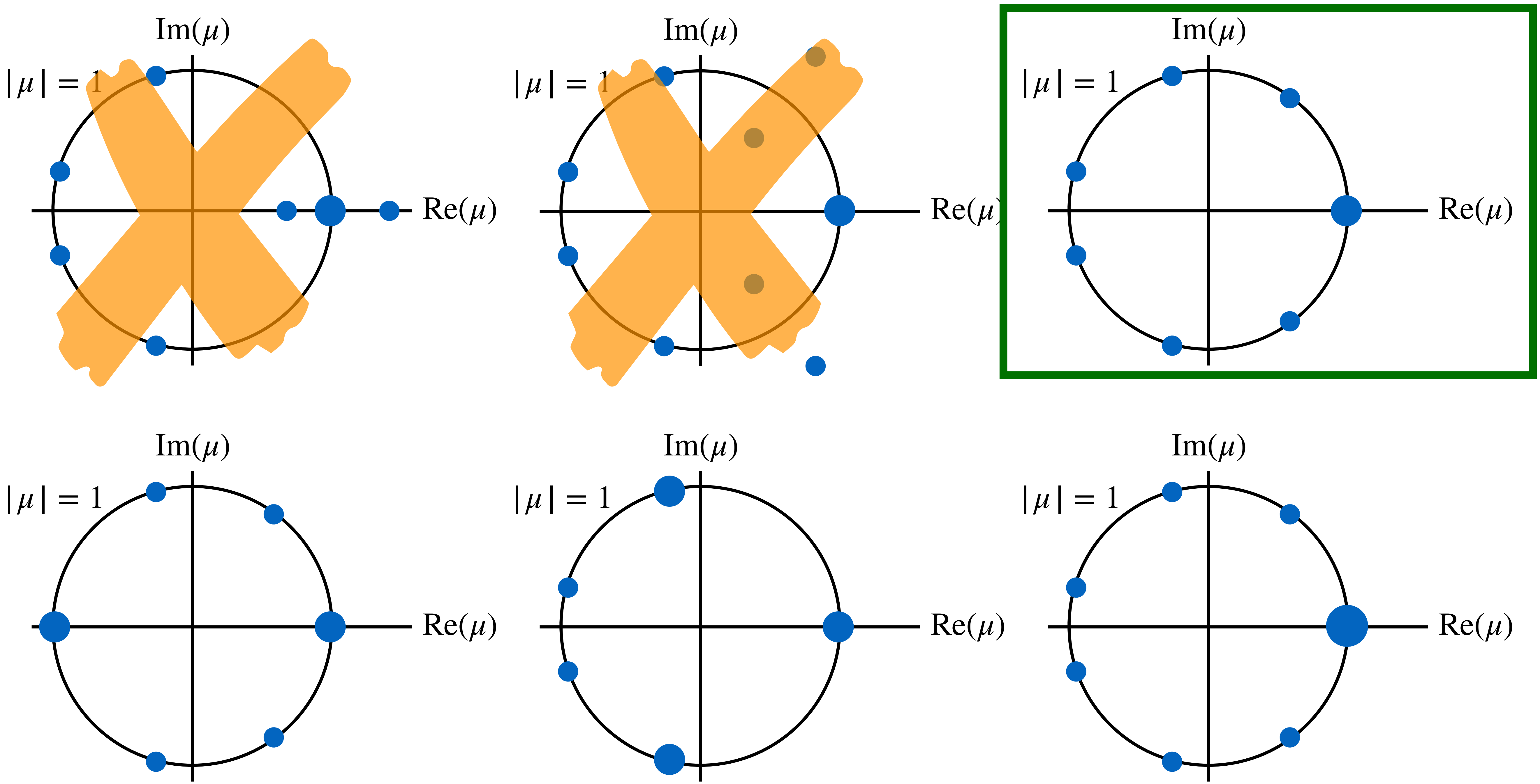
$$\dot{X} = JD^2H(x(t))X + \varepsilon D_x g(x(t), t; \Omega, \varepsilon)$$
$$X(0) = I$$

$\Pi_0 = X_0(m\tau_0)$ is the solution at $\varepsilon = 0$

Arnold (1989), Meyer and Offin (2017)

Towards Stability

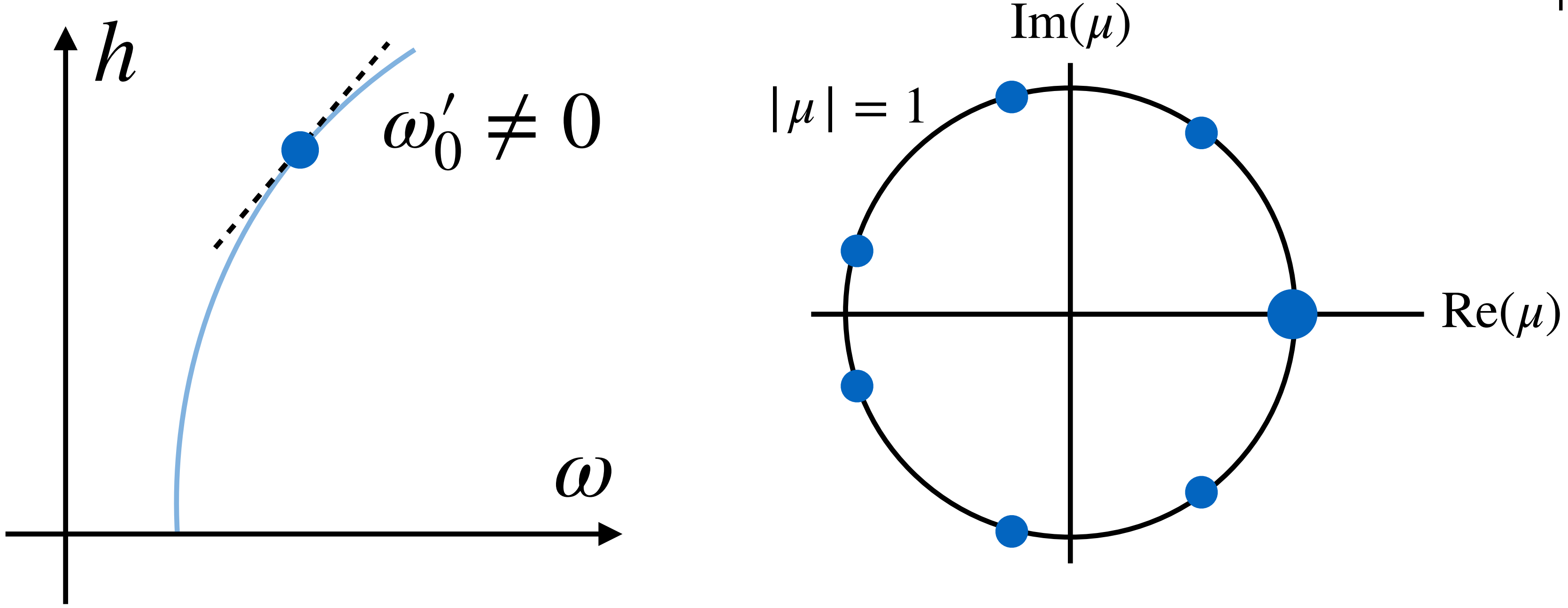
The conservative limit has always at least 2 eigenvalues of Π_0 equal to $+1$. Possible configurations of the unperturbed spectrum



Towards Stability

- We consider a conservative limit that satisfies

Note: the eigenvalue +1 is not regular



- For each of the n couples of eigenvalues, define the **nonlinear damping rate**

$$C_i = -\frac{1}{m\tau_0}\int_0^{m\tau_0} \mathrm{trace}\left(S_i X_0^{-1}(t) D_x g(x_0(t), t; l\omega_0/m, 0) X_0(t) R_i\right) dt$$

$\mathrm{span}(R_i)$ *is the i-th eigenspace,* $S_i = \left(R_i^\top J R_i\right)^{-1} R_i^\top J$ *and* $\dot{X}_0 = J D^2 H(x_0(t)) X_0, \;\; X_0(0) = I$

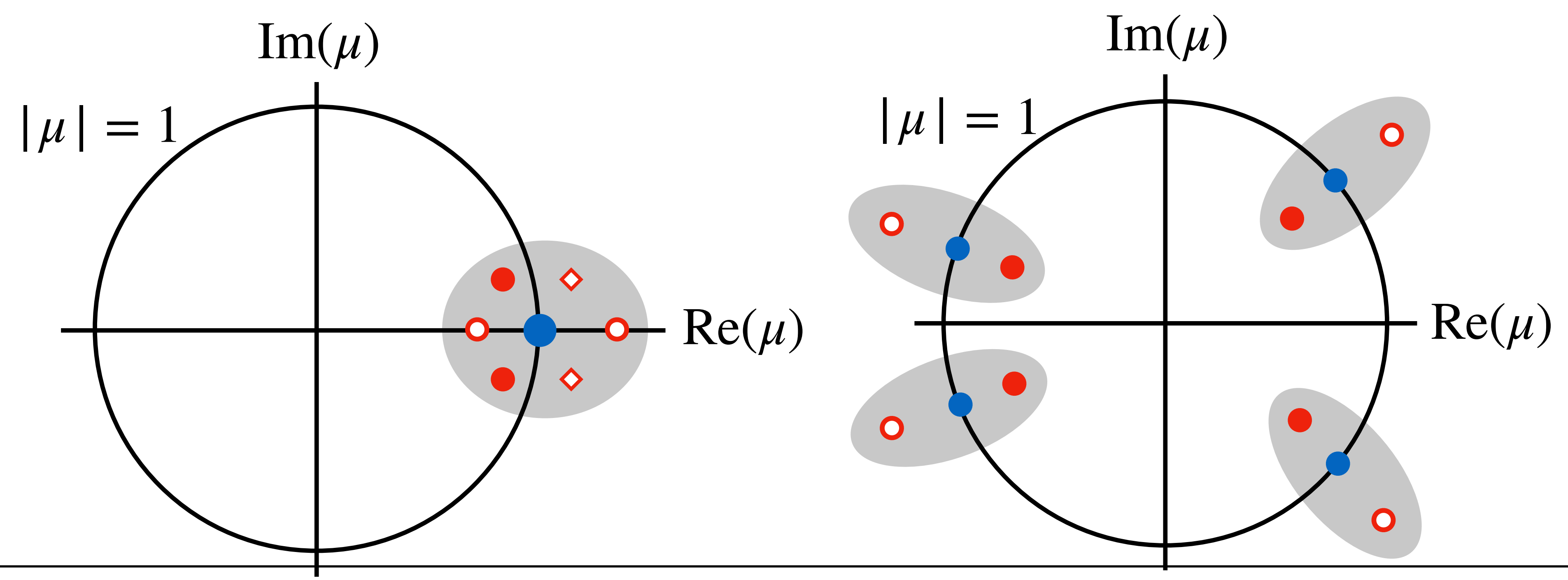
Main Result: Stability

The forced-damped periodic orbit is **unstable** if

$$\mathscr{M}'_{m:l}(s_0)\omega'_0 < 0 \qquad \underline{\text{or}} \qquad \exists\, i \in \{1,\ldots n\} : C_i < 0$$

The forced-damped periodic orbit is **asymptotically stable** if

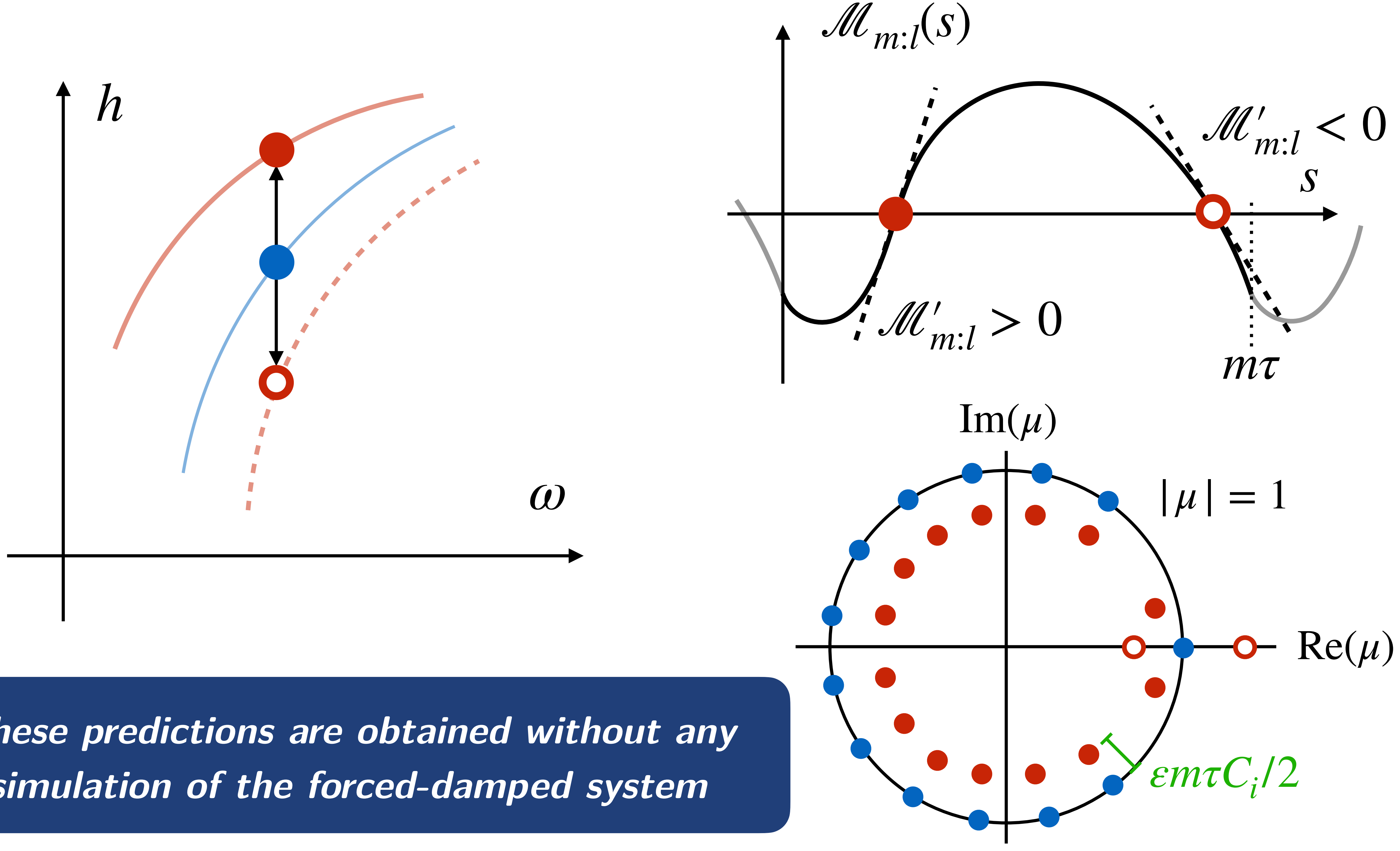
$$\mathscr{M}'_{m:l}(s_0)\omega'_0 > 0 \qquad \underline{\text{and}} \qquad C_i > 0\ \forall\, i \in \{1,\ldots n\}$$



Cenedese & Haller, *Stability of forced-damped response in mechanical systems from a Melnikov analysis*, Chaos (2020)

Connection with Experimental Observations

- Assume that the nonlinear damping rates are positive



These predictions are obtained without any simulation of the forced-damped system

Remarks

- The formula for the nonlinear damping rate is complex.

$$C_i = -\frac{1}{m\tau_0}\int_0^{m\tau_0} \text{trace}\left(S_i X_0^{-1}(t) D_x g(x_0(t), t; l\omega_0/m, 0) X_0(t) R_i\right) dt$$

- For $n = 1$, $C_1 = -\frac{1}{m\tau_0}\int_0^{m\tau_0} \text{trace}\left(D_x g(x_0(t), t; l\omega_0/m, 0)\right) dt$

- For $Q = F(t) - \alpha M(q)p$, then $C_i = \alpha \; \forall \, i \in \{1, \dots n\}$

- Instability conditions can be formulated for other cases

Example: Subharmonics in a Gyro

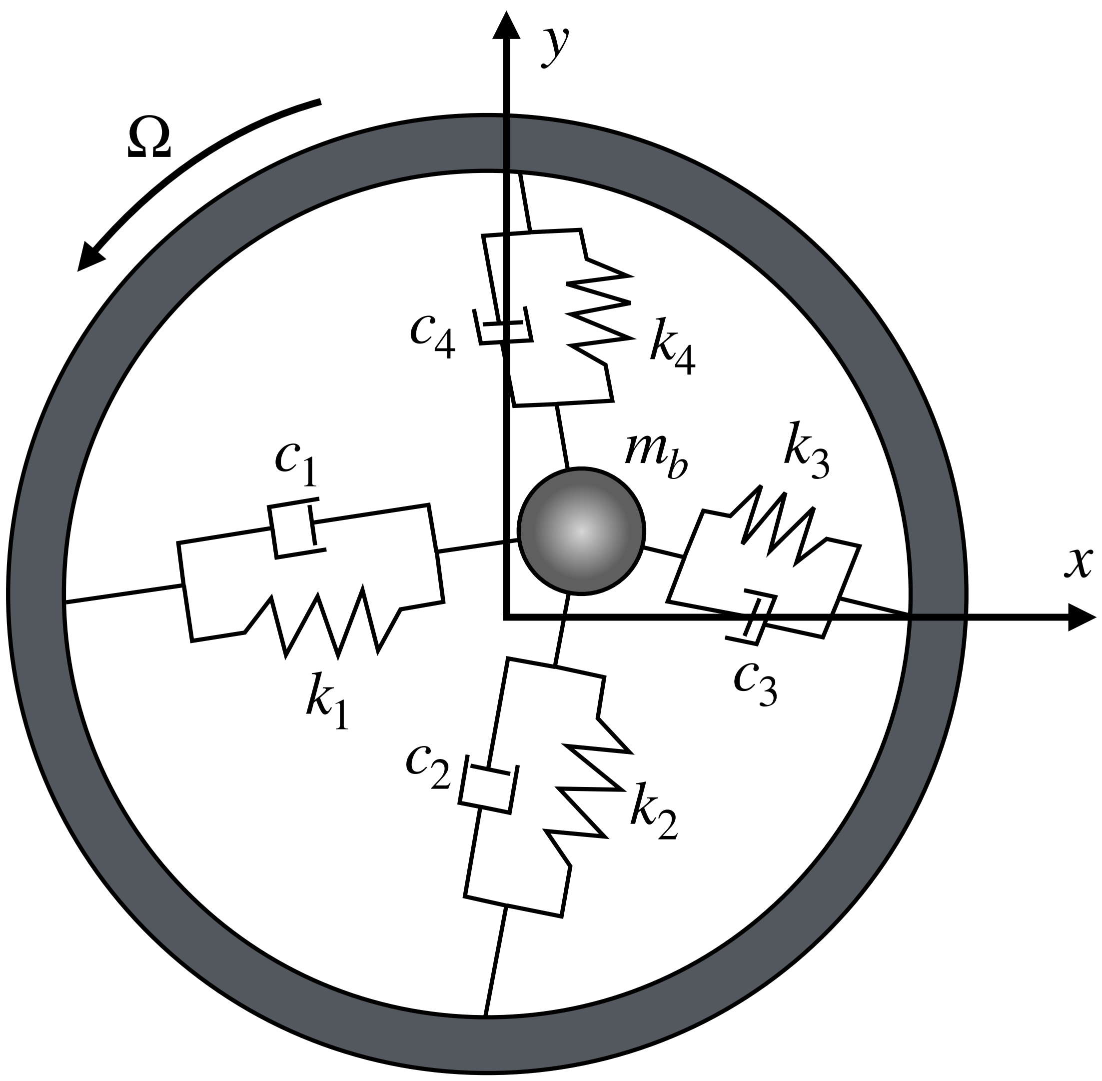
$$m_b\ddot{q} + 2G\dot{q} - m_b\Omega^2 q + DV(q) = \hat{Q}(q, \dot{q}, t)$$

$$G = m_b \begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix},$$

$$V(q) = \frac{1}{2}\sum_{j=1}^{4} k_j\big(l_j(x,y) - l_0\big)^2,$$

$$l_{1,3}(x,y) = \sqrt{(l_0 \pm x)^2 + y^2},$$

$$l_{2,4}(x,y) = \sqrt{x^2 + (l_0 \pm y)^2},$$



$$\Omega = 0.942,\ l_0 = 1,\ k_1 = 1,\ k_2 = 4.08,\ k_3 = 1.37,\ k_4 = 2.51$$

Example: Subharmonics in a Gyro

$$m_b\ddot{q} + 2G\dot{q} - m_b\Omega^2 q + DV(q) = \hat{Q}(q, \dot{q}, t)$$

$$\hat{Q}(q, \dot{q}, t) = \varepsilon\big(Q_{d,\alpha}(q, \dot{q}) + Q_{d,\beta}(q, \dot{q}) + Q_f(t)\big)$$

- Damping linearly depending on the absolute velocities the mass m_b (e.g. air damping) $\varepsilon Q_{d,\alpha}(q, \dot{q}) = -\varepsilon\alpha m_b(\dot{q} + m_b^{-1}Gq)$;
- Stiffness-proportional damping for the spring-damper elements, i.e. $c_j = \varepsilon\beta k_j$ for $j = 1,\ldots 4$ and $\varepsilon Q_{d,\beta}(q, \dot{q}) = -\varepsilon\beta C(q)\dot{q}$,

$$C(q) = \sum_{j=1}^{4} k_j \begin{bmatrix} \big(\partial_x l_j(x, y)\big)^2 & \partial_x l_j(x, y)\partial_y l_j(x, y) \\ \partial_x l_j(x, y)\partial_y l_j(x, y) & \big(\partial_y l_j(x, y)\big)^2 \end{bmatrix}$$

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- Mono-harmonic forcing of frequency $l\Omega$

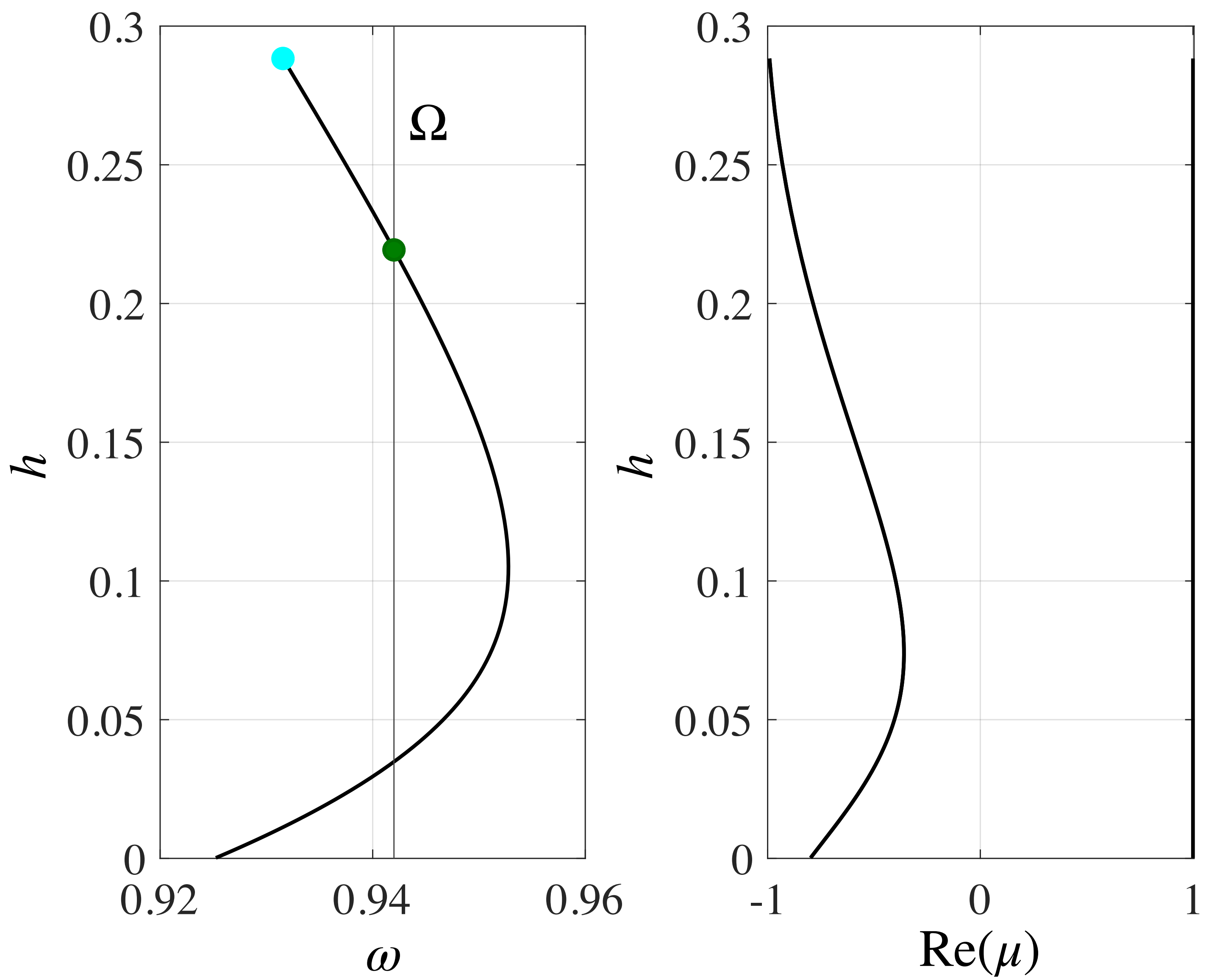
$$\varepsilon Q_f(t) = \varepsilon\begin{pmatrix} +\cos(l\Omega t) \\ -\sin(l\Omega t)\end{pmatrix}, \quad l \in \mathbb{N}\,.$$

Example: Subharmonics in a Gyro

- Equations of motion in Hamiltonian form

$$\dot{q} = -Gq + p,$$
$$\dot{p} = -DV(q) - Gp + \varepsilon\big(Q_f(t) - \alpha p - \beta C(q)(p - Gq)\big)\,.$$

- Conservative limit:
  - Linearized frequencies at the equilibrium 0.92513 and 3.1431
  - Focus on the first NNM



Example: Subharmonics in a Gyro

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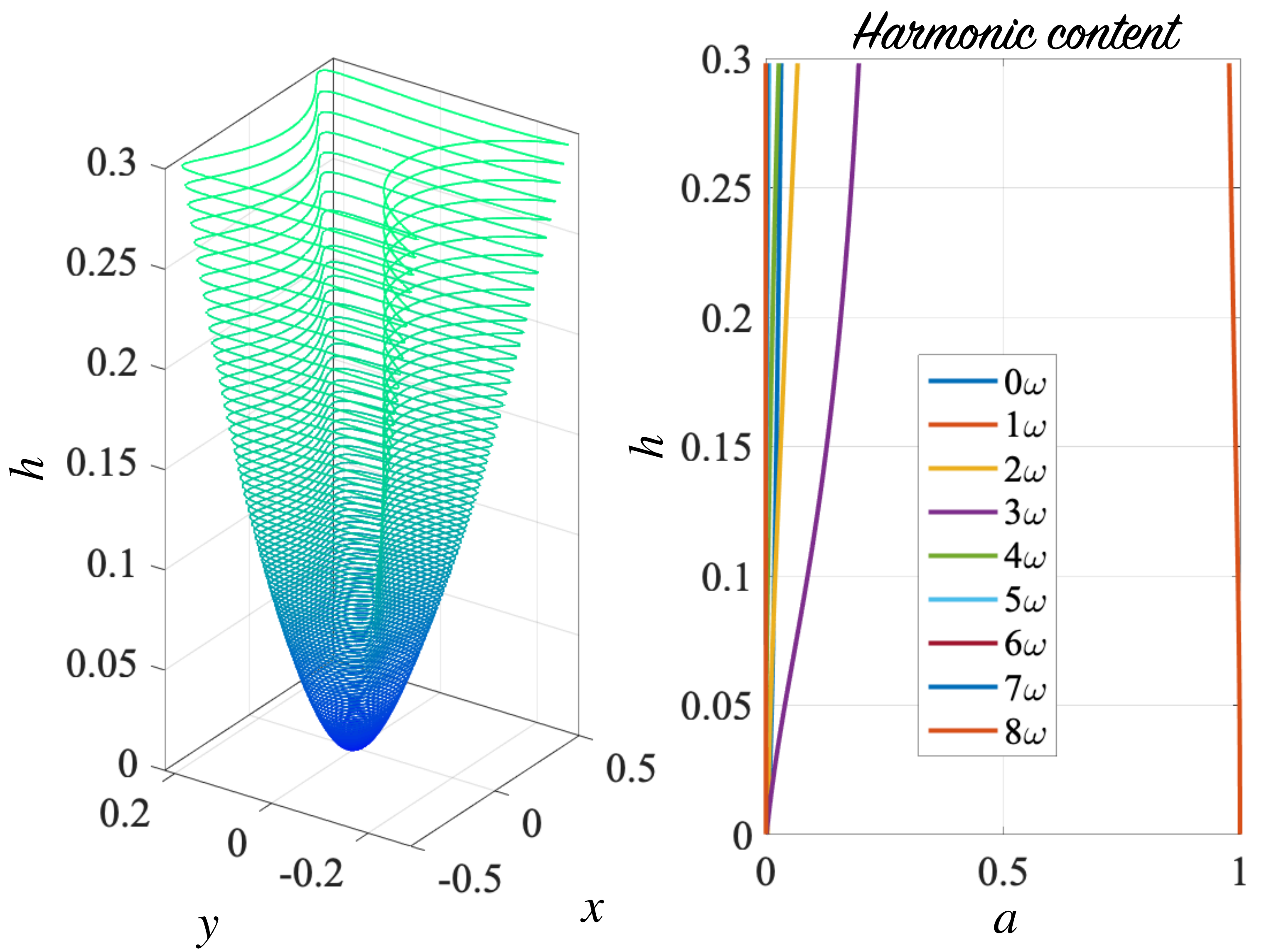
- Conservative limit:
  - Linearized frequencies at the equilibrium 0.92513 and 3.1431
  - Focus on the first NNM
  - Set $l = 3$

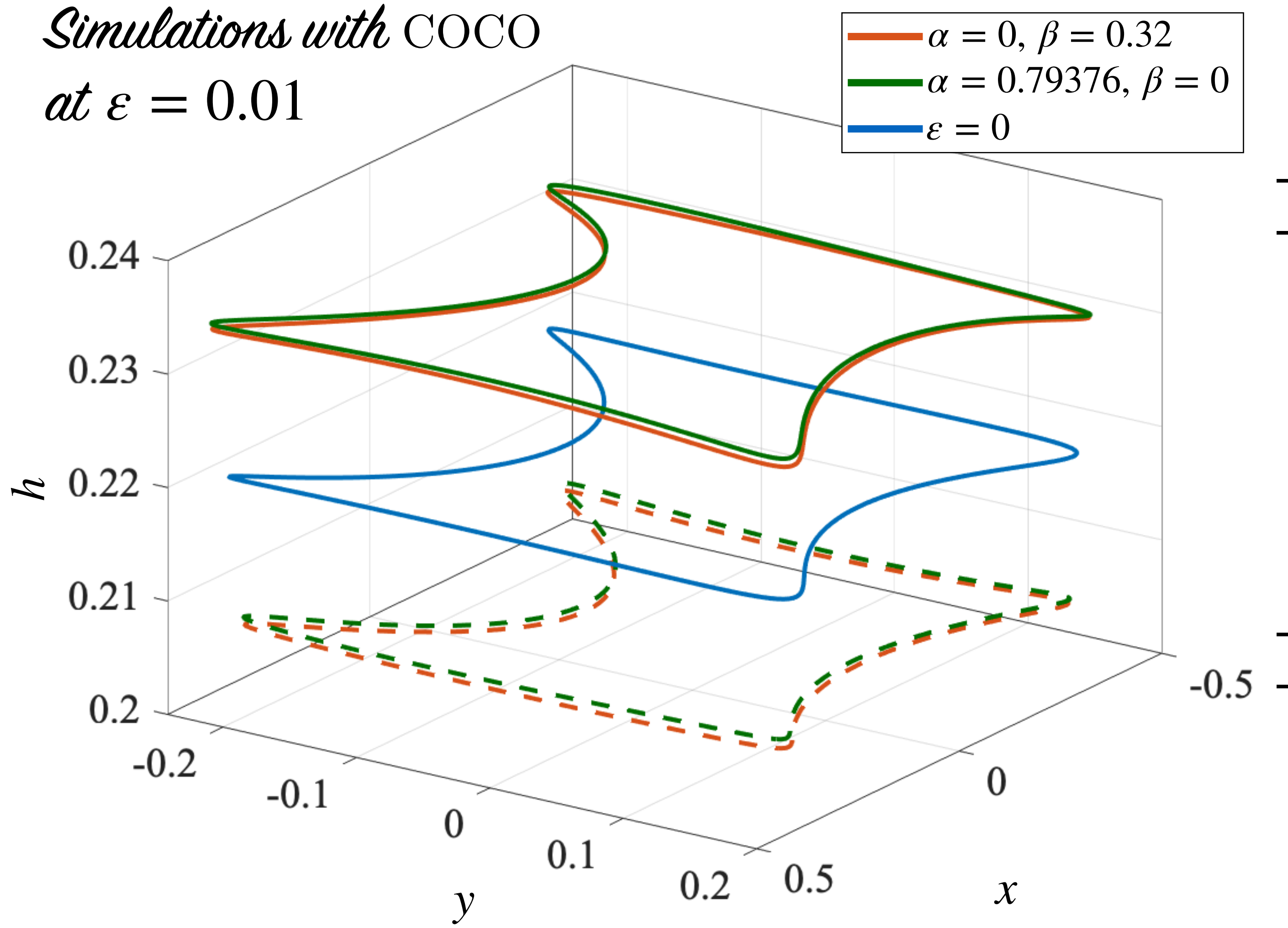


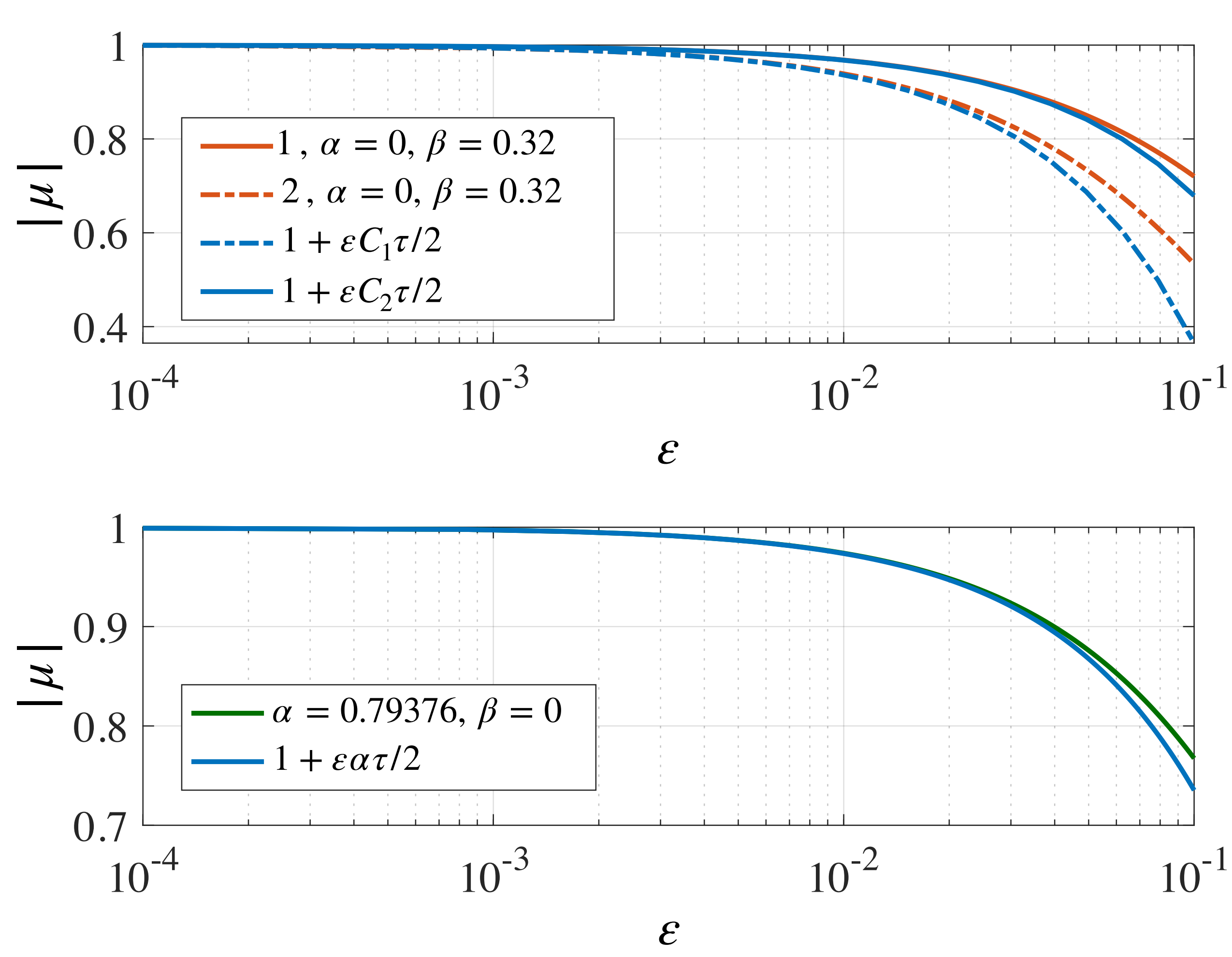
Example: Subharmonics in a Gyro

- Analysis of the two separate damping mechanisms

- The Melnikov function is $\mathscr{M}_{1:3}(s) = 1.4402\cos(3\Omega s) - 1.1553$ for $\alpha = 0.76376,\ \beta = 0$ and $\alpha = 0,\ \beta = 0.32$

Simulations with COCO *at* $\varepsilon = 0.01$





From Single Orbits to Orbits Families

- We have a framework to study eventual **singular behaviors** when varying a parameter κ

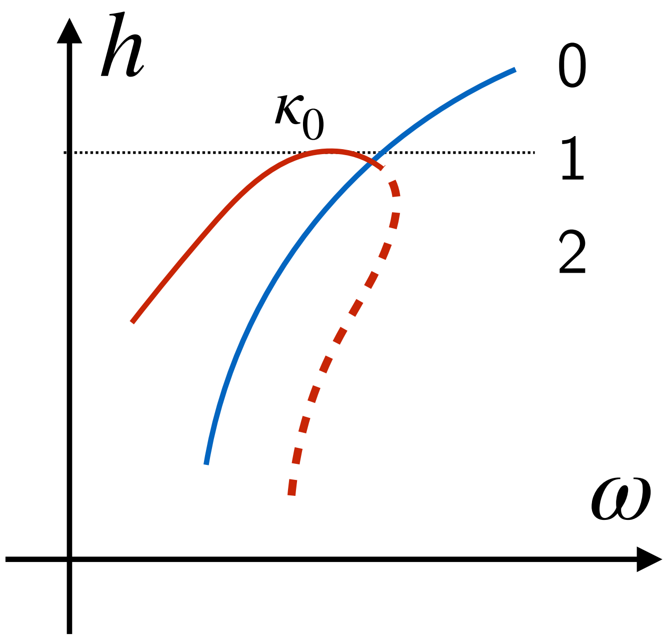
- We focus on quadratic zeros, defined as:

$$\mathscr{M}_{m:l}(s_0, \kappa_0) = D_s \mathscr{M}_{m:l}(s_0, \kappa_0) = 0 \qquad D_{ss} \mathscr{M}_{m:l}(s_0, \kappa_0) \neq 0$$

- The simplest case is the one of **limit point** (codim. 0)

$$D_\kappa \mathscr{M}_{m:l}(s_0, \kappa_0) \neq 0$$

Detection of maximal responses

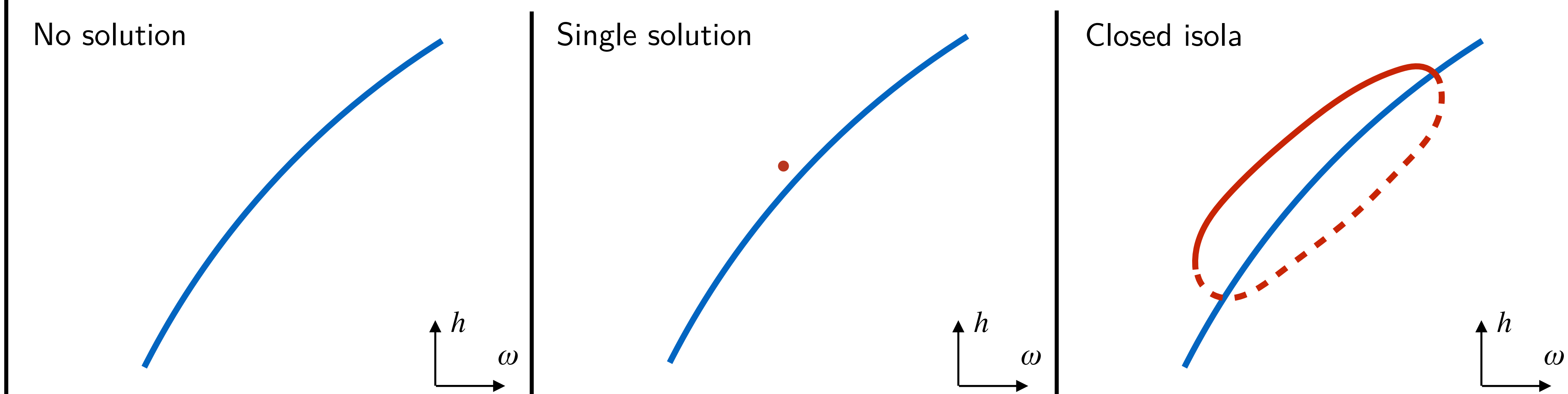


A Zoo of Bifurcations

Defining conditions: $\mathscr{M}_{m:l}(s_0, \omega_0) = D_s\mathscr{M}_{m:l}(s_0, \omega_0) = D_\omega\mathscr{M}_{m:l}(s_0, \omega_0) = 0$

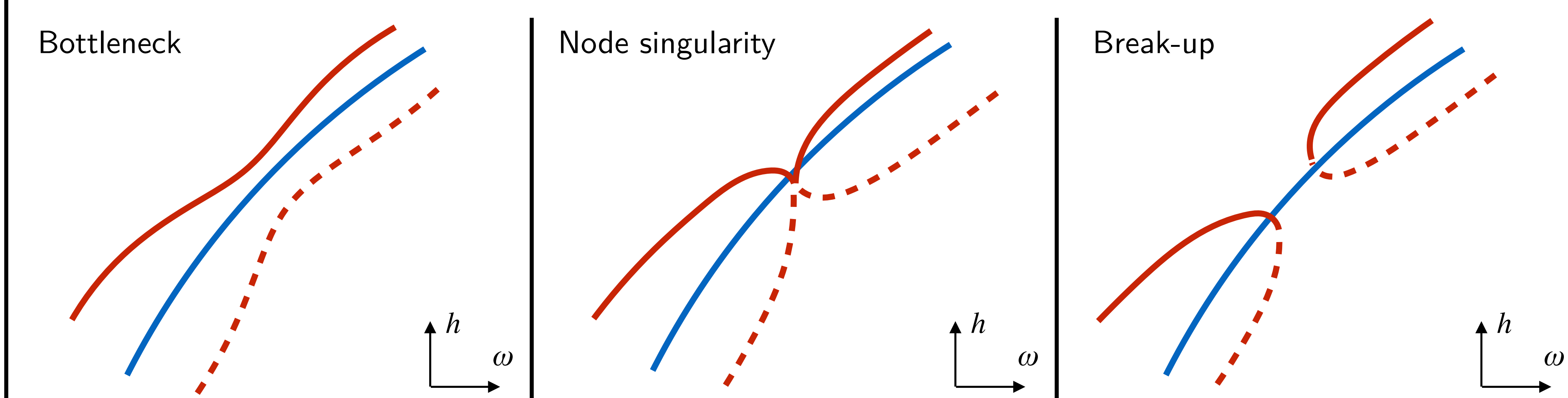
Isola Center

Nondegeneracy condition: $\det(D^2\mathscr{M}_{m:l}(s_0, \omega_0)) > 0$



Simple Bifurcation

Nondegeneracy condition: $\det(D^2\mathscr{M}_{m:l}(s_0, \omega_0)) < 0$

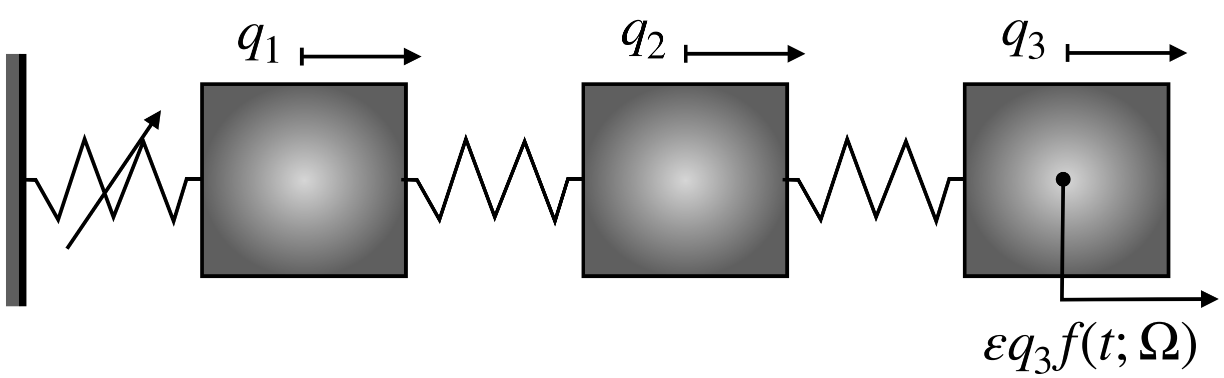


Cenedese & Haller, *How do conservative backbone curves perturb into frequency responses? A Melnikov function analysis*, P.R.S.A (2020)

Example: Parametric Forcing and Isolas

$$\begin{aligned}
\dot{q} &= p, \\
\dot{p}_1 &= -k(q_1 - q_2) - k/3q_1 - aq_1^2 - bq_1^3 - \varepsilon\alpha p_1, \\
\dot{p}_2 &= -k(q_2 - q_1) - k(q_2 - q_3) - \varepsilon\alpha p_2, \\
\dot{p}_3 &= -k(q_3 - q_2) + \varepsilon(q_3 f(t;\Omega) - \alpha p_3),
\end{aligned}$$

$$\begin{aligned}
q, p &\in \mathbb{R}^3 \\
k &= 1, \\
a &= -0.5, \\
b &= 1,
\end{aligned}$$

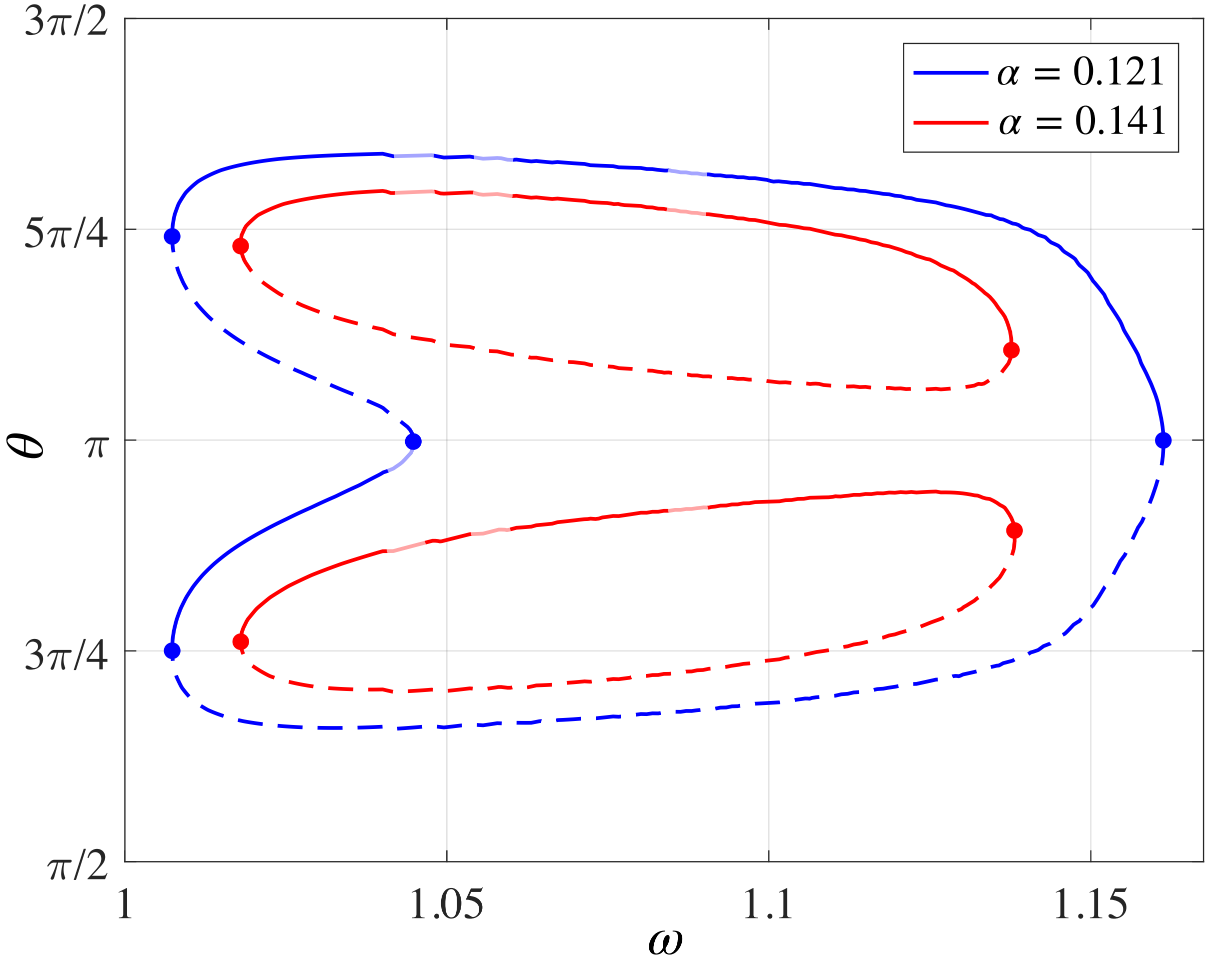


Approximation of a square-wave up to the 6-th harmonic

$$f(t;\Omega) = \frac{4}{\pi}\sum_{j=1}^{3}\frac{1}{2j-1}\sin\big((2j-1)\Omega t\big),$$

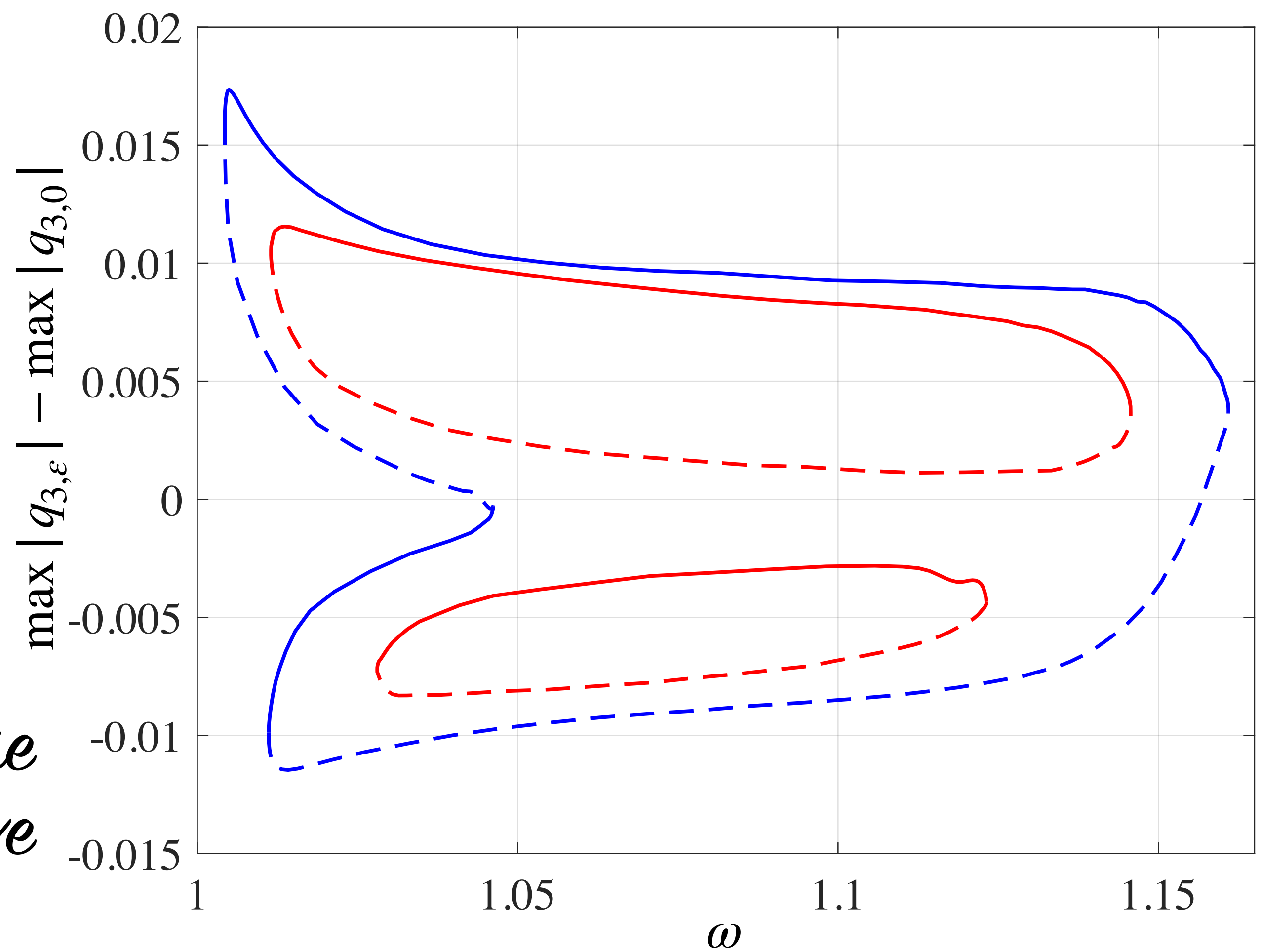
Example: Parametric Forcing and Isolas

- Assume a 1:1 resonance and evaluate $\mathscr{M}_{1:1}$ along the family



Loci of zeros of the Melnikov function as function of the frequency and the phase shift of the orbit

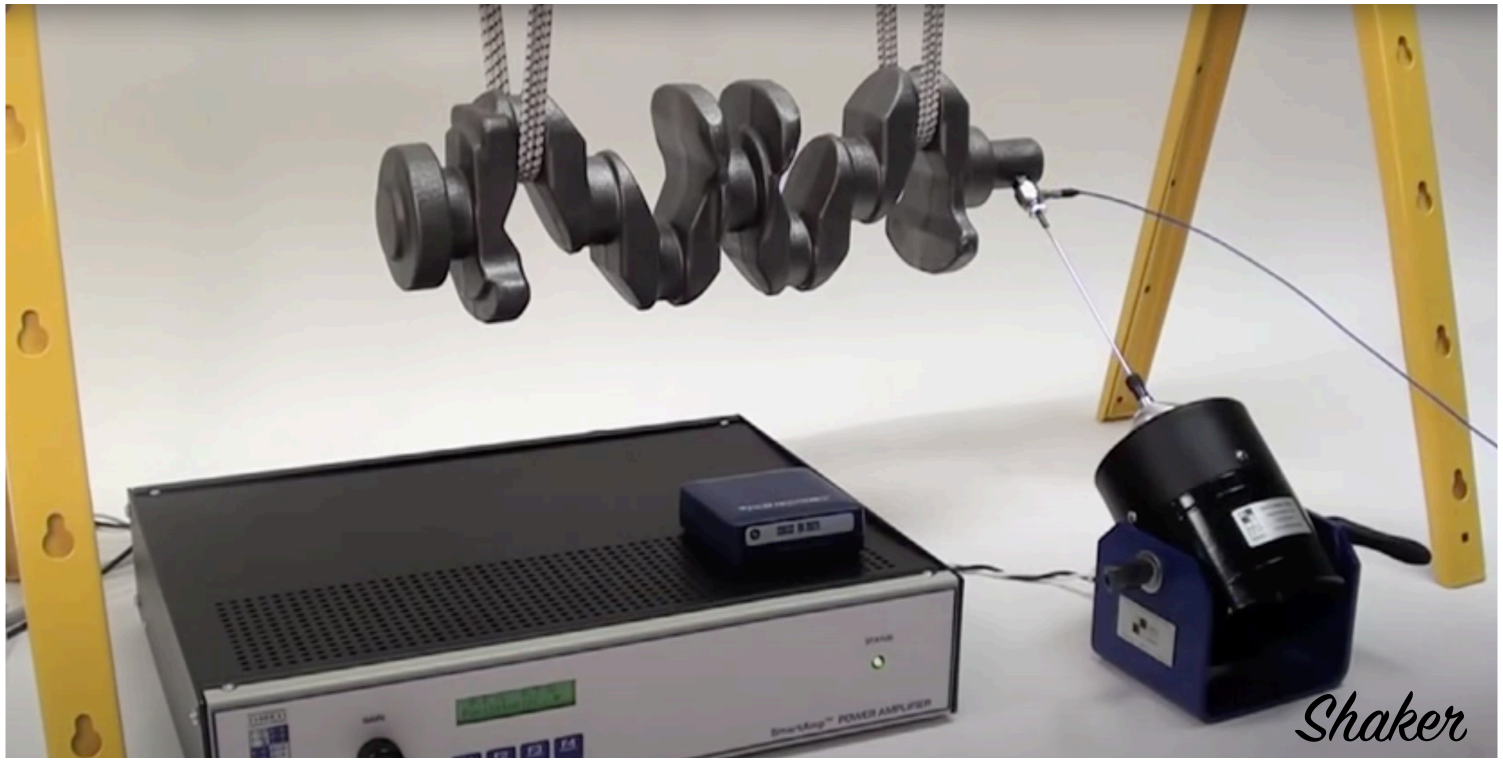
$\varepsilon = 0.0025$



Distance of the frequency response from the backbone curve

Experimental Applications

Testing for backbone curve extraction



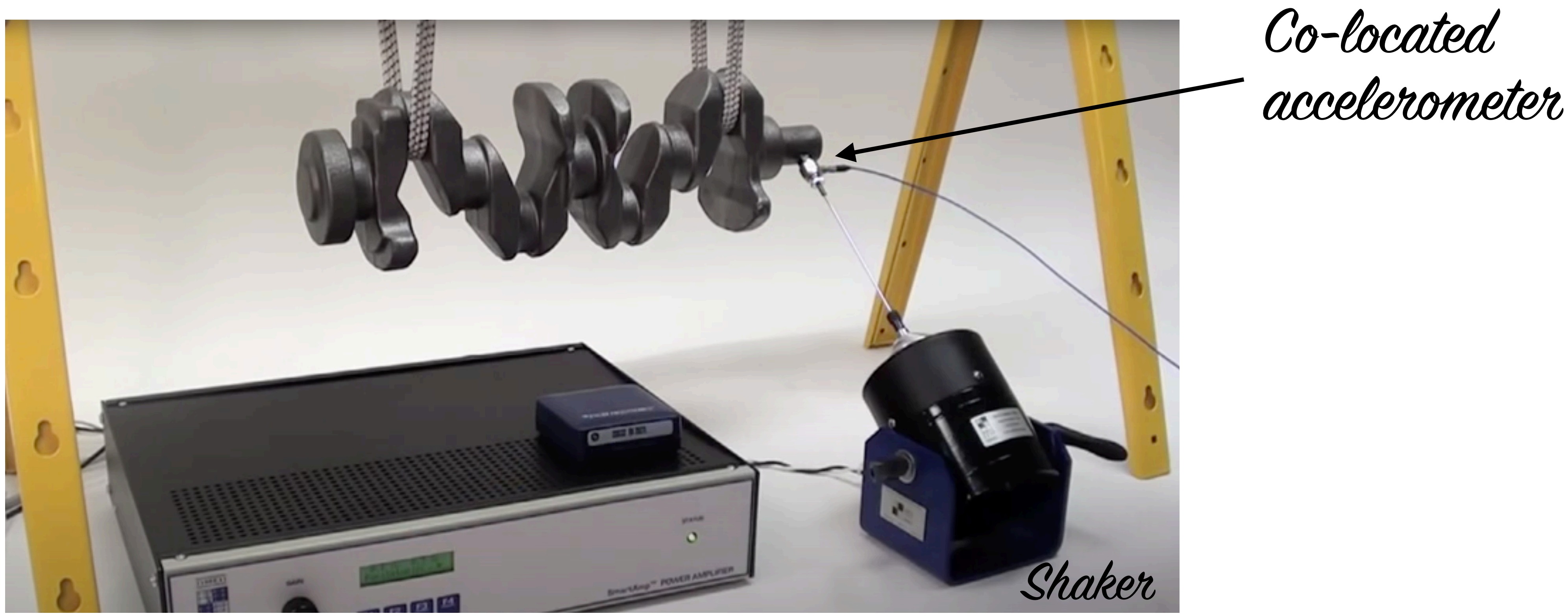
Phase-lag quadrature criterion: forcing is exactly balancing the damping if the phase lag between forcing and measurement is 90°

This was show for: synchronous motions and linear damping

Peeters, G. Kerschen, & Golinval (2011), Renson, Gonzalez-Buelga, Barton & Neild (2017), Peter, Scheel, Krack, & Leine (2018)

Experimental Applications

Testing for backbone curve extraction



Phase-lag quadrature criterion: forcing is exactly balancing the damping if the phase lag between forcing and measurement is 90°

Using our Melnikov analysis one can show that this is valid, when forcing is mono-harmonic, for arbitrary motions and damping shapes, but only for co-located measurements!

Summary and Future Directions

- An energy balance is sufficient to establish the existence of weakly forced-damped vibrations from the conservative limit, while their stability can be studied with nonlinear damping rates

- These analytical results matches with available ones for single-degree-of-freedom oscillators and with real life observations

- Our approach offers significant advantages both for numerical and experimental studies

- What about the survival of tori?