



# Existence of Spectral Submanifolds in Time Delay Systems

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## Abstract

Spectral submanifolds (SSMs) are invariant manifolds of a dynamical system, defined by the property of being tangent to a spectral subspace of the linearized dynamics at a steady state. We show existence, along with certain desirable properties such as smoothness, attractivity and conditional uniqueness, of SSMs associated to a large class of spectral subspaces in time delay systems. Building on these results, we generalize the criteria for existence of inertial manifolds – defined as globally exponentially attracting Lipschitz invariant manifolds of finite dimension – and show that they need not have dimension equal to that of the physical configuration, in contrast to previous accounts. We then demonstrate the applicability of these results on a few simple examples.

**Keywords** Invariant manifolds · Spectral submanifolds · Inertial manifolds · Time delay systems · Sun-star calculus

**Mathematics Subject Classification** 34K19

## 1 Introduction

The potential existence of lower-dimensional representations of evolutionary systems is central to many application-oriented subdisciplines of mathematical physics. Perhaps the most natural way to achieve dimensional reduction is by restricting attention to lower-dimensional attracting invariant manifolds. Our core interest here is to uncover a certain class of such invariant manifolds in the phase space of delay differential equations (DDEs), with the objective of characterizing their internal asymptotic dynamics. Delay equations are abundant in the applied sciences, with examples ranging from

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epidemic theory (Diekmann and Heesterbeek 2000), population dynamics (Cushing 2013) and other areas of mathematical biology (Glass et al. 2021; Milton et al. 2009; Zhang et al. 2018) to a large variety of fields on the engineering side (Insperger and Stépán 2011), including control theory (Richard 2003; Habib et al. 2022), machine tool vibrations (Molnár et al. 2017) and traffic dynamics (Orosz et al. 2010).

The evolution in a delay equation with locally Lipschitz continuous right-hand side can be characterized by a semiflow  $\{\varphi_t\}_{t \geq 0}$  acting on an open subset of the space of continuous functions,  $X = C([-h, 0]; \mathbb{R}^n)$ . Here  $n$  is the physical dimension of the space on which the delay equation is posed and  $h$  is the maximal delay. The invariant manifolds of interest to us are spectral submanifolds (or SSMs), i.e., manifolds that act as nonlinear continuations of dominant eigenspaces of a linearized delay equation (see Haller and Ponsioen 2016 and Haller 2025). This is a local concept based at a stationary state  $u_0 \in X$  (i.e.,  $\varphi_t(u_0) = u_0$  for all  $t \geq 0$ )<sup>1</sup> of the phase space, i.e. SSMs need only be constructed on a neighbourhood of  $u_0$ . We may suppose without loss of generality that  $u_0 = 0$  (see Sect 3). Suppose the dynamics, linearized about 0, are driven by a generator  $A$ . An SSM associates to a spectral subset  $\Sigma \subset \sigma(A)$  a manifold  $W^\Sigma$ , that is locally invariant under  $\varphi_t$  and tangent to the spectral subspace  $\text{im}(P_\Sigma)$  at 0.<sup>2</sup> The overarching objective of SSMs is to subdivide the dynamics (preserving its nonlinear structure) according to asymptotic properties of trajectories in the vicinity of the stationary state, akin to spectral projections for linear systems. Early versions of this concept date back to the stable, unstable and center manifold theorems (Liapounoff 1907; Hirsch and Pugh 1970; Kelley 1966), where the spectral subsets are confined to either a half-space of the complex plane or the imaginary axis. In practical applications, especially to dissipative systems, it is useful to maintain generality and consider arbitrary spectral subsets. Of main interest are of course pseudo-unstable and slow manifolds characterized by the rightmost part of the spectrum (Irwin 1980; Chen et al. 1997), since they correspond to the asymptotically prevalent part of the dynamics. There exist, however, scenarios in which an alternative choice of  $\Sigma$  proves more beneficial (e.g., the buckling beam example of Haller et al. 2023).

The theoretical foundations of SSMs were laid by the parameterization method of Cabre et al. 2003; Cabré et al. 2003, 2005; Haro and de la Llave 2006b, a (with assumptions optimized for the stable case). The classical linearization results of Sternberg (Sternberg 1957, 1958) allow for more general hyperbolic equilibria but under more stringent nonresonance assumptions (Haller 2025). In recent years, the evolution of SSM theory, as well as its equation- and data-driven implementations, were motivated by contemporary problems of increasing physical complexity. Specifically, soft robotics inspired an extension of the theory to temporally aperiodically forced systems (Haller and Kaundinya 2024; Kaundinya et al. 2025); the identification of edge states in fluid dynamics (Kaszás et al. 2022) and the aforementioned beam buckling example motivated the introduction of fractional and mixed-mode SSMs (Haller et al. 2023); the precise identification of nonlinearities in tabletop gravity experiments prompted the development of oblique projection to SSMs along their stable foliation (Bettini et al. 2025) (see also Szalai 2020 on general results on such foliations); nonlinear

<sup>1</sup> More general invariant sets are of course permitted, but in this work we focus on the case of a fixed point.

<sup>2</sup> Here,  $P_\Sigma$  is the spectral projection associated to  $\Sigma$ , assuming that it is well defined.

structural vibration problems motivated the development of SSM theory for random dynamical systems (Xu et al. 2025). Advancements on the practical side include a more direct equation-driven formulation for finite element models (Jain and Haller 2022; Vizzaccaro et al. 2022; Opreni et al. 2023), and a data-driven approach for constructing SSMs (Cenedese et al. 2022; Axås et al. 2023), enabling treatment of large-scale problems. All these advances along with further ones are discussed in the book of Haller (2025).

Given the prevalence of time delay systems in applied sciences, it is pressing to extend the theory to encompass such systems. A pathway in this direction has already been initiated by the recent work of Szaksz et al. (2024, 2025) (see also Tang and Li 2025), who demonstrate, importantly, that the setting of delay equations permit the SSM reduction to be performed in the function space (serving as the phase space) itself without the need for a priori discretization. Here, we prove existence and regularity results in order to provide the rigorous foundations facilitating such a reduction.

The idea to seek finite-dimensional approximations of delay equations is by no means novel: several distinct techniques have been developed over the years. Numerical methods aimed at discretizing the full equation are surveyed in the work of Bellen and Zennaro (2013), with a particularly popular example being the semi-discretization method of Insperger and Stépán (2002, 2011); a survey with specific emphasis on how (and whether) dynamical qualities of the full, infinite-dimensional system are preserved under such discretizations was conducted by Garay (2005). Works on model reduction thus far have typically focused on linear systems, achieving lower dimensionality by means of an optimal choice of linear projection that best resembles the original dynamics (Michiels et al. 2011; Beattie and Gugercin 2009). An alternative approach permitting a certain class of nonlinearities is proposed by van de Wouw et al. (2015). Techniques aimed at invariant manifold computations have focused on the center and unstable manifolds for the most part (Krauskopf and Green 2003; Green et al. 2004; Sahai and Vladimirovsky 2009; Scarcioiti and Astolfi 2015) (see also Farkas 2001, 2002 for an early initiative providing rigorous theoretical results for the convergence of such approximations even for center-unstable manifolds). These works compute the invariant manifolds after having discretized the equation, as opposed to the aforementioned approach of Szaksz et al. (2025) and that of Groothedde and James (2017), the latter being the first instance in which the parameterization method was applied to invariant manifold computation in delay equations (see also de Wolff 2018; Hénot 2021; Hénot et al. 2022 for subsequent developments). Notably, the authors in these works derive the manifold coefficients from the infinitesimal version of the invariance equation, in contrast to what we do herein (see Remark 27). Also notable are the works of He and de la Llave He and De la Llave (2017); He and de la Llave (2016) that extract quasiperiodic solutions of DDEs with state-dependent delay via the parameterization method. Subsequent work by Yang et al. (2021) focuses on limit cycles but also extracts their slow stable manifolds.

On the theoretical side of the delay equations literature, stable ( $\Sigma = \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\}$ ), unstable ( $\Sigma = \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda > 0\}$ ) and center ( $\Sigma = i\mathbb{R} \cap \sigma(A)$ ) manifolds were already subject of discussion in the monograph of Diekmann et al. (2012) (see also the earlier references Hale and Perello 1964 and Diekmann and van Gils 1984). Subsequent theoretical advancements prioritized

encompassing a larger variety of problems rather than types of manifolds (i.e.,  $\Sigma$  remained confined to the same subsets). Particular attention has been given to the notoriously difficult problem of state-dependent delays (Hartung et al. 2006) and center(stable) manifolds thereof (Krisztin 2006; Qesmi and Walther 2009). Moreover, center manifold theory has been extended to cover the case of infinite delays (Matsunaga et al. 2015) and that of impulsive DDEs (Church and Liu 2018). In contrast, Example 5.2 of Chen et al. (1997) considers global pseudo-unstable manifolds ( $\Sigma = \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq \gamma\}$  for some  $\gamma \in \mathbb{R}$ ) of neutral functional differential equations (thus including DDEs) in the functional setting of Hale and Lunel (2013)<sup>3</sup>. The localization of this approach, which we carry out here to obtain SSMs, is more natural to perform in the setting known as sun-star calculus due to Diekmann et al. (2012), utilizing perturbation theory of dual semigroups. The idea for this first came about in the sequence of papers Diekmann (1987); Clément et al. (1987, 1988, 1989, 2020); Diekmann et al. (2023).

A related concept, an inertial manifold (IM) is defined as a globally exponentially attracting, finite dimensional, invariant Lipschitz manifold (Foiias et al. 1988; Constantin et al. 2012; Titi 1990). These properties render IMs the most desirable invariant manifolds for model reduction purposes. Their existence, however, is generally difficult (if not impossible) to prove. The term inertial manifold was initially coined by Foiias et al. (1988) in the context of parabolic PDEs; the question of their existence in the two-dimensional Navier-Stokes equations, however, has remained an open problem since then. In all classical proofs, the existence of IMs hinges on the ratio of the global Lipschitz constant of the nonlinearity and the spectral gap being sufficiently small. In Navier-Stokes flows, this ratio cannot be controlled in an arbitrary fashion by moving down the sequence of eigenvalues of the Laplacian due to the presence of a spatial derivative in the nonlinearity. (This is unlike in reaction-diffusion equations, for which this procedure yields IMs (Robinson 2001)).

In contrast, existence of IMs in the small-delay regime of DDEs has been observed as early as the 60s in a sequence of works by Ryabov (1960a,b, 1961, 1963) (see also Driver 1968, 1976; Chicone 2003), who termed trajectories on IMs special solutions. Their method of proof is focused on extracting such trajectories directly, hence circumventing any mention of spectral gaps or even invariant manifolds, but it has since been shown that these trajectories actually form a  $C^1$  manifold (Chicone 2003). Recently, these results were extended to neutral functional differential equations (Chen and Shen 2022) with improved smoothness (see also Anikushin 2023). All of these works assume a small delay  $h$ , and the dimension of the inertial manifold is always equal to the dimension of the physical space  $n$ .

In this work, we show that small delays are not a necessity for inertial manifolds to exist. We reformulate the problem using the results of Chen et al. (1997) to the familiar smallness condition on the ratio the global Lipschitz constant of the nonlinearity to the spectral gap. This also has the geometric advantage of identifying the IM as an extension of an SSM; that is, we obtain its tangency to a spectral subspace at a fixed point under appropriate conditions. In particular, we demonstrate, through examples

<sup>3</sup> The setup described in Chen et al. (1997) is akin to our Remark 25 – hence a subsequent localization is needed to dispose of the global Lipschitz assumption and obtain tangency results, at the expense of constraining the domain.

as well, that there exist inertial manifolds in delay equations with dimensions differing from  $n$ . We then show explicitly how shrinking the delay is a suitable way to obtain arbitrarily large spectral gaps (see also Chen and Shen 2020), and hence arbitrary smoothness and rate of attraction towards the IM, thus recovering the results of Chen and Shen (2022) for the case of DDEs. Roughly speaking, a spectral gap obtained in this way arises between eigenvalues of the system in the limit  $h \rightarrow 0$  and the infinitely many new ones emanating from  $-\infty$  due to the addition of the delay. In this case, therefore, the IM has to be of dimension  $n$ , which is sometimes called the case of negligible delays (Kurzweil 2006).

## 1.1 Overview

The ideal setting for the purposes herein is the sun-star calculus of Diekmann et al. (2012), since it provides a convenient way to linearize about a fixed point using variation of constants formulae. We recollect the basics of this theory in Sections 2 and 3.

As already noted, the convention in the delay equations literature is to take as phase space  $X := C([-h, 0]; \mathbb{R}^n)$ , the Banach space of continuous functions equipped with the supremum norm  $|\cdot|_X := |\cdot|_\infty$ , for  $h > 0$  (the maximal delay) and  $n \in \mathbb{N}$ . Let  $f : O \rightarrow \mathbb{R}^n$  be a locally Lipschitz continuous function on the open set  $O \subset X$ . We consider autonomous DDEs of the form

$$\dot{x}(t) = f(x_t), \quad (1a)$$

$$x_0 = u, \quad (1b)$$

where  $x_t : \theta \mapsto x(t + \theta) \in X$  denotes the state variable and  $u \in O$  the initial condition. A solution to (1) is a function  $x : [-h, t^*) \rightarrow \mathbb{R}^n$ ,  $t^* \in (0, \infty) \cup \{\infty\}$ , which satisfies (1b), is continuously differentiable on  $(0, t^*)$  such that (1a) holds; and which moreover possesses a right derivative equal to  $f(x_0)$  at 0. Classical results assert (see, for example, Hale and Lunel 2013; Diekmann et al. 2012) that under the local Lipschitz assumption on  $f$  there exists a unique solution to (1) for each initial condition  $u \in O$ . Let us denote by  $\{\varphi_t\}_{t \geq 0}$  the semiflow (1) generates. Regularity properties of  $\varphi$  will be discussed more thoroughly in Section 2.2 (see Proposition 4).

The main results for a DDE of the form (1) are formulated in Section 4. Here, we give a brief overview of what they comprise. As in the Introduction, let us suppose that a stationary state of the semiflow  $\varphi$  is attained at  $0 \in O$ , i.e.,  $f(0) = 0$ . Let  $A$  denote the generator of the semigroup  $t \mapsto D\varphi_t(0)$ , and let  $\Sigma \subset \sigma(A)$  be a nonempty bounded spectral subset with spectral projection  $P_\Sigma$ . For these preliminary statements we shall also assume that  $f$  is  $C^k$  with  $k \geq 1$ , although it is not strictly necessary (see the footnote to assumption (A.1)).

**Theorem 1** *Suppose that the spectral subset  $\Sigma$  is of the form  $\Sigma = \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq \gamma\}$  for some  $\gamma \in \mathbb{R}$ . Then, there exists a neighbourhood  $U$  of the stationary state  $0 \in O$  and a locally invariant (under the semiflow  $\varphi$ )  $C^\ell$  manifold  $W^\Sigma \subset U$  (an SSM) tangent to the spectral subspace  $\operatorname{im}(P_\Sigma)$  at 0. Here the integer  $1 \leq \ell \leq k$  is determined by*

the spectral gap condition

$$\sup \operatorname{Re} (\sigma(A) \setminus \Sigma) < \ell \inf \operatorname{Re} \Sigma.$$

Moreover,  $W^\Sigma$  attracts trajectories that remain in  $U$  at an exponential rate of  $\gamma$ , synchronized along the leaves of a  $C^0$  foliation.

The precise version of this result is Theorem 12. We remark that the  $C^1$  version of Theorem 1 is to be expected to hold based on Chen et al. (1997) and Diekmann et al. (2012). In Section 6, we nevertheless carry out the proof in detail. As required in the proof, we also show the smoothness of the semiflow in the sun-star setting in Appendix B, which has apparently been unavailable in the literature.

The following theorem is a less direct consequence of existing results:

**Theorem 2** Suppose  $\Sigma \subset \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\}$  and that  $\Sigma$  satisfies the additional non-resonance conditions specified in (A.4). Set  $\gamma := \inf \operatorname{Re} \Sigma$  and choose an integer  $1 \leq \ell \leq k$  such that

$$\sup\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < \gamma\} < \ell\gamma.$$

Then, there exists a neighbourhood  $U$  of  $0 \in O$  and a locally invariant (under the semiflow  $\varphi$ )  $C^\ell$  manifold  $W^\Sigma \subset U$  (an SSM) tangent to the spectral subspace  $\operatorname{im}(P_\Sigma)$  at 0.

For the precise statement, see Theorem 16. The proof constructs the desired manifolds by first constructing the spectral foliation (Szalai 2020; Buza 2025) tangent to the spectral subspace corresponding to the spectral subset  $\Sigma \cup \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < \inf \operatorname{Re} \Sigma\}$  within the stable manifold of  $0 \in O$ ; then intersecting the leaf passing through the origin with a manifold produced in Theorem 1. A result analogous to Theorem 2 holds for  $\Sigma \subset \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda > 0\}$  as well, whose proof follows from an application of the parameterization method (Cabre et al. 2003) to the unstable manifold under the assumptions listed in Theorem 18.

Our results on inertial manifolds are described in Section 4.1. The main advancements were described in the final paragraph of the Introduction; we give a summary statement below.

**Theorem 3** Suppose that  $f$ , the right hand-side of the DDE (1), is globally Lipschitz, and hence the semiflow  $\varphi$  is globally defined. Set  $L := \operatorname{Lip}(f - Df(0))$ . Suppose that there exists a large enough spectral gap  $\nu$  (made precise in Theorem 20 and Remark 23) along the negative real axis such that

$$\frac{L}{\nu} < C \tag{2}$$

for some fixed constant  $C > 0$  (see (33)). Then there exists an inertial manifold  $W^\Sigma$  tangent to  $\operatorname{im}(P_\Sigma)$  at  $0 \in O$  (uniquely characterized by the Lyapunov exponents of its trajectories, see (71)-(72)), where  $\Sigma$  is defined as the subset of  $\sigma(A)$  to the right of this spectral gap. The manifold  $W^\Sigma$  attracts trajectories along a globally defined  $C^0$  foliation at an exponential rate faster than  $\inf \operatorname{Re} \Sigma$ .

The corresponding precise statement is Theorem 20. The smoothness and rate of attraction of the inertial manifold can be improved by means of decreasing the ratio in (2). Corollary 24 then asserts that one way to exhibit a small ratio in (2) is via shrinking the delay.

In Section 5, we demonstrate how these results can be applied through three examples. In Section 5.1, a nonlinear version of the Cushing equation (Cushing 1977) (a particularly simple case of distributed delay) is treated, where we construct inertial manifolds based solely on the spectral gap condition at a moderate delay  $h = 1$ , thereby establishing IMs with dimension not equal to that of the physical system. Section 5.2 provides an even simpler example which showcases how an arbitrarily large spectral gap can be achieved by means of shrinking the delay  $h$ . The final example (Section 5.2.1) revisits the equation treated by Szaksz et al. (2025), and shows how inertial manifolds can be obtained in the absence of global control over the nonlinearity, if a bound on the size of the global attractor is known.

The remainder of the paper consists of the proofs of the main theorems in Sections 6, 7 and 8 with smoothness results deferred to appendices. In particular, smoothness of the semiflow is discussed in Appendix B; whereas smoothness of invariant manifolds is obtained in Appendix C (akin to Chapter IX of Diekmann et al. 2012). Some preliminaries are recalled in Appendix A.

## 2 Delay Equations

In this section we recollect the basics of DDE theory following the book of Diekmann et al. (2012).

### 2.1 DDE as an Evolution Equation

From the perspective of dynamical systems, the most favorable way to handle (1) is through the lens of the evolution equation it generates on  $X$ . To achieve this rigorously in a unified fashion for all DDEs of the form (1), we follow the approach set out in Diekmann et al. (2012) utilizing perturbation theory of dual semigroups.

We begin by considering the simplified problem

$$\dot{x}(t) = 0, \quad (3a)$$

$$x_0 = u, \quad u \in X, \quad (3b)$$

which generates a strongly continuous semigroup<sup>4</sup>  $t \mapsto T_0(t)$  called the shift semigroup on  $X$ , given explicitly as

$$[T_0(t)u](\theta) = \begin{cases} u(t + \theta), & \text{if } -h \leq t + \theta \leq 0, \\ u(0), & \text{if } t + \theta \geq 0. \end{cases} \quad (4)$$

<sup>4</sup> Appendix A.2 summarizes the concepts we use here from the theory of semigroups.

Its generator is a closed densely defined operator  $A_0 : \text{dom}(A_0) \rightarrow X$ ,  $\text{dom}(A_0) \subset X$ , given by

$$\text{dom}(A_0) = \left\{ u \in X \mid \dot{u} \in X, \dot{u}(0) = 0 \right\}, \quad A_0 u = \dot{u}. \quad (5)$$

The peculiar detail about (5) is that the update rule (3a) only manifests through the definition of the domain  $\text{dom}(A_0)$ , which would therefore have to be altered for each DDE (1a) on a case-by-case basis. To circumvent the technical complications this would induce (e.g., in nonlinear problems, in particular when linearizing about a fixed point), Diekmann et al. (2012) proceeds by exploiting the principles of duality. The dual space of  $X = C([-h, 0]; \mathbb{R}^n)$  is isomorphic to the space of *normalized bounded variations*<sup>5</sup>,  $X^* \cong \text{NBV}([0, h]; \mathbb{R}^n)$ , via the pairing

$$\langle \zeta, u \rangle_{X^*, X} = \int_0^h d\zeta(t)u(-t), \quad \zeta \in \text{NBV}([0, h]; \mathbb{R}^n). \quad (6)$$

The integral in (6) is to be interpreted in the Riemann-Stieltjes sense (see Appendix A.1), or equivalently, through the measure  $\mu$  corresponding to  $\zeta$ .

The dual semigroup  $t \mapsto T_0^*(t) := T_0(t)^*$  need not be strongly continuous; but a theorem of Phillips (1955); Hille and Phillips (1996) asserts that there exists a closed linear subspace  $X^\odot$  of  $X^*$  such that  $T_0^*(t)X^\odot \subset X^\odot$  and  $T_0^\odot(t) := T_0^*(t)|_{X^\odot}$  is strongly continuous; let us denote its generator by  $A_0^\odot$ . The space  $X^\odot$  is called the *sun subspace* of  $X$ . Explicitly,  $X^\odot = \text{dom}(A_0^*)$ , where the closure is taken in the norm topology of  $X^*$  (see also (92), Appendix A.2). Furthermore, we have<sup>6</sup>

$$X^\odot \cong \mathbb{R}^n \times L^1([0, h], \lambda; \mathbb{R}^n), \quad |(c, v)|_{X^\odot} = |c| + |v|_{L^1}. \quad (7)$$

Here  $c \in \mathbb{R}^n$  represents the value at 0 through the isomorphism  $(c, v) \mapsto w$ ,  $w(t) = c + \int_0^t v$  mapping  $\mathbb{R}^n \times L^1([0, h], \lambda; \mathbb{R}^n)$  to  $X^\odot$ . We refer to Theorem II.5.2, Diekmann et al. (2012), for the details. It is perhaps more illuminating to interpret (7) in terms of measures:  $X^\odot$  consists of Borel measures  $\mu$  that can be written as a sum  $\mu = \nu + c\delta_0$  for  $c \in \mathbb{R}^n$  and  $\nu$  absolutely continuous with respect to the Lebesgue measure  $\lambda$ ,  $\nu \ll \lambda$ .

We may perform the same procedure once more, now for the strongly continuous semigroup  $T_0^\odot$  on  $X^\odot$ . The dual space of  $X^\odot$  can be characterized, using the representation (7), as

$$X^{\odot*} = \mathbb{R}^n \times L^\infty([-h, 0], \lambda; \mathbb{R}^n), \quad |(b, w)|_{X^{\odot*}} = \sup\{|b|, |w|_{L^\infty}\} \quad (8)$$

<sup>5</sup> For the relevant definitions, see Appendix A.1. The reader might be more used to the interpretation of  $X^*$  through the modern version of Riesz's representation theorem, which assigns to each  $x^* \in X^*$  a signed finite  $\mathbb{R}^n$ -valued Borel measure  $\mu$  on  $[-h, 0]$ . Historically, the literature concerning delay equations has adopted the convention to declare  $\zeta : [0, h] \rightarrow \mathbb{R}^n$  via  $\zeta(t) = \mu([-t, 0])$ , which constitutes an isomorphic correspondence between  $X^*$  and  $\text{NBV}([0, h]; \mathbb{R}^n)$ . (In fact, Riesz's original result is posed in terms of BV functions Riesz 1909.)

<sup>6</sup> Here  $\lambda$  refers to the Lebesgue measure.

with norm induced by the pairing

$$\langle (b, w), (c, v) \rangle_{X^{\odot*}, X^{\odot}} = \langle b, c \rangle_{\mathbb{R}^n} + \int_0^h w(-\theta)v(\theta) d\lambda(\theta).$$

Seeking the sun subspace of (8) on which  $T_0^{\odot*}$  is strongly continuous, one arrives at  $X^{\odot\odot} = \overline{\text{dom}(A_0^{\odot*})} = \iota(X)$  (Theorem II.5.5 of Diekmann et al. (2012)), where  $\iota : X \hookrightarrow X^{\odot*}$  is the continuous embedding  $u \mapsto (u(0), u)$  defined by the pairing between  $X$  and  $X^{\odot}$  (in the future, we shall occasionally identify  $X$  with its image under  $\iota$ ). We remark that it is precisely the form of this embedding that permits the transition of the update rule (1a) from the domain into the evolution equation below in (10). The property  $X = X^{\odot\odot}$  is called  $\odot$ -reflexivity, which can also be equivalently characterized by weak compactness of the resolvent  $(\lambda \text{id} - A_0)^{-1}$  (de Pagter 1989).

We may now write (1) as an evolution equation, exploiting the extra flexibility provided by the enlarged space  $X^{\odot*}$ . We separate (1) into two parts: one given by the shift semigroup  $T_0$  from (3) and one by the update rule in (1a). This allows one to view all delay equations of the form (1) as finite rank perturbations of the shift semigroup. Specifically, the canonical basis  $(e_1, \dots, e_n)$  of  $\mathbb{R}^n$  determines  $n$  elements

$$r_i^{\odot*} = (e_i, 0) \in X^{\odot*}, \quad 1 \leq i \leq n,$$

which can be used to construct the finite rank nonlinear map  $O \rightarrow X^{\odot*}$ ,

$$F(u) = \sum_{i=1}^n \langle f(u), e_i \rangle_{\mathbb{R}^n} r_i^{\odot*}. \quad (9)$$

We arrive at the formal<sup>7</sup> evolution equation

$$\iota \frac{d}{dt} x_t = A_0^{\odot*} \iota x_t + F(x_t), \quad (10a)$$

$$x_0 = u \in O, \quad (10b)$$

where, explicitly,

$$\begin{aligned} \text{dom}(A_0^{\odot*}) &= \{(c, u) \mid u \text{ is Lipschitz with } u(0) = c\}, \\ A_0^{\odot*}(c, u) &= (0, \dot{u}), \end{aligned}$$

where we recall that Lipschitz functions admit derivatives in the  $L^\infty$  sense. Provided  $f$  is locally Lipschitz (which implies, through (9), that  $F$  is locally Lipschitz), the variation of constants formula

$$x_t = T_0(t)u + \iota^{-1} \int_0^t T_0^{\odot*}(t-s)F(x_s) ds, \quad u \in O, \quad t \geq 0, \quad (11)$$

<sup>7</sup> Equation (10) is formal because (10a) is only satisfied by solutions of (1) for  $t > 0$ , for general  $u \in O$ . If  $u \in C^1([-h, 0]; \mathbb{R}^n) \cap O$ , then (10a) is satisfied for all  $t \geq 0$ , with  $d/dt$  interpreted as a right derivative at  $t = 0$ . We shall hence mostly work with (11), which corresponds to solutions of (1).

provides solutions which are in one-to-one correspondence with solutions of (1), by Proposition VII.6.1 of Diekmann et al. (2012). The integral in (11) is to be interpreted in the weak\* sense (see Appendix A.3 for details).

## 2.2 Properties of Semiflows Generated by Delay Equations

Delay equations, through (11), generate semiflows – for the general definitions and notational conventions, we refer the reader to Appendix A.2; for notions of smoothness, to Appendix A.4. We gather the properties of such semiflows in the following proposition.

**Proposition 4** (Results of Sections VII.3, VII.4 and VII.6, Diekmann et al. 2012) *Suppose  $f$  is locally Lipschitz on  $O$ . Then, there exists a unique maximal semiflow  $\varphi : \mathcal{D}^\varphi \rightarrow O$  defined on the open domain  $\mathcal{D}^\varphi \subset [0, \infty) \times O$  associated to the maps  $\varphi_t : u \mapsto x_t$  given by (11). Moreover, the semiflow  $\varphi$  satisfies the following:*

- (i)  $\varphi$  is jointly continuous on  $\mathcal{D}^\varphi$ .
- (ii) If  $f$  is  $C^k$ , then for each  $t \geq 0$  fixed,  $\varphi_t : \mathcal{D}_t^\varphi \rightarrow O$  is  $C^k$ .
- (iii) If  $f$  is  $C^1$ , then  $\varphi$  is jointly  $C^1$  on  $\{(t, u) \in \mathcal{D}^\varphi \mid t > h\}$ .
- (iv) If  $f$  is bounded,  $t \geq h$  and  $B \subset O$  is a bounded set, then  $\varphi([h, t] \times (B \cup \mathcal{D}_t^\varphi))$  is precompact in  $X$ .

**Proof** This follows from results of Sections VII.3, VII.4 and VII.6, Diekmann et al. (2012), with the exception of (ii) for  $k \geq 2$ . The proof of the  $C^k$  case,  $k \geq 2$ , is given in Appendix B (in the perturbed setting, described in Section 3).  $\square$

## 3 Linearization About a Stationary State

Suppose  $w \in O$  is a stationary point of (1), i.e.,  $\varphi_t(w) = 0$  for all  $t \geq 0$ , which, through the lens of (10), implies  $A_0^{\odot*} w + F(w) = 0$ . Without loss of generality, by relabeling  $w$  to  $0 \in X$ , adjusting  $O$  and  $\varphi$  accordingly, we may assume that the stationary state is attained at 0 and that  $O$  is a neighbourhood of 0 (Corollary VII.5.3, Diekmann et al. 2012). In this setting, we have  $F(0) = 0$ . Assume  $f$  is  $C^1$ , which implies  $F$  is also  $C^1$ . Therefore,

$$F(u) = DF(0)u + R(u), \quad u \in O, \quad (12)$$

where  $R : O \rightarrow X^{\odot*}$  is a  $C^1$  map satisfying  $R(v) = o(|v|_X)$  as  $|v|_X \rightarrow 0$ . In particular,  $R(0) = 0$  and  $DR(0) = 0$ .

Hence, the formal evolution equation (10) reads

$$\begin{aligned} {}_t \frac{d}{dt} x_t &= (A_0^{\odot*} {}_t + DF(0))x_t + R(x_t), \\ x_0 &= u \in O. \end{aligned}$$

The linear part on  $X$ , given by

$$\text{dom}(A) = \left\{ u \in \iota^{-1}(\text{dom}(A_0^{\odot*})) \mid A_0^{\odot*} \iota u + DF(0)u \in \iota(X) \right\}, \quad (13a)$$

$$Au = \iota^{-1}(A_0^{\odot*} \iota u + DF(0)u), \quad (13b)$$

generates a strongly continuous semigroup on  $X$ , given explicitly as (Theorem III.2.4 and Corollary III.2.13, Diekmann et al. 2012)

$$T(t)u = T_0(t)u + \iota^{-1} \int_0^t T_0^{\odot*}(t-s)DF(0)T(s)u \, ds, \quad u \in X, \, t \geq 0. \quad (14)$$

Note that  $X$  is also  $\odot$ -reflexive with respect to  $T$  and that the  $\odot$  and  $\odot^*$  spaces with respect to  $T$  remain the same as those with respect to  $T_0$  (this is a consequence of  $DF(0)$  being a bounded perturbation). The full semiflow, as in (11), can then be given as a nonlinear perturbation of (14) (Proposition VII.5.4, Diekmann et al. 2012),

$$\varphi_t(u) = T(t)u + \iota^{-1} \int_0^t T^{\odot*}(t-s)R \circ \varphi_s(u) \, ds, \quad (t, u) \in \mathcal{D}^\varphi. \quad (15)$$

Note that  $T(t) = D\varphi_t(0)$  for all  $t \geq 0$  (see also Proposition VII.4.5, Diekmann et al. 2012).

### 3.1 Spectrum of $A$

Since our original equation (1) was posed on the real space  $\mathbb{R}^n$  for physical reasons, the operator  $A : \text{dom}(A) \rightarrow X$  acts on a real Banach space  $X = C([-h, 0]; \mathbb{R}^n)$  – its spectrum is hence defined via the device of complexification (see Sect III.7 of Diekmann et al. 2012). This is done by declaring a complex structure on  $X_{\mathbb{C}} := X \otimes_{\mathbb{R}} \mathbb{C}$  via  $b(u \otimes c) = u \otimes (bc)$  for  $b \in \mathbb{C}$ , then equipping  $X_{\mathbb{C}}$  with an admissible norm induced by  $|\cdot|_X$  (Definition III.7.5 of Diekmann et al. 2012). The resulting space, termed the complexification  $X_{\mathbb{C}}$  of  $X$ , can be identified with  $X \oplus iX$ , on which one may consider complexifications of operators  $A$  originally acting on  $X$  via  $A_{\mathbb{C}}(u + iv) := Au + iAv$ , with  $\text{dom}(A_{\mathbb{C}}) = \text{dom}(A) \oplus i \text{dom}(A)$ . Since all admissible norms are equivalent, this gives rise to a well-defined notion of spectrum via  $\sigma(A; X) := \sigma(A_{\mathbb{C}}; X_{\mathbb{C}})$ . Note, in particular, that  $X_{\mathbb{C}} \cong C([-h, 0]; \mathbb{C}^n)$  in our example. We remark also that  $A_{\mathbb{C}}$  generates  $(T(t))_{\mathbb{C}}$ , which will be used implicitly in what follows.

The spectrum of delay equations is easiest to characterize via the eventual compactness of the linearization semigroup  $T(t)$ . This property can be inferred as a consequence of Proposition 4, applied to the linearized equation

$$\dot{x}(t) = Df(0)x_t, \quad (16a)$$

$$x_0 = u \in X. \quad (16b)$$

Indeed, item (iv) implies that  $T(t)$ , as defined in (14), is a compact operator for each  $t \geq h$  (recall that  $Df(0) \in \mathcal{L}(X, \mathbb{R}^n)$ ). The spectral theory of eventually compact semigroups then implies the following.

**Theorem 5** (Theorem B.IV.2.1, Arendt et al. 1986) *Suppose  $A$  is the generator of an eventually compact semigroup  $T(t)$  on a Banach space  $X$ . The spectrum of  $A$  consists of a countable set, possibly empty or finite, of eigenvalues (i.e.,  $\sigma(A; X) = \sigma_p(A; X)$ ) of finite algebraic multiplicity. The set  $\{\lambda \in \sigma(A; X) \mid \operatorname{Re} \lambda \geq \gamma\}$  is finite for all  $\gamma \in \mathbb{R}$ . Moreover, the following spectral mapping property holds*

$$\sigma(T(t); X) \setminus \{0\} = e^{t\sigma(A; X)}. \quad (17)$$

**Proof** This is just a restatement of Theorem B.IV.2.1, Arendt et al. (1986), apart from the last statement (17), which can be inferred from Theorem IV.2.2, Diekmann et al. (2012), but we give a proof here as well. The spectral mapping property (17) is a consequence of  $\sigma(A; X) = \sigma_p(A; X)$  and the spectral mapping property of point spectra for strongly continuous semigroups (IV.3.7, Engel et al. 2000), which states

$$\sigma_p(T(t); X) \setminus \{0\} = e^{t\sigma_p(A; X)}.$$

To conclude (17), we observe, through eventual compactness of  $T$  and the formula  $T(t)^n = T(nt)$  (for  $n \in \mathbb{N}$  large enough so that  $T(nt)$  is compact) that  $\sigma(T(t))^n \setminus \{0\} = \sigma(T(t)^n) \setminus \{0\} = \sigma_p(T(t)^n) \setminus \{0\} = \sigma_p(T(t))^n \setminus \{0\}$ ; hence  $\sigma(T(t)) \setminus \{0\} = \sigma_p(T(t)) \setminus \{0\}$ .  $\square$

The following lemma recalls the existence of spectral projections.

**Lemma 6** (Spectral projections) *Maintaining the assumptions of Theorem 5, let  $\Sigma$  denote a bounded subset of  $\sigma(A; X)$ ; denote by  $\Sigma' = \sigma(A; X) \setminus \Sigma$ . There exists a projection  $P_\Sigma$  on  $X_\mathbb{C}$  satisfying the following:<sup>8</sup>*

- (i)  $X_\mathbb{C}$  admits a splitting  $X_\mathbb{C} = X_\Sigma \oplus X_{\Sigma'}$  of  $A_\mathbb{C}$ -invariant closed linear subspaces  $X_\Sigma = \operatorname{im}(P_\Sigma)$  and  $X_{\Sigma'} = \ker(P_\Sigma)$ , with  $X_\Sigma$  finite dimensional.
- (ii) The operator  $A_\Sigma := \tilde{P}_\Sigma A_\mathbb{C} \iota_\Sigma : X_\Sigma \rightarrow X_\Sigma$  is bounded, its spectrum is  $\sigma(A_\Sigma; X_\Sigma) = \Sigma$ .
- (iii) The unbounded operator  $A_{\Sigma'} := \tilde{P}_{\Sigma'} A_\mathbb{C} |_{X_{\Sigma'} \cap \operatorname{dom}(A_\mathbb{C})} : X_{\Sigma'} \cap \operatorname{dom}(A_\mathbb{C}) \rightarrow X_{\Sigma'}$  has spectrum  $\sigma(A_{\Sigma'}; X_{\Sigma'}) = \Sigma'$ .
- (iv) If  $\Sigma$  is invariant under complex conjugation, then  $P_\Sigma$  preserves the real subspace, i.e.,  $P_\Sigma X \subset X$ , and all points above can be reformulated in terms of ' $X$ ' and ' $A$ ' instead of ' $X_\mathbb{C}$ ' and ' $A_\mathbb{C}$ '.

**Proof** Points (i)–(iii) follow directly from holomorphic functional calculus, see e.g. Theorem 6.19, Bühler and Salamon (2018). Note that in order to conclude finite dimensionality of  $X_\Sigma$ , we used the first two assertions of Theorem 5, in particular, that  $\Sigma$  is hence a finite set consisting of eigenvalues of finite algebraic multiplicity.

<sup>8</sup> Here and elsewhere,  $\iota_\Sigma : X_\Sigma \hookrightarrow X$  denotes the inclusion and  $\tilde{P}_\Sigma : X \rightarrow X_\Sigma$  the projection with restricted range. We also denote by  $P_{\Sigma'} = \operatorname{id}_X - P_\Sigma$ .

Point (iv) follows from Corollary IV.2.19 of Diekmann et al. (2012), which states the result only for a single pair of complex conjugate eigenvalues, but trivially extends to the present case in view of the observation above, which implies  $P_\Sigma = \sum_{\lambda \in \Sigma} P_{\{\lambda\}}$ .  $\square$

Since our treatment of delay equations assumes  $A$  is real, its eigenvalues appear in complex conjugate pairs, i.e.,  $\sigma(A; X) = \sigma_p(A; X)$  is invariant under conjugation. Hence, we can always select  $\Sigma$  in a way such that  $P_\Sigma$  preserves the real subspace by replacing  $\Sigma$  with  $\Sigma \cup \text{conj}(\Sigma)$ . In the future, we shall restrict our attention to projections as such.

We shall also assume that  $\Sigma' = \sigma(A; X) \setminus \Sigma \neq \emptyset$  for almost the entirety of this paper, with the exception of Remark 25 concerning inertial manifolds.<sup>9</sup> We remark that this condition does not automatically hold because  $\sigma(A; X)$  may be finite in Theorem 5. However, the only interesting example for which the condition fails is the following in Remark 7. It will only be used for Remark 25, whose purpose is to simplify some assumptions for inertial manifolds.

**Remark 7** Consider the operator  $A_0 : \text{dom}(A_0) \rightarrow X$  given in (5). Knowing that  $\sigma(A_0; X) = \sigma_p(A_0; X)$  by Theorem 5, one can deduce from the eigenvalue equation

$$A_0 u = \dot{u} = \lambda u, \quad u \in \text{dom}(A_0),$$

that  $\sigma(A_0; X) = \{0\}$  (recall that  $\dot{u}(0) = 0$  for  $u \in \text{dom}(A_0)$ ). Moreover, the eigenvalue 0 has algebraic and geometric multiplicity equal to  $n$ ; the eigenspace  $X_{\{0\}} \subset X$  is spanned by basis elements of  $\mathbb{R}^n$  interpreted as constant functions over  $[-h, 0]$ . The associated projection  $P_{\{0\}}$  can be identified as follows. Since spectral projections commute with the operator that defines them, we must have

$$P_{\{0\}} A_0 = A_0 P_{\{0\}}, \quad \text{on } \text{dom}(A_0),$$

which implies that  $P_{\{0\}} \dot{u} = 0$  for all  $u \in \text{dom}(A_0)$ . Hence,  $P_{\{0\}} u = u(0)$ .

Next, we consider the exponential dichotomy that  $P_\Sigma$  produces. Denote the *growth bound* of a strongly continuous semigroup  $t \mapsto T(t)$  with generator  $A$  by

$$\omega(A) := \inf \left\{ \omega \in \mathbb{R} \mid \exists M > 0 \text{ such that } \|T(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}, \forall t \geq 0 \right\}. \quad (18)$$

Note that this is well defined:  $\omega(A) < \infty$  for strongly continuous semigroups. Recall that for all  $t > 0$  we have (see, e.g., van Neerven 1996, Proposition 1.2.2)

$$\omega(A) = \frac{\log r_\sigma(T(t))}{t}, \quad (19)$$

where  $r_\sigma(T(t)) = \sup_{\lambda \in \sigma(T(t))} |\lambda|$  denotes the spectral radius.

<sup>9</sup> To extend the results herein to the case when  $\Sigma' = \emptyset$ , one should simply interpret  $\beta$  in what follows as  $-\infty$  and  $-\beta + \varepsilon_2$  as some (large) negative number (specifically in Lemma 8). See also Remark 25. The assumption is made here mostly for notational convenience.

**Lemma 8** (Exponential dichotomy) *Assume the setting of Theorem 5, and let  $\Sigma$  and  $P_\Sigma$  be as in Lemma 6, with  $\Sigma$  invariant under complex conjugation and  $\Sigma' \neq \emptyset$ . Set  $\alpha = \inf\{\operatorname{Re} \lambda \mid \lambda \in \Sigma\}$  and  $\beta = \sup\{\operatorname{Re} \lambda \mid \lambda \in \Sigma'\}$ . Then, for any  $\varepsilon_1, \varepsilon_2 > 0$ , there exists  $K_1, K_2 > 0$  such that*

$$\|T(t)P_\Sigma\|_{\mathcal{L}(X)} \leq K_1 e^{(\alpha - \varepsilon_1)t} \|P_\Sigma\|_{\mathcal{L}(X)}, \quad t \leq 0, \quad (20a)$$

$$\|T(t)P_{\Sigma'}\|_{\mathcal{L}(X)} \leq K_2 e^{(\beta + \varepsilon_2)t} \|P_{\Sigma'}\|_{\mathcal{L}(X)}, \quad t \geq 0. \quad (20b)$$

**Proof** It is a consequence of (91) that  $T(t)$  can be factored into two semigroups according to the decomposition  $X = X_\Sigma \oplus X_{\Sigma'}$ .<sup>10</sup> In particular,  $A_\Sigma$  generates  $\tilde{P}_\Sigma T(t)|_{X_\Sigma} : X_\Sigma \rightarrow X_\Sigma$ ; similarly for  $A_{\Sigma'}$ . Since  $A_\Sigma$  is bounded,  $t \mapsto \tilde{P}_\Sigma T(t)|_{X_\Sigma}$  is norm continuous, hence forming a group (Lemma 7.17, Bühler and Salamon 2018). In particular,  $-A_\Sigma$  generates the semigroup  $\{\tilde{P}_\Sigma T(-t)|_{X_\Sigma}\}_{t \geq 0}$ ; hence, by (19),

$$\begin{aligned} \omega(-A_\Sigma) &= \frac{1}{t} \log r_\sigma(\tilde{P}_\Sigma T(-t)|_{X_\Sigma}) \\ &= \frac{1}{t} \sup_{\lambda \in \Sigma} \log |e^{-t\lambda}| \\ &= -\alpha. \end{aligned}$$

By definition (18), for any  $\varepsilon_1 > 0$ , there exists  $K_1 > 0$  such that

$$\|\tilde{P}_\Sigma T(-t)|_{X_\Sigma}\|_{\mathcal{L}(X_\Sigma)} \leq K_1 e^{(-\alpha + \varepsilon_1)t}, \quad t \geq 0.$$

Therefore,

$$\begin{aligned} \|T(t)P_\Sigma\|_{\mathcal{L}(X)} &= \|\tilde{P}_\Sigma T(t)|_{X_\Sigma} \tilde{P}_\Sigma\|_{\mathcal{L}(X)} \\ &\leq \|\tilde{P}_\Sigma T(-t)|_{X_\Sigma}\|_{\mathcal{L}(X_\Sigma)} \|P_\Sigma\|_{\mathcal{L}(X)} \\ &\leq K_1 e^{(\alpha - \varepsilon_1)t} \|P_\Sigma\|_{\mathcal{L}(X)}, \quad t \leq 0. \end{aligned}$$

Similar considerations lead to (20b).  $\square$

### 3.1.1 Characteristic Equation

The spectrum of  $A$  is usually computed via the *characteristic equation* associated to the linear system (16). To derive the characteristic equation, one first complexifies (16) so that  $Df(0) \in \mathcal{L}(X_{\mathbb{C}}, \mathbb{C}^n)$ , then transforms it into a more manageable form via the Riesz representation theorem, treating  $Df(0)$  as  $n$  linear functionals on  $X_{\mathbb{C}}$ . In particular, there exists a unique  $\zeta \in \operatorname{NBV}([0, h]; \mathbb{C}^{n \times n})$  for which

$$Df(0)u = \int_0^h d\zeta(\theta)u(-\theta), \quad u \in X_{\mathbb{C}}, \quad (21)$$

<sup>10</sup> The subspaces are equipped with the norm inherited from  $X$ ; in this case,  $\|\tilde{P}_\Sigma\|_{\mathcal{L}(X, X_\Sigma)} = \|P_\Sigma\|_{\mathcal{L}(X)}$ .

where the integral is interpreted in the Riemann-Stieltjes sense (see Appendix A.1).

The *characteristic matrix* is defined by

$$\Delta(z) = z \operatorname{id}_{\mathbb{C}^{n \times n}} - \int_0^h e^{-z\tau} d\zeta(\tau), \quad z \in \mathbb{C}, \quad (22)$$

the corresponding characteristic equation reads

$$\det(\Delta(z)) = 0. \quad (23)$$

Equation (23) is usually obtained from (16) and (21) via a Laplace transform (see Sect I.3, Diekmann et al. 2012), or by direct substitution of the model solution  $ce^{zt}$ ,  $c \in \mathbb{C}^n$ .

It is almost immediate from the latter approach that roots of (23) are contained in  $\sigma_p(A)$ ; one only needs to check that the eigenfunction thus produced,  $w : \theta \mapsto ce^{\lambda\theta}$  on  $X_{\mathbb{C}}$  for which  $\Delta(\lambda)c = 0$ , is contained in  $\operatorname{dom}(A_{\mathbb{C}})$ , i.e.,  $(Df(0)w, \dot{w}) \in \iota(X_{\mathbb{C}})$ . Thus, we need only verify  $Df(0)w = \dot{w}(0)$ , which follows from the observation

$$\dot{w}(0) - Df(0)w = \Delta(\lambda)c.$$

The reverse inclusion requires a careful assessment of the resolvent operator of  $A$  (see Theorem IV.3.1, Diekmann et al. 2012). The overall statement, given in Corollary IV.3.3, Diekmann (1987), is

$$\{\lambda \in \mathbb{C} \mid \det(\Delta(\lambda)) = 0\} = \sigma_p(A; X) = \sigma(A; X). \quad (24)$$

Due to their significance in determining the asymptotic behaviour of linear systems, characteristic equations of functional differential equations have received widespread attention throughout the years (Stépán 1989; Michiels and Niculescu 2007; Cooke 1963). In the special case of delay differential equations, the asymptotic stability of linear systems is completely determined by the roots of (23), which is a simple consequence of Lemma 8 and (24). For functional equations of arbitrary type, spectral stability is a necessary but generally not sufficient condition (Stépán 1989). We remark also that, while the asymptotic behaviour is completely characterized by the spectrum in the linear delayed case, there might exist *small solutions* (defined as solutions decaying faster than  $e^{at}$  for all  $a \in \mathbb{R}$ ) that are not captured by spectral expansions (see Chapter V, Diekmann et al. 2012).

### 3.1.2 Asymptotics of the Spectrum, Barriers to $C^k$ Smoothness of Pseudo-Unstable Manifolds

For pseudo-unstable manifolds of interest herein,  $C^k$  smoothness,  $k \geq 2$ , of the manifolds hinges on spectral gap conditions of the form

$$\beta < k\alpha < 0, \quad \alpha = \inf_{\lambda \in \Sigma} \operatorname{Re} \lambda, \quad \beta = \sup_{\lambda \in \Sigma'} \operatorname{Re} \lambda, \quad (25)$$

where  $\Sigma \subset \sigma(A)$  is a bounded subset of the spectrum and  $\Sigma' = \sigma(A) \setminus \Sigma$ , as in the statement of Lemma 8. In this subsection, we show that (25) is violated whenever  $|\alpha|$  is large enough, i.e., for all but finitely many choices of  $\Sigma$ , under the mild assumption  $\zeta \in \text{SBV}$  (for the relevant definition, see Appendix A.1).

For this, let us recall two results about the roots of (23).

**Proposition 9** (Theorem V.7.11, Diekmann et al. 2012) *Suppose  $\zeta \in \text{SBV}$ . There exist constants  $C, C_0, C_1 > 0$  such that all roots  $\lambda \in \mathbb{C}$  of (23) with  $|\lambda| > C$  must be contained in the strip*

$$\left\{ z \in \mathbb{C} \mid C_0 e^{-h \operatorname{Re} z} \leq |z| \leq C_1 e^{-h \operatorname{Re} z} \right\}.$$

**Proposition 10** (Theorem 4.12, Lunel 1989) *The number of roots of (23) with modulus less than  $r$ , denoted by  $n(r)$ , satisfies<sup>11</sup>*

$$\lim_{r \rightarrow \infty} \frac{n(r)}{r} = \frac{E(\det \Delta)}{\pi}. \quad (26)$$

**Lemma 11** *Let  $\zeta \in \text{SBV}$  and suppose  $\sigma(A)$  has infinite cardinality. Then, (25) fails for  $k = 2$  (and hence for all  $k \geq 2$ ) for  $|\alpha|$  large enough.*

**Proof** Supposing the contrary, we may extract a decreasing sequence  $(\alpha_j)_{j \in \mathbb{N}} \subset \operatorname{Re} \sigma(A)$ ,  $\alpha_j \rightarrow -\infty$ , such that

$$\beta_j < 2\alpha_j < 0, \quad \beta_j := \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A), \operatorname{Re} \lambda < \alpha_j\}. \quad (27)$$

Choose  $\delta > 0$  small enough so that  $2\delta < -\alpha_1$ . Set  $r_{1,j} := C_1 e^{-(\alpha_j - \delta)h}$  and  $r_{2,j} := C_0 e^{-(\beta_j + \delta)h}$ , where  $C_0, C_1$  are as in Proposition 9. Then, for  $j$  large enough so that  $r_{2,j} > r_{1,j} > C$ , we have by Proposition 9 that

$$n(r_{2,j}) - n(r_{1,j}) \leq \#\{z \in \sigma(A) \mid \operatorname{Re} z \in (\beta_j + \delta, \alpha_j - \delta)\}.$$

By definition of  $\beta_j$ , we must have  $n(r_{2,j}) = n(r_{1,j})$ . On the other hand, Proposition 10 states that

$$\left| \frac{n(r)}{r} - \frac{E(\det \Delta)}{\pi} \right| < \varepsilon_j, \quad \text{for all } r \geq r_{1,j},$$

with  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow \infty$ . Hence, for  $j$  sufficiently large as above,

$$n(r_{2,j}) - n(r_{1,j}) > \left( \frac{E(\det \Delta)}{\pi} - \varepsilon_j \right) C_0 e^{-(\beta_j + \delta)h} - \left( \frac{E(\det \Delta)}{\pi} + \varepsilon_j \right) C_1 e^{-(\alpha_j - \delta)h}$$

<sup>11</sup> In (26),  $E(\det \Delta)$  refers to the exponential type of the function  $z \mapsto \det(\Delta(z))$ . For the precise definition, see Definition 4.1, Lunel (1989), but for all purposes herein, one could just interpret it as a positive constant determined by  $\zeta$ .

$$> e^{-(\alpha_j - \delta)h} \left[ \left( \frac{E(\det \Delta)}{\pi} - \varepsilon_j \right) C_0 e^{-(\alpha_j + 2\delta)h} - \left( \frac{E(\det \Delta)}{\pi} + \varepsilon_j \right) C_1 \right],$$

where the last line used (27). It is clear that the right hand side is positive (in fact, arbitrarily large) for  $j$  large enough, which yields the desired contradiction.  $\square$

## 4 Statement of Results

In this section, we formulate precisely our main results. For this, we first group the notational conventions and hypotheses that shall be used throughout.

(A.1) (Setup) Let  $X$  denote the Banach space  $C([-h, 0]; \mathbb{R}^n)$  equipped with the supremum norm. Let  $O \subset X$  be an open neighbourhood of 0; and suppose  $f : O \rightarrow \mathbb{R}^n$  is of class  $C^1$ .<sup>12</sup> Let  $\varphi : \mathcal{D}^\varphi \rightarrow O$  denote the semiflow generated by (1) (see Proposition 4). Suppose 0 is a stationary state of  $\varphi$ , and  $F$  and  $R$  are as in (12). Suppose  $\Sigma \subset \sigma(A; X)$  is a nonempty bounded subset of the spectrum invariant under complex conjugation, where  $A$  is defined in (13), and set  $\Sigma' = \sigma(A; X) \setminus \Sigma \neq \emptyset$ . Denote the associated projection by  $P_\Sigma$ , its image by  $X_\Sigma$  and its kernel by  $X_{\Sigma'}$  (as in Lemma 6). Set  $P_{\Sigma'} = \text{id}_X - P_\Sigma$ .

For statements concerning higher-degree smoothness, we shall utilize the following assumption on  $f$ .

(A.2) (Regularity) Suppose  $f : O \rightarrow \mathbb{R}^n$  from (A.1) is of class  $C^k$ ,  $k \in \mathbb{N}^{\geq 2} \cup \{\infty\}$ .

For the first result, we assume the following form of  $\Sigma$ .

(A.3) (Spectral subset) Suppose  $\Sigma$  from (A.1) is of form

$$\Sigma = \{\lambda \in \sigma(A; X) \mid \text{Re } \lambda \geq \gamma\}$$

for some  $\gamma \in \mathbb{R}$ . Let  $\alpha := \inf_{\lambda \in \Sigma} \text{Re } \lambda$ , and  $\beta := \sup_{\lambda \in \Sigma'} \text{Re } \lambda$ .

For a given set  $U$ , let us denote by  $\mathcal{U}(U)$  the set of pairs  $(t, u)$  for which a trajectory initiated at  $u$  remains in  $U$  for all times up to  $t$ , i.e.,

$$\mathcal{U}(U) = \{(t, u) \in \mathcal{D}^\varphi \mid \varphi([0, t] \times \{u\}) \subset U\},$$

and let  $\mathcal{U}_u(U) = \{t \in \mathcal{D}_u^\varphi \mid (t, u) \in \mathcal{U}(U)\}$ .

We begin with a pseudo-unstable manifold result, an application of (Chen et al., 1997, Theorem 1.1).

**Theorem 12** Assume (A.1) and (A.3). The following hold:

- (i) (Invariant manifold) There exists an open neighbourhood  $U \subset O$  of 0 and a  $C^1$  submanifold  $W^\Sigma \subset U$  tangent to  $X_\Sigma$  at 0, which is locally invariant under  $\varphi_t$ : If  $u \in W^\Sigma$ , then  $\varphi_t(u) \in W^\Sigma$  for all  $(t, u) \in \mathcal{U}(U)$ .  $\{\varphi_t|_{W^\Sigma}\}_{t \geq 0}$  extends to a jointly  $C^1$  flow on  $\mathcal{U}(U \cap W^\Sigma)$ .

<sup>12</sup> For the Lipschitz version of Theorem 12, one could drop the  $C^1$  assumption provided the equation is already in the form (12) and  $\text{Lip}_\rho(R) \rightarrow 0$  as  $\rho \rightarrow 0$  (in the notation of Lemma 28).

- (ii) (Smoothness) Suppose in addition (A.2). If  $\beta < \ell\alpha$  for some integer  $\ell \leq k$ , then  $W^\Sigma$  is a  $C^\ell$  submanifold of  $U$ . (If  $\gamma > 0$ , then  $W^\Sigma$  is of class  $C^k$ .) In either case, the restricted semiflow  $\varphi_t|_{W^\Sigma}$  extends to a jointly  $C^\ell$  (resp.  $C^k$ ) flow on  $\mathcal{U}(U \cap W^\Sigma)$ .
- (iii) (Attractivity) There exists a continuous map  $\pi : U \rightarrow W^\Sigma$  such that

$$\pi \circ \varphi_t(u) = \varphi_t|_{W^\Sigma} \circ \pi(u), \quad \text{for all } (t, u) \in \mathcal{U}(U). \quad (28)$$

Moreover, for any  $u \in U$ , there exists  $C > 0$  such that

$$|\varphi_t(u) - \varphi_t \circ \pi(u)|_X \leq C e^{\gamma t}, \quad \text{for all } t \in \mathcal{U}_u(U). \quad (29)$$

The preimages  $\{\pi^{-1}(w)\}_{w \in W^\Sigma}$  are Lipschitz submanifolds of  $U$ .

- (iv) (Pseudo-uniqueness) If  $\gamma > 0$ ,  $W^\Sigma$  is uniquely determined by its tangency to  $X_\Sigma$  at 0. If  $\gamma < 0$ ,  $\pi^{-1}(0)$  is uniquely determined by its tangency to  $X_{\Sigma'}$  at 0. Otherwise, if  $W^\Sigma$  and  $\widetilde{W^\Sigma}$  are two invariant manifolds tangent to  $X_\Sigma$  at 0, there exists a neighbourhood  $V \subset O$  on which the reduced dynamics are topologically conjugate, i.e., there exists a homeomorphism  $g : W^\Sigma \cap V \rightarrow \widetilde{W^\Sigma} \cap V$  such that

$$\varphi_t \circ g(u) = g \circ \varphi_t(u), \quad \text{for all } (t, u) \in \mathcal{U}(W^\Sigma \cap V).$$

**Proof** see Sect 6. □

**Remark 13** It is possible to replace (29) by the stronger statement

$$|\varphi_t(u) - \varphi_t \circ \pi(u)|_X \leq C e^{\gamma t} |u - \pi(u)|_X, \quad (t, u) \in \mathcal{U}(U),$$

where now the constant  $C > 0$  is independent of  $u$ . This follows by repeating the proof of Lemma 4.4 in Buza (2024) for the case of delay equations, and using its result in place of (78) in the course of the proof.

**Remark 14** (Comparison with the usual parameterization/SSM assumptions) The assumption  $\beta < \ell\alpha$  is standard for the added smoothness of pseudo-unstable manifolds (see, e.g., Irwin 1980); it essentially stems from Lemma 44 below. Note that this implies (and is strictly stronger than) the usual condition (Cabr   et al., 2005, Theorem 3.2(3)) on outer nonresonances up to order  $\ell$ . However, the spectral squeeze assumption, (Cabr   et al., 2005, Theorem 3.2(4)), is not satisfied for any  $L$ , since the spectrum  $\sigma(A; X)$  of the linear operator  $A$  generally grows unbounded in negative real part. This essentially corresponds to the non-invertibility of  $A_{\Sigma'}$ , which precludes (Cabr   et al., 2003, Theorem 1.1) from being applicable.

**Remark 15** (Strong-stable/unstable manifolds) It is well known that  $W^\Sigma$  is uniquely characterized by Lyapunov exponents of negative semiorbits extendable for all times if  $\gamma > 0$  (see statement (i) of Theorem 31). In this case,  $W^\Sigma$  is called a strong unstable manifold; it is a consequence of the above properties that it is of class  $C^k$ . If  $\gamma \leq 0$ , then  $\pi^{-1}(0)$  represents the strong stable manifold, which is again  $C^k$  smooth and can be characterized by forward Lyapunov exponents of its trajectories. These are known results, but for a (partial) proof of the strong unstable case, see Appendix C.

Next, we generalize parts of this statement to spectral subsets not of the form (A.3). The following assumption (replacing (A.3)) specifies the form of stable spectral subsets that are admissible.

(A.4) (Nonresonance) Suppose  $\Sigma$  is a bounded spectral subset satisfying

$$\Sigma \subset \{\lambda \in \sigma(A; X) \mid \operatorname{Re} \lambda < 0\}$$

and invariant under complex conjugation. Let

$$\tilde{\Sigma} = \{\lambda \in \sigma(A; X) \setminus \Sigma \mid \inf \operatorname{Re} \Sigma \leq \operatorname{Re} \lambda < 0\},$$

and  $\tilde{\Sigma}' = \{\lambda \in \sigma(A; X) \mid \operatorname{Re} \lambda < 0\} \setminus \tilde{\Sigma}$ . Denote by  $r$  the smallest integer for which

$$(r+1) \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A; X), \operatorname{Re} \lambda < 0\} < \inf\{\operatorname{Re} \lambda \mid \lambda \in \tilde{\Sigma}\}.$$

Suppose that  $\Sigma$  satisfies the non-resonance condition

$$((i-j)\tilde{\Sigma} + j\tilde{\Sigma}') \cap \tilde{\Sigma} = \emptyset$$

for all pairs  $(j, i)$  of integers such that  $2 \leq i \leq r$  and  $1 \leq j \leq i$ .

**Theorem 16** Suppose (A.1), (A.2) and (A.4) hold such that  $k > r$ . Set  $\gamma := \inf \operatorname{Re} \Sigma$  and choose an integer  $1 \leq \ell \leq k$  such that

$$\sup\{\lambda \in \sigma(A; X) \mid \operatorname{Re} \lambda < \gamma\} < \ell\gamma.$$

There exists an open neighbourhood  $U \subset O$  of 0 and a  $C^\ell$  submanifold  $W^\Sigma \subset U$  tangent to  $X_\Sigma$  at 0, which is locally invariant under  $\varphi_t$ : If  $u \in W^\Sigma$ , then  $\varphi_t(u) \in W^\Sigma$  for all  $(t, u) \in \mathcal{U}(U)$ .  $\{\varphi_t|_{W^\Sigma}\}_{t \geq 0}$  extends to a jointly  $C^\ell$  flow on  $\mathcal{U}(U \cap W^\Sigma)$ .

**Proof** see Sect 7. □

**Remark 17** The proof of Theorem 16 proceeds by extracting infinite-dimensional manifolds associated to nonresonant, stable parts of the spectrum (as in de la Llave 1997), then taking the intersection of these manifolds with the pseudo-unstable manifolds constructed in Theorem 12. The nonresonance conditions listed in (A.4) reflect this. In contrast to Remark 14, the reason these stable spectral manifold results are applicable is due to the fact that we are looking at the complementary problem, which requires only the invertibility of  $A_\Sigma$  rather than that of  $A_{\Sigma'}$ .

We have a similar result for the case of unstable spectral subsets.

(A.5) (Nonresonance) Suppose  $\Sigma$  is a spectral subset satisfying

$$\Sigma \subset \{\lambda \in \sigma(A; X) \mid \operatorname{Re} \lambda > 0\}$$

and invariant under complex conjugation. Denote by  $r$  the smallest integer for which

$$(r + 1) \inf\{\operatorname{Re} \lambda \mid \lambda \in \Sigma\} > \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A; X) \setminus \Sigma\}.$$

Suppose that  $\Sigma$  satisfies the (non-resonance) condition

$$j\Sigma \cap \{\lambda \in \sigma(A; X) \setminus \Sigma \mid \operatorname{Re} \lambda > 0\} = \emptyset$$

for all integers  $j$  such that  $2 \leq j \leq r$ .

**Theorem 18** Suppose (A.1), (A.2) and (A.5) hold such that  $k > r$ . There exists an open neighbourhood  $U \subset O$  of 0 and a  $C^k$  submanifold  $W^\Sigma \subset U$  tangent to  $X_\Sigma$  at 0, which is locally invariant under  $\varphi_t$ : If  $u \in W^\Sigma$ , then  $\varphi_t(u) \in W^\Sigma$  for all  $(t, u) \in \mathcal{U}(U)$ .  $\{\varphi_t|_{W^\Sigma}\}_{t \geq 0}$  extends to a jointly  $C^k$  flow on  $\mathcal{U}(U \cap W^\Sigma)$ .

**Proof** The proof consists of an application of Theorem 1.2 of Cabre et al. (2003) to the time-one map of the  $C^k$  flow obtained upon restricting  $\varphi$  to the finite-dimensional unstable manifold of 0. The details would be analogous to the proof of Theorem 16 and are hence omitted.  $\square$

## 4.1 Inertial Manifolds

This section concerns the existence of inertial manifolds in system (1). In the small delay regime, a positive answer to this question came as early as the 60s, through a series of papers by Ryabov (1960a, b, 1961, 1963); see also Driver (1968, 1976); Chicone (2003). Here we derive a different condition not based on the delay, but instead on the existence of a large enough spectral gap (Theorem 20). We do so utilizing the results of Chen et al. (1997) used in the proof of Theorem 12. These not only show the existence of a smooth inertial manifold, but also the existence of a corresponding invariant foliation, whose leaves characterize the approach of trajectories towards the inertial manifold (c.f. (73)). We consequently show that large enough spectral gaps exist in systems with sufficiently small delays – compared to the Lipschitz constant of the nonlinearity – thus recovering also the classical results (Corollary 24 and Remark 25).

As a prerequisite, we shall henceforth assume that  $f$  is globally Lipschitz on the whole of  $X$ . This is not strictly necessary if the existence of a compact global attractor is known a priori, in which case cutoff functions encompassing the attractor can be used to ease this assumption (as in Foias et al. 1988).

The term *inertial manifold* was initially coined by the authors of Foias et al. (1988), who define inertial manifolds as follows.<sup>13</sup>

**Definition 19** (*Inertial manifolds, Foias et al. 1988; Robinson 2001*) Consider a semiflow  $\{\varphi_t\}_{t \geq 0}$  on a Banach space  $X$ , with domain  $\mathcal{D}^\varphi = \mathbb{R}^{\geq 0} \times X$ . A subset  $\mathcal{M} \subset X$  is said to be an inertial manifold for  $\varphi_t$  if it has the following three properties

<sup>13</sup> Note that Definition 19 only makes sense for globally defined semiflows – but this is certainly the case in our setting when  $f$  is globally Lipschitz, see Lemma 29.

- (1)  $\mathcal{M}$  is a finite-dimensional Lipschitz manifold,
- (2)  $\mathcal{M}$  is positively invariant, i.e.,  $\varphi_t(\mathcal{M}) \subset \mathcal{M}$ , for all  $t \geq 0$ ,
- (3)  $\mathcal{M}$  attracts all trajectories exponentially; that is, there exists  $\kappa < 0$  (the rate of attraction) such that

$$\text{dist}_X(\varphi_t(u), \mathcal{M}) \leq C(|u|_X)e^{\kappa t}, \quad u \in X, \quad t \geq 0,$$

where  $C(|u|_X)$  is a constant that depends only on the norm of  $u \in X$ .

The following theorem states that, for a sufficiently large spectral gap, inertial manifolds exist for delayed equations. For the precise statement, we fix the following constants. Let  $M > 0$  and  $\omega > \omega(A)$ ,  $\omega \neq 0$ , be such that (c.f. (18))  $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$  for all  $t \geq 0$ .

**Theorem 20** Suppose (A.1), (A.3) with  $\alpha \leq 0$  and that  $f$  is globally Lipschitz on the whole of  $X$ . Let  $\varepsilon_1, \varepsilon_2 > 0$  be real numbers and let  $K_1$  and  $K_2$  be the constants from Lemma 8 associated to  $\alpha, \beta, \varepsilon_1$  and  $\varepsilon_2$ . If there exists a positive real number

$$\nu < \frac{(\alpha - \beta - \varepsilon_1 - \varepsilon_2)}{\ln 2} \quad (30)$$

such that

$$\text{Lip}(R) < \frac{\omega}{2M(e^{\omega/\nu} - 1)} \quad (31)$$

and

$$(\sqrt{K_1} + \sqrt{K_2})^2 4M^2 \text{Lip}(R) < \omega \frac{e^{(\alpha - \varepsilon_1 - \omega)/\nu}}{e^{\omega/\nu} - 1} \quad (32)$$

are satisfied, then the conclusions of Theorem 12 hold for  $U = X$ ; where the rate of attraction in (29) is replaced by some value  $\kappa \in (\beta, \alpha)$  (we no longer have full control over this), and moreover, assertion (iv) can be replaced by genuine uniqueness, characterized by the property (71)–(72) in Theorem 31(i). In particular,  $W^\Sigma$  is a  $C^1$  inertial manifold.

**Proof** see Sect 8. □

**Remark 21** (Precise rate of decay) The precise value of the decay rate  $\kappa$  guaranteed by Theorem 20 can be computed directly as the smaller solution of  $\varrho(e^{s/\nu})\text{Lip}(\varphi_{1/\nu} - T(1/\nu)) = 1$  within  $s \in (\beta + \varepsilon_2, \alpha - \varepsilon_1)$ , with  $\varrho$  as in (75). In fact, since the proof of Theorem 20 only used  $\alpha \leq 0$  to conclude attractivity; it is possible to construct inertial manifolds if  $\beta < 0 < \alpha$  provided the above defined  $\kappa$  is negative. This is certainly the case if  $(\sqrt{K_1}e^{(\beta + \varepsilon_2)/\nu} + \sqrt{K_2}e^{(\alpha - \varepsilon_1)/\nu})/(\sqrt{K_1} + \sqrt{K_2}) < 1$ , which is obtained upon examining the minimum of the above function in  $(\beta, \alpha)$ .

**Remark 22** The smoothness of the inertial manifold can be improved to  $C^\ell$ , provided that (A.2) holds with  $k = \ell$  and the two solutions  $\gamma_2 \leq \gamma_1$  of  $\varrho(e^{s/\nu})\text{Lip}(\varphi_{1/\nu} - T(1/\nu)) = 1$  within  $s \in (\beta + \varepsilon_2, \alpha - \varepsilon_1)$  satisfy  $\gamma_2 < \ell\gamma_1$ . (This follows from Appendix C upon noting that  $[\gamma_2, \gamma_1]$  here must be contained in the permissible interval  $[\tilde{\eta}, \bar{\eta}]$  in (103) with  $\text{Lip}(R)$  fixed; or otherwise via a slight alteration of Theorem 5 in Irwin 1980.)

**Remark 23** Supposing  $\nu > |\omega|$ , it suffices to check

$$\left(\sqrt{K_1} + \sqrt{K_2}\right)^2 8M^2 \text{Lip}(R) e^{(\omega + \varepsilon_1 - \alpha)/|\omega|} < \nu \quad (33)$$

in place of the condition (32). This follows by elementary properties of the exponential function ( $\text{sign}(x)(e^x - 1) \leq 2|x|$  for  $|x| < 1$ ).

The next corollary shows that the spectral gap assumption of Theorem 20 can be replaced by a smallness assumption on the delay. We thus recover, albeit with alternative assumptions, the results of Driver (1968). What we are able to conclude is slightly stronger due to the foliation result of Chen et al. (1997).

The proof of Corollary 24 is based on the observation that we can exhibit an arbitrarily large spectral gap for small enough delays. For this, we trace the dependence of the characteristic matrix  $\Delta(z)$ , from (22), on  $h$ . It will be convenient to use the representation

$$\det(\Delta(z)) = z^n - \sum_{j=1}^n z^{n-j} \left( \prod_{i=1}^j \int_0^h e^{-zt} d\eta_{ji}(t) \right), \quad (34)$$

where each  $\eta_{ji}$  is a linear combination of entries of  $\zeta$ . This follows directly from taking the determinant of (22) (Lemma I.4.3, Diekmann et al. 2012). To make the statement more concise, we define<sup>14</sup>

$$Q := \sum_{j=1}^n \frac{j}{n} \left( \prod_{i=1}^j \text{TV}(\eta_{ji}) \right)^{\frac{n}{j}} (2(n-j))^{\frac{n}{j}-1}. \quad (35)$$

**Corollary 24** Assume (A.1) and that  $f$  is globally Lipschitz on the whole of  $X$ . Set  $\gamma := -(2Q)^{\frac{1}{n}} - 1$ <sup>15</sup> and produce the set  $\Sigma$  as in (A.3). If there exist  $\varepsilon_1, \varepsilon_2 > 0$  such that

$$h < \min \left\{ \frac{1}{r} \ln \left( (2Q)^{-\frac{1}{n}} r \right), \frac{1}{|\gamma|} \ln \left( (2Q)^{-\frac{1}{n}} |\gamma| \right) \right\} \quad (36)$$

with

$$r := \max \left\{ \left( \sqrt{K_1} + \sqrt{K_2} \right)^2 8M^2 \text{Lip}(R) e^{(\omega + \varepsilon_1 - \alpha)/|\omega|}, |\omega|, \frac{\omega}{\ln \left( \frac{\omega}{2M \text{Lip}(R)} + 1 \right)} \right\} \ln 2 - \gamma + \varepsilon_1 + \varepsilon_2, \quad (37)$$

( $K_1, K_2$  depending on  $\varepsilon_1, \varepsilon_2$  as in the statement of Lemma 8) then the conclusions of Theorem 20 hold.

<sup>14</sup> Here, TV refers to the total variation defined in (89). In formula (35), the convention  $0^0 = 1$  is used.

<sup>15</sup> The result could be stated for arbitrary  $\gamma < -(2Q)^{\frac{1}{n}}$  for which the conditions hold, it is merely fixed here for concreteness and ease of use.

**Proof** see Sect 8. □

For  $\tau \leq h$ , let us consider any smooth map  $g_\tau : [-h, 0] \rightarrow [-\tau, 0]$  squeezing the interval (e.g.,  $\theta \mapsto \theta\tau/h$ ). Consider  $g_\tau^* : C([- \tau, 0]; \mathbb{R}^n) \rightarrow C([-h, 0]; \mathbb{R}^n)$  given by  $g_\tau^*u := u \circ g_\tau$ . A brief inspection of the characteristic eq. (23), shows that if the right-hand side  $f$  of (1a) is replaced by  $f \circ g_\tau^*$ , then the number of roots (including multiplicity) of (23) within a compact subset  $K \subset \mathbb{C}$  is constant in a neighbourhood of a fixed  $\tau$  for which there are no roots on  $\partial K$  by Hurwitz's theorem. The limit  $\tau \rightarrow 0$  yields the characteristic matrix  $\Delta(z) = z \text{id} - \zeta(h)$ , whose roots correspond to eigenvalues of the non-delayed system. In this setting, the proof of Corollary 24 constructs the spectral gap between (perturbations of) the set of eigenvalues present in the non-delayed system and the new ones appearing due to the delay. This is not explicitly apparent from the proof, but for small enough delays it certainly must be the case due to the uniqueness result Theorem 3 of Driver (1968) and the above remarks. We thus gain explicit understanding of the inertial manifold for small delays: it must be tangent at the fixed point to the spectral subspace spanned by the perturbed eigenvalues also present in the non-delayed system. This gives further confirmation to the natural intuition that sufficiently small delays are negligible, in the sense that all solutions decay exponentially fast (with exponent depending on the delay – c.f. the role of  $r$  in Corollary 24) to a finite-dimensional invariant manifold of the same dimension and characteristics as the non-delayed system. A different, perhaps more direct perspective on negligible delays is given in Kurzweil (2006).

**Remark 25** If  $f(0) = 0$ , one could also apply Theorem 20 to the fixed point 0 with  $A = A_0$  and  $R = F$ , i.e., without carrying out the linearization procedure of Section 3 and keeping the linear part within the nonlinearity  $F$ . Indeed, Theorem 31 – see also Chen et al. (1997) – itself does not assume that the nonlinearity is of higher order. We lose, however, the tangency conclusion of Theorem 12(i). In this case, our linear operator is  $A_0$ , whose spectral properties were discussed in Remark 7. For any  $\gamma < 0$ , we have  $\alpha = 0$ , and that  $\Sigma' = \emptyset$ . The projection  $P_{\Sigma'} = \text{id}_X - P_{\{0\}}$  is given by  $P_{\Sigma'}u = u - u(0)$ . Lemma 8 continues to hold in this setting, we can in fact deduce this explicitly: For any  $\beta_2 < 0$  (taking the place of  $\beta + \varepsilon_2$ ),

$$\|T_0(t)P_{\Sigma'}\|_{\mathcal{L}(X)} \leq K_2 e^{\beta_2 t} \|P_{\Sigma'}\|_{\mathcal{L}(X)}$$

holds trivially since  $T_0(t)P_{\Sigma'}u = 0$  for all  $t \geq h$  (recalling (4)). The only caveat is that  $K_2$  thus depends on  $\beta_2$  and the delay  $h$  through  $K_2 e^{\beta_2 h} = 1$ . The other part of the dichotomy, (20a), remains unaffected and we may safely take  $K_1 = 1$  for any  $\varepsilon_1 > 0$ . In this setting, (30) translates to  $v < (-\beta_2 - \varepsilon_1)/\ln 2$ , which we may take as large as we want at the expense of enlarging  $K_2$ . We also note that  $M = 1$  and any  $\omega > 0$  works for the case of  $T_0$  – let us fix  $\omega = 1$  and henceforth assume  $\beta_2 < -1$ . The condition (31) can be easily satisfied by the choice of  $\beta_2$  alone. The other condition, (33), translates (with  $\varepsilon_1 \ll 1$ ) to

$$16 \left(1 + e^{-\beta_2 h/2}\right)^2 \text{Lip}(F) < |\beta_2|, \quad (38)$$

which is perhaps more in line with the original assumptions required for inertial manifolds in, e.g., Driver (1968) and Chicone (2003). What we gain on top of classical results is that the rate of attraction  $\kappa$  towards the inertial manifold can be controlled via increasing  $\beta_2$  and shrinking  $h$ , that the  $C^\ell$  smoothness of  $W^\Sigma$  can be improved via  $\beta_2$  and  $h$  once more, and that trajectories are confined to a  $C^0$  foliation along their approach to the inertial manifold.

The precise way in which the rate of decay  $\kappa$  can be controlled via Remark 21 is best understood via examples. We will give a more elaborate description during the course of Example 2 in Section 5.2.

We also remark that, by squeezing the delay via  $g_\tau$  as above, (38) can always be achieved, since  $\text{Lip}(F \circ g_\tau^*) = \text{Lip}(F)$ .

## 5 Examples

The local invariance property of  $W^\Sigma$  (Theorem 12(i)) can be alternatively characterized as

$$\varphi_t \circ K(u_1) = K \circ \psi_t(u_1), \quad u_1 \in V \text{ such that } (t, K(u_1)) \in \mathcal{U}(U), \quad (39)$$

where  $K : V \rightarrow X$  (with  $V \subset X_\Sigma$  open, containing 0) is the embedding with image  $W^\Sigma$  and  $\psi_t$  is the  $C^\ell$  flow representing the reduced dynamics. In particular,  $\psi$  is generated by a  $C^{\ell-1}$  vector field  $H$ .

In practice, one seeks the desired manifold and reduced dynamics in the form of polynomial expressions for  $K$  and  $H$ . For the purposes herein, we shall content ourselves with third order reduced dynamics – this requires the manifold to be computed to second order. The explicit computation of the polynomial coefficients has been carried out in Diekmann et al. (2012), Chapter X, in the context of the Hopf bifurcation on a center manifold, and more recently in Szaksz et al. (2025) in the context of spectral submanifolds. Here, we give a brief recollection of the results in the present nomenclature. (The projections  $P_\Sigma^{\odot*}$ ,  $P_{\Sigma'}^{\odot*}$  are formally introduced in Appendix C.)

**Lemma 26** *Suppose Theorem 12 holds with  $W^\Sigma \subset C^\ell$ ,  $\ell \geq 3$ . Then,*

$$\begin{aligned} DK(0) &= \iota_\Sigma, \\ D^2K(0) &= \iota^{-1} \int_0^\infty T^{\odot*}(s) P_{\Sigma'}^{\odot*} D^2R(0) [T(-s)\iota_\Sigma \cdot, T(-s)\iota_\Sigma \cdot] ds, \end{aligned}$$

where  $\iota_\Sigma : X_\Sigma \hookrightarrow X$  is the inclusion. Moreover,  $H = A_\Sigma + \tilde{P}_\Sigma^{\odot*} R \circ K$  and

$$\begin{aligned} DH(0) &= A_\Sigma, \\ D^2H(0) &= \tilde{P}_\Sigma^{\odot*} D^2R(0) [\iota_\Sigma, \iota_\Sigma], \\ D^3H(0)[u, v, w] &= \tilde{P}_\Sigma^{\odot*} D^3R(0) [\iota_\Sigma u, \iota_\Sigma v, \iota_\Sigma w] + \tilde{P}_\Sigma^{\odot*} D^2R(0) [D^2K(0)[u, v], \iota_\Sigma w] \\ &\quad + \tilde{P}_\Sigma^{\odot*} D^2R(0) [D^2K(0)[w, u], \iota_\Sigma v] + \tilde{P}_\Sigma^{\odot*} D^2R(0) [D^2K(0)[v, w], \iota_\Sigma u], \end{aligned}$$

where  $u, v, w \in X_\Sigma$ .

**Proof** See Appendix D.  $\square$

In what follows, we use  $(\cdot)^\leq$  to denote the 3-jet of a map, e.g.,  $H^\leq(u) = DH(0)u + \frac{1}{2}D^2H(0)[u, u] + \frac{1}{6}D^3H(0)[u, u, u]$ .

We remark that the form of  $K$ ,  $K = (\text{id}, \phi)$ , is of no coincidence – our method of proof assumed that the manifold can be written as a graph of some function  $\phi : X_\Sigma \rightarrow X_{\Sigma'}$ . In the language of Haro et al. (2016), this is the graph style of parameterization. The parameterization, however, is not unique (even if the manifold  $W^\Sigma$  is) – there is a degree of freedom given by the invariance of (105) under  $K \mapsto K \circ \Theta$  and  $\psi_t \mapsto \Theta^{-1} \circ \psi_t \circ \Theta$  (hence  $H \mapsto D\Theta^{-1}[H \circ \Theta]$ ) for any diffeomorphism  $\Theta : M \rightarrow X_\Sigma$  for a manifold  $M$  modeled on  $X_\Sigma$ . We shall make use of this degree of freedom in the examples, so as to put the reduced dynamics in a more manageable and familiar form.

The formula for  $D^2K(0)$  can be significantly simplified if  $\Sigma$  consists of algebraically simple eigenvalues. If  $w_1, \dots, w_m$  is a basis for  $X_\Sigma$  with respect to which  $A_\Sigma$  is diagonal<sup>16</sup> and we denote by  $\lambda_i$  the eigenvalue corresponding to  $w_i$ , then for  $1 \leq i, j \leq m$ ,

$$D^2K(0)[w_i, w_j] = \iota^{-1} \int_0^\infty e^{-(\lambda_i + \lambda_j)s} T^{\odot*}(s) P_{\Sigma'}^{\odot*} D^2R(0)[w_i, w_j] ds.$$

By the assumption on  $\ell$ ,  $\text{Re}(\lambda_i + \lambda_j) > \beta$ . Therefore,

$$\iota^{-1} \int_0^\infty e^{-(\lambda_i + \lambda_j)s} T^{\odot*}(s) P_{\Sigma'}^{\odot*} ds = \iota^{-1} ((\lambda_i + \lambda_j) \text{id} - A^{\odot*})^{-1} P_{\Sigma'}^{\odot*}. \quad (40)$$

For strongly continuous semigroups, (40) is well known and is usually termed the resolvent identity (see, e.g., Lemma 7.23 of Bühler and Salamon 2018). It continues to hold in the present setting due to the properties  $A^{\odot*}$  and  $T^{\odot*}$  satisfy (in particular, that  $\|T^{\odot*}(t)P_{\Sigma'}^{\odot*}\| \leq Me^{\omega t}$  for some  $\omega > \beta$ ,  $M > 0$  and all  $t \geq 0$ , see Lemma 41), as can be directly checked by going through the proofs of Lemmas 7.13 and 7.23 in Bühler and Salamon (2018). See also Exercise IX.2.8, Diekmann et al. (2012). We hence arrive at the formula

$$D^2K(0)[w_i, w_j] = P_{\Sigma'} \iota^{-1} ((\lambda_i + \lambda_j) \text{id} - A^{\odot*})^{-1} D^2R(0)[w_i, w_j],$$

from which the full operator  $D^2K(0)$  can be reconstructed.

We recall from Diekmann et al. (2012), IV.(3.3), the following formula for a projection  $P_{\{\lambda\}}$  associated to a simple eigenvalue  $\lambda$ :

$$[P_{\{\lambda\}}u](\theta) = e^{\lambda\theta} \xi(\lambda) \left[ u(0) + \int_0^h d\zeta(s) \int_0^s e^{-\lambda\tau} u(\tau - s) d\tau \right], \quad (41)$$

<sup>16</sup> This can only be achieved over the complexification of  $X_\Sigma$  and  $A_\Sigma$ ; we omit this from the notation, otherwise it would get too cluttered. Once the full operator  $D^2K(0)$  is reconstructed, it will of course preserve the real subspace.

where  $\xi(z) := \Delta(z)^{-1}(z - \lambda)$ . The obvious modifications – replacing  $u(0)$  above with the first component of an element of  $X^{\odot*}$  – give the extended projection  $P_{\{\lambda\}}^{\odot*}$ . We will also need the formula for  $(z \operatorname{id} - A^{\odot*})^{-1}$ , but only the part pertaining to elements of the form  $(c, 0) \in X^{\odot*}$ . This is given in Corollary IV.5.4, Diekmann et al. (2012), as

$$(z \operatorname{id} - A^{\odot*})^{-1}(c, 0) = (\Delta(z)^{-1}c, \theta \mapsto e^{z\theta} \Delta(z)^{-1}c).$$

Combining these together, one obtains the explicit formula

$$D^2K(0)[w_i, w_j] = \left[ e^{(\lambda_i + \lambda_j)[\cdot]} - \sum_{\lambda \in \Sigma} e^{\lambda[\cdot]} \xi(\lambda) \left( 1 + \int_0^h d\zeta(s) \frac{e^{-\lambda s} - e^{-(\lambda_i + \lambda_j)s}}{\lambda_i + \lambda_j - \lambda} \right) \right] \\ \times \Delta(\lambda_i + \lambda_j)^{-1} \operatorname{proj}_1(D^2R(0)[w_i, w_j]), \quad (42)$$

where  $\operatorname{proj}_1 : X^{\odot*} \rightarrow \mathbb{R}^n$  is the projection onto the first factor. Higher order terms can be computed analogously – for instance, if  $D^2K(0) = 0$ , the coefficient  $D^3K(0)$  is obtained by replacing  $D^2R(0)$  by  $D^3R(0)$  in the above and using three eigenvectors/eigenvalues.

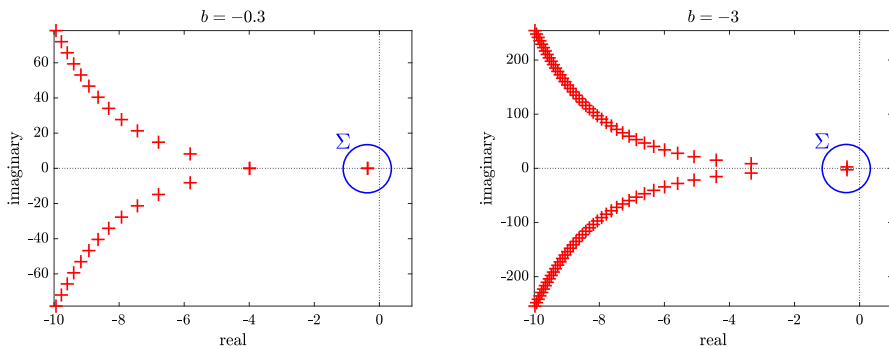
We remark if  $\ell$  does not meet the conditions of Lemma 26, one may still compute the right-hand sides of the coefficients given therein, so long as  $R$  is sufficiently smooth and ‘nonresonance’ conditions are met (so that  $\lambda_i + \lambda_j \notin \Sigma'$  in the above). However, one may no longer associate them to Taylor coefficients of  $K$  and  $H$ .

**Remark 27** According to Theorem 12(ii), it makes sense to differentiate (39) in time at  $t = 0$ . The resulting infinitesimal version of the invariance equation reads

$$f(K(u_1)) = DK(u_1)[H(u_1)]. \quad (43)$$

Several works utilizing the parameterization method prefer to treat this version of the invariance equation (Groothedde and James 2017; Hénot 2021; Hénot et al. 2022), the approaches of which could simplify much of the calculations performed below. In fact, if one seeks the embedding corresponding to linear reduced dynamics, the expressions for the embedding coefficients become particularly simple (see (Groothedde and James, 2017, page 9)). Assuming a priori the existence of an invariant manifold, this procedure was made rigorous in de Wolff 2018, Lemma 3.2.1 within the sun-star setting. Higher-order reduced dynamics are considered in Hénot (2021). Herein, we opted not to follow this approach mainly because the present exposition highlights better how the computed coefficients result from the theoretical proof. The resulting coefficients agree no matter whether (39) or (43) is used, once the style of parameterization has been fixed.

The examples given below are illustrative academic examples that are meant to clarify the theoretical results, particularly those concerning inertial manifolds; we do not touch on many of the otherwise practically important aspects. For a practical assessment of SSM reduction in the context of DDEs, we refer the reader to the recent works (Szaksz et al. 2025, 2026; Abbasciano et al. 2026a, b), alongside the ones listed



**Fig. 1** Roots of (45) in the complex plane

in Remark 27 above. A data-driven implementation using the computational package SSMLearn (Cenedese et al. 2022) is considered in Abbasciano et al. (2026a, b).

### 5.1 The Cushing Equation

We consider the system (with its linear part taken from Insperger and Stépán 2011, originally from Cushing 1977)

$$\dot{x}(t) = b \int_0^h x(t - \theta) d\theta + a(x(t) - \sin(x(t))) \quad (44)$$

on  $\mathbb{R}$ , for some  $b \in \mathbb{R}$  and  $a > 0$ . Equation (44) admits an equilibrium state at 0. Throughout, we shall fix the delay  $h = 1$ . In the setting of Section 3, we have  $\zeta(t) = bt$ ,  $t \in [0, h]$ , and

$$R(u) = (au(0) - a \sin u(0), 0).$$

We have immediately that  $\text{Lip}(R) = 2a$ .

The characteristic matrix (22) reads

$$\Delta(z) = z - b \frac{1 - e^{-zh}}{z}. \quad (45)$$

Note that  $\lim_{z \rightarrow 0} \Delta(z) = -bh$ , that is, roots can only cross the imaginary axis away from the origin, so long as  $b \neq 0$ . It can be shown that (44) is linearly stable for  $b \in (-\pi^2/(2h^2), 0)$  (see Sect 2.1.2 of Insperger and Stépán 2011 or plug in  $z = i\varpi$ ,  $\varpi \in \mathbb{R} \setminus \{0\}$ , into (45) and seek the smallest-in-modulus solution for  $b$ ).

We perform SSM computations at two values of  $b$ ,  $b = -0.3$  and  $b = -3$  within the stable domain. As a first step, we need to compute the corresponding spectra, which is shown in Fig 1.<sup>17</sup> For later reference, we note here the values of the slowest

eigenvalues:

$$\begin{aligned}\lambda_1 &\approx -0.361, & \lambda_2 &\approx -3.99, & \text{for } b = -0.3; \\ \lambda_{1,2} &\approx -0.387 \pm i2.66, & \lambda_{3,4} &\approx -3.33 \pm i8.67, & \text{for } b = -3.\end{aligned}$$

We take  $\Sigma = \{\lambda_1\}$  for the  $b = -0.3$  case and  $\Sigma = \{\lambda_1, \lambda_2\}$  for the  $b = -3$  case. The values above determine  $\alpha = -0.361$  and  $\beta = -3.99$  for  $b = -0.3$ ; and  $\alpha = -0.387$  and  $\beta = -3.33$  for  $b = -3$ . Our manifold  $W^\Sigma$  will thus have sufficient smoothness,  $\ell \geq 3$ , to recover third order dynamics (Lemma 26).

Next we show under what conditions on  $a$  there exist inertial manifolds, which will be uniquely determined by their dimension in this example. An inspection of (45) shows that the equations  $\Delta(z) = 0$  and  $\Delta'(z) = 0$ , when considered simultaneously, admit no solutions if  $b = -0.3$  or  $b = -3$  (and  $h = 1$ ). This implies, that all roots of  $\Delta$  are simple. In turn, Corollary V.6.4 of Diekmann et al. (2012) implies that the solution to the linear part of (44) can be written as the convergent series  $\sum_j e^{\lambda_j t} c_j$  for some  $c_j \in \mathbb{C}$ . Thus, for any  $\varepsilon_1, \varepsilon_2 > 0$  and  $\omega > \alpha$ , we may take  $K_1 = K_2 = M = 1$  in the notation of Theorem 20. Therefore, using that  $\text{Lip}(R) = 2a$ , condition (33) boils down to the existence of  $\omega > \alpha$  for which

$$\max \left\{ 64a \ln 2 e^{(\omega - \alpha)/|\omega|}, \ln 2|\omega| \right\} < \alpha - \beta$$

holds. Since, in both cases,  $\ln 2|\alpha| < \alpha - \beta$ , we may take  $\omega$  sufficiently close to  $\alpha$ , so that the only remaining condition is

$$a < \frac{\alpha - \beta}{64 \ln 2}.$$

Next, we compute the reduced dynamics. We first note that  $R = g \circ \text{ev}_0$  for  $g: \mathbb{R} \rightarrow X^{\odot*}$ ,  $y \mapsto (a \sin(y) - ay, 0)$ , and  $\text{ev}_0: u \mapsto u(0)$  the (bounded, linear) evaluation map  $X \rightarrow \mathbb{R}$ . Hence,

$$D^j R(0) = D^j g(0)[\text{ev}_0(\cdot), \dots, \text{ev}_0(\cdot)].$$

We have that  $D^2 R(0) = 0$ , which implies by  $D^2 H(0)$  and  $D^2 K(0) = 0$  by Lemma 26. The third coefficient is simply  $D^3 R(0) = (a[\text{ev}_0(\cdot)]^3, 0)$ .

Consider the case  $b = -0.3$ . Then  $A_\Sigma = \lambda_1$ ; what remains to compute is  $D^3 H(0)$ . By (41), we have

$$D^3 H(0) = \theta \mapsto e^{\lambda_1 \theta} \xi(\lambda_1) a[\text{ev}_0(\cdot)]^3.$$

The third order dynamics are therefore given by

$$\frac{d}{dt} w = H^\leq(w) = \lambda_1 w + \frac{1}{6} (\theta \mapsto e^{\lambda_1 \theta} \xi(\lambda_1) a[\text{ev}_0(w)]^3), \quad w \in X_\Sigma. \quad (46)$$

<sup>17</sup> The eigenvalues were computed by separating (45) into real and imaginary parts, solving one analytically and then verifying the compatibility condition stemming from the other via the Matlab function `fsolve`. The computed spectrum was then cross-validated via the publicly available package (Appeltans et al. 2022, 2023).

In the present setting,  $X_\Sigma$  is a one-dimensional vector space spanned by  $\theta \mapsto e^{\lambda_1 \theta}$  (see Sect 3.1.1). We may use the linear isomorphism  $\Theta : \mathbb{R} \rightarrow X_\Sigma$ ,  $\Theta(y) = (\theta \mapsto ye^{\lambda_1 \theta})$  to pull (46) back to  $\mathbb{R}$ . The desired dynamics are  $\tilde{H} := D\Theta^{-1}[H^\leq \circ \Theta] = \Theta^{-1} \circ H^\leq \circ \Theta$ , or explicitly,

$$\frac{d}{dt}y = \tilde{H}(y) = \lambda_1 y + \frac{1}{6}\xi(\lambda_1)ay^3. \quad (47)$$

The coefficient  $\xi(\lambda_1)$  can be explicitly computed via L'Hôpital's rule as

$$\xi(\lambda_1) = \lim_{z \rightarrow \lambda_1} \frac{z - \lambda_1}{z - b \frac{1 - e^{-zh}}{z}} = \frac{1}{1 + \frac{b}{\lambda_1^2}(1 - e^{-\lambda_1 h}) - \frac{bh}{\lambda_1}e^{-\lambda_1 h}}.$$

We now turn to  $b = -3$ , the case in which  $\Sigma$  comprises a pair of a complex conjugate eigenvalues. The relevant projection is  $\tilde{P}_\Sigma^{\odot*} = \tilde{P}_{\{\lambda_1\}}^{\odot*} + \tilde{P}_{\{\lambda_2\}}^{\odot*}$ , whose image defines the (two dimensional, real) subspace

$$X_\Sigma = \{\theta \mapsto ze^{\lambda_1 \theta} + \bar{z}e^{\lambda_2 \theta} \in X \mid z \in \mathbb{C}\}. \quad (48)$$

An alternative characterization, via  $z = x + iy$ , is

$$X_\Sigma = \left\{ \theta \mapsto 2e^{\operatorname{Re} \lambda_1 \theta} (x \cos(\operatorname{Im} \lambda_1 \theta) - y \sin(\operatorname{Im} \lambda_1 \theta)) \in X \mid (x, y) \in \mathbb{R}^2 \right\}. \quad (49)$$

The third order dynamics on  $X_\Sigma$  are given by

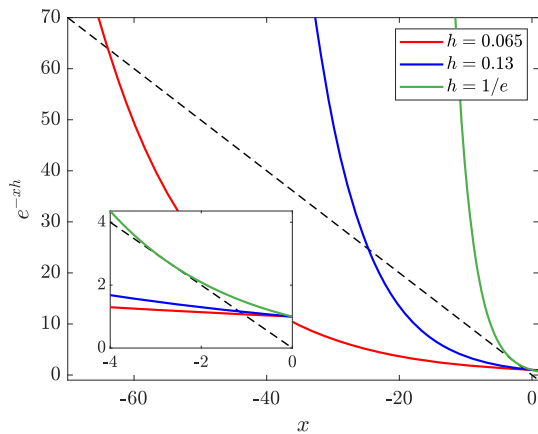
$$\frac{d}{dt}w = H^\leq(w) = A_\Sigma w + \frac{1}{6}(\theta \mapsto (e^{\lambda_1 \theta} \xi(\lambda_1) + e^{\lambda_2 \theta} \xi(\lambda_2))a[\operatorname{ev}_0(w)]^3), \quad w \in X_\Sigma.$$

The second characterization of  $X_\Sigma$ , (49), suggests the parameterization  $\Theta : \mathbb{R}^2 \rightarrow X_\Sigma$ ,  $\Theta(x, y) = \theta \mapsto 2e^{\operatorname{Re} \lambda_1 \theta} (x \cos(\operatorname{Im} \lambda_1 \theta) - y \sin(\operatorname{Im} \lambda_1 \theta))$ . By linearity once more,  $\tilde{H} = \Theta^{-1} \circ H^\leq \circ \Theta$ , resulting in

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \operatorname{Re} \lambda_1 & -\operatorname{Im} \lambda_1 \\ \operatorname{Im} \lambda_1 & \operatorname{Re} \lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{4}{3}x^3a \begin{pmatrix} \operatorname{Re} \xi(\lambda_1) \\ \operatorname{Im} \xi(\lambda_1) \end{pmatrix}. \quad (50)$$

It is perhaps more suitable to parameterize dynamics characterized by complex conjugate pairs of eigenvalues on the half-cylinder  $\mathbb{R}^{\geq 0} \times \mathbb{S}$ , where  $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ , corresponding to polar coordinates. This can be achieved via  $\Theta : \mathbb{R}^{\geq 0} \times \mathbb{S} \rightarrow X_\Sigma$  given by  $\Theta(r, \phi) = \theta \mapsto re^{i\phi}e^{\lambda_1 \theta} + re^{-i\phi}e^{\lambda_2 \theta}$ . (Note that  $\Theta$  is a diffeomorphism on  $\{r > 0\}$  but not at the origin, as the preimage of  $\{0\}$  is  $\{0\} \times \mathbb{S}$ ). The reduced dynamics would then be obtained as  $\tilde{H} = D\Theta^{-1}[H^\leq \circ \Theta]$  – we will only carry out the explicit computation for the following example, however.

**Fig. 2** The real roots of (52), obtained as the intersection of  $x \mapsto e^{-xh}$  and  $x \mapsto -x$  for three different values of  $h$ . The black dashed line shows  $x \mapsto -x$



## 5.2 Small Delay Inertial Manifolds For a One-Dimensional Example

Consider now

$$\dot{x}(t) = -x(t-h) + x(t) - \sin x(t). \quad (51)$$

We examine the vicinity of the steady state at  $x = 0$ . As before, we consider the  $x(t)$  term as part of the nonlinearity, so as to make it third order. The linear part hence agrees with the problem treated in Szaksz et al. (2025).

In the setting of Section 3,

$$\zeta(\theta) = \begin{cases} 0, & \text{for } \theta \in [0, h) \\ -1, & \text{for } \theta = h \end{cases}$$

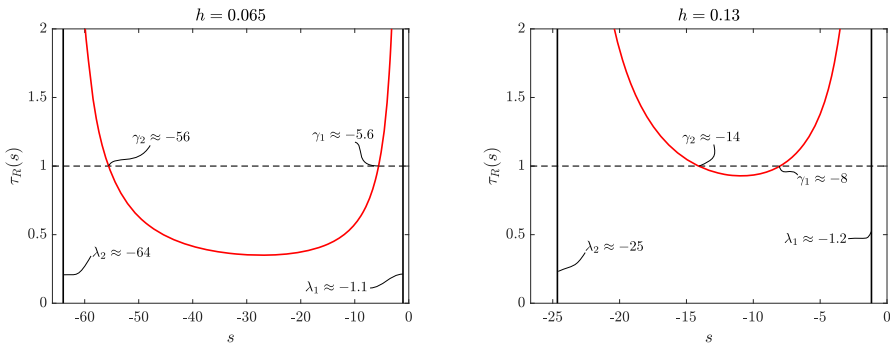
and  $R(u) = (u(0) - \sin u(0), 0)$ , thus  $\text{Lip}(R) = 2$ . The characteristic equation takes the simple form

$$\Delta(z) = z + e^{-zh} = 0. \quad (52)$$

For  $h < 1/e$ , (52) admits two real solutions  $\lambda_1(h), \lambda_2(h) \in \mathbb{R}$ ,  $\lambda_2(h) < \lambda_1(h)$ . Here  $\lambda_1(h)$  corresponds to the eigenvalue that perturbs from  $\lambda_1(0) = -1$ ; whereas  $\lim_{h \rightarrow 0} \lambda_2(h) = -\infty$  (see also Fig 2). All other roots must be complex and have real part less than  $\lambda_2(h)$  (clearly there is no root with  $\text{Re } z \in (\lambda_2(h), \lambda_1(h))$  – now the continuity argument preceding Remark 25 along with Theorem 5 implies the desired conclusion).

Next, we derive conditions on the delay  $h$  under which there exists a unique inertial manifold associated to  $\Sigma = \{\lambda_1(h)\}$ . Via the same argument as in Example 5.1, we may take for the constants appearing in Theorem 20  $M = K_1 = K_2 = 1$  for any  $\varepsilon_1, \varepsilon_2 > 0$  and  $\omega > \lambda_1(h)$ . The constant  $Q$  from (35) is simply  $Q = \text{TV}(\zeta) = 1$ , hence  $\gamma = -3$  in the statement of Corollary 24. We compute  $r$  from (37) as

$$r = 3 + \ln 2 \max \left\{ 64e^{(\omega - \lambda_1(h))/|\omega|}, |\omega|, \frac{\omega}{\ln(\omega/4 + 1)} \right\}.$$



**Fig. 3** The graph of (53) for two values of  $h$

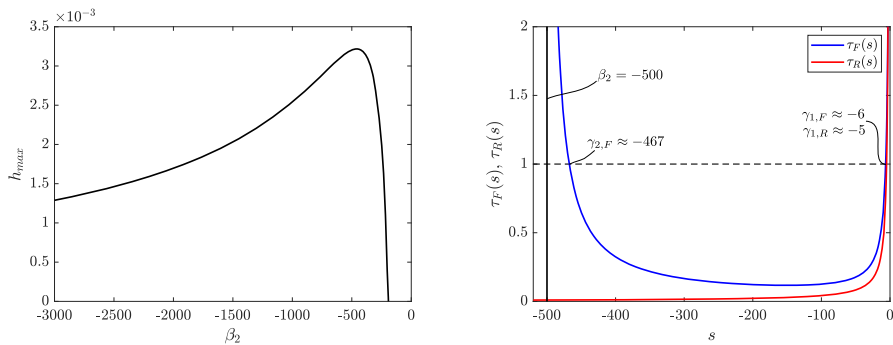
Noting that  $\lambda_1(h) \in (-5, -1)$  for all  $h \in (0, 1/e)$  (see Fig 3 of Szaksz et al. 2025), we may safely take  $\omega$  arbitrarily close to  $\lambda_1(h)$  (for each  $h$ ) and still have the first term dominate – we may hence use  $r = 3 + 64 \ln 2$ . Plugging this in (36), Corollary 24 now asserts, for  $h < 0.066$ , the existence of a unique inertial manifold  $W^\Sigma$  tangent to the space  $X_\Sigma = \{\theta \mapsto ce^{\lambda_1(h)\theta} \in X \mid c \in \mathbb{R}\}$  at the origin.

Next, we estimate (from above) the rate of attraction  $\kappa$ . We do this via the initial version of the inertial manifold result, Theorem 20, more specifically, via the route proposed in Remark 21. For a given  $h < 1/e$ , we may compute the two rightmost eigenvalues  $\lambda_1, \lambda_2 \in \mathbb{R}$  and set  $\varepsilon = 10^{-5}$  (arbitrary),  $v = (\lambda_1 - \lambda_2 - 3\varepsilon)/\ln(2)$ ,  $\alpha_1 = e^{(\lambda_1 - \varepsilon)/v}$  and  $\alpha_2 = e^{(\lambda_2 + \varepsilon)/v}$ . We estimate the Lipschitz constant of the nonlinearity  $\text{Lip}(\varphi_{1/v} - T(1/v))$  according to (87) (with  $\omega = \lambda_1 + \varepsilon$ ). The function of interest (from Remark 21) is

$$\tau_R(s) := \left( \frac{1}{\alpha_1 - e^{s/v}} + \frac{1}{e^{s/v} - \alpha_2} \right) \frac{2}{\omega} (e^{\omega/v} - 1) e^{\omega/v} \text{Lip}(R), \quad s \in (\lambda_2 + \varepsilon, \lambda_1 - \varepsilon), \quad (53)$$

plotted in Fig 3 for  $h = 0.065$  and  $h = 0.13$ . The intersection of (53) with the line  $s \mapsto 1$  give  $\gamma_2$  and  $\gamma_1$  from the statement of Theorem 31. In particular, the first intersection point,  $\gamma_2$ , approximates the decay rate  $\kappa$ . This is an upper bound on  $\kappa$ , since the estimate on the Lipschitz constant (possibly) pushes the curve (53) upwards.

We also observe that, while Corollary 24 necessitated  $h < 0.066$ , the initial version in Theorem 20 shows existence and uniqueness of the inertial manifold  $W^\Sigma$  for up to  $h = 0.13$  (see Fig 3). The difference roughly gauges the loss incurred through the estimates of Remark 23 and the proof of Corollary 24. Let us compare this with what Remark 25 would suggest. For this, we verify (38). First, we note that  $|\beta_2|$  must be chosen larger than the left hand side of (38) evaluated at  $h = 0$ . In our case, this translates to  $\beta_2 < -192$  (using  $\text{Lip}(F) = 3$ ). Beyond this value, we may always find  $h$  satisfying (38) – the dependence of the maximal such  $h$  on  $\beta_2$  is depicted in Fig 4 (left). Recall that Remark 25 applied Theorem 31 with the linear part in the ‘nonlinearity’ – it hence does not carry any tangency property. Similarly to the case above, we may compute  $\varrho$  from (75) to obtain the Lyapunov exponents characterizing the manifold (and thus also the decay rate). In the present context,  $\omega = 1$ ,  $v = (-\beta_2 - 2\varepsilon)/\ln 2$ ,



**Fig. 4** (Left) dependence of the maximal delay  $h_{\max}$  on  $\beta_2$  – according to the (suboptimal) prediction of (38). (Right) The graph of (54) compared with (53) for  $h = 0.002$  and  $\beta_2 = -500$

$\alpha_1 = e^{-\varepsilon/\nu}$ ,  $\alpha_2 = e^{\beta_2/\nu}$ , and  $C_2 = e^{-h\beta_2}$ . We plot

$$\tau_F(s) := \left( \frac{1}{\alpha_1 - e^{s/\nu}} + \frac{C_2}{e^{s/\nu} - \alpha_2} \right) \frac{2}{\omega} (e^{\omega/\nu} - 1) e^{\omega/\nu} \text{Lip}(F), \quad s \in (\beta_2, -\varepsilon) \quad (54)$$

alongside the prior curve (53) in Fig 4 (right), for the value  $h = 0.002$  (and  $\beta_2 = -500$ ). Comparing the two, we may conclude via the negative semiorbit characterization of Theorem 31(i) that the two manifolds must coincide.

### 5.2.1 The Example of Szaksz et al. (2025)

Next, we revisit the example considered in Szaksz et al. (2025). In particular, we modify the nonlinearity in (51) to  $-x(t)^3$ , that is,

$$\dot{x}(t) = -x(t-h) - x(t)^3. \quad (55)$$

This differs from the equation in Szaksz et al. (2025) by a sign (on the nonlinearity) – we performed this change to make the Hopf bifurcation occurring at the critical delay  $h = \pi/2$  supercritical. Nonetheless, all coefficients of the parameterization will be comparable to Szaksz et al. (2025) up to a sign difference.

We take  $\Sigma = \{\lambda_1\}$  when the dominant eigenvalue is real (when  $h < 1/e$ ) and  $\Sigma = \{\lambda_1, \lambda_2\}$  when the dominant eigenvalues form a complex conjugate pair (when  $h > 1/e$ ),  $\bar{\lambda}_2 = \lambda_1$ . Despite the nonlinearity not being globally Lipschitz, the local theory (i.e., Theorems 12, 16, 18) continues to hold, we thus obtain local existence of  $W^\Sigma$  about 0. Next, we carry out the explicit computation of the parameterization up to third order via Lemma 26 and a normal form transformation for the complex case to recover the findings of Szaksz et al. (2025). Note that the use of Lemma 26 necessitates a spectral gap of the form  $\lambda_2 < 4\lambda_1$  in the case of a dominant real eigenvalue, which is satisfied for  $h < \sqrt[3]{2} \ln(2)/3$ . It is also certainly satisfied in the complex conjugate case near the bifurcation.

When the dominant eigenvalue is real,  $\Sigma = \{\lambda_1\}$ , the reduced dynamics are

$$\dot{y} = \tilde{H}(y) = \lambda_1 y - \xi(\lambda_1) y^3, \quad (56)$$

where  $\tilde{H} := \Theta^{-1} \circ H \circ \Theta$  with  $\Theta : \mathbb{R} \rightarrow X_\Sigma$  denoting the isomorphism  $y \mapsto (\theta \mapsto y e^{\lambda_1 \theta})$  as before. This is computed entirely analogously to (47), now with  $R(u) = (-u(0)^3, 0)$  – and hence  $D^3 R(0) = (-6[\text{ev}_0(\cdot)]^3, 0)$ , noting that  $D^2 K(0)$  vanishes just like in Section 5.1. As for the parameterization of the manifold,  $\tilde{K} := K \circ \Theta$ , it remains to compute the third coefficient. For this, the third order analogue of (42) is employed, which yields

$$D^3 K(0)[\Theta(y), \Theta(y), \Theta(y)] = -6 \left[ e^{3\lambda_1[\cdot]} - e^{\lambda_1[\cdot]} \xi(\lambda_1) \frac{\Delta(3\lambda_1)}{2\lambda_1} \right] \Delta(3\lambda_1)^{-1} y^3. \quad (57)$$

Noting that  $\xi$  is given explicitly as

$$\xi(\lambda_1) = \lim_{z \rightarrow \lambda_1} \frac{z - \lambda_1}{\Delta(z)} = \frac{1}{1 - h e^{-\lambda_1 h}} = \frac{1}{1 + \lambda_1 h},$$

we observe that (56) and (57) recover (89)–(92) of Szaksz et al. (2025) (with a sign difference).

In the complex conjugate case, when  $\Sigma = \{\lambda_1, \lambda_2\}$ , the reduced dynamics are obtained analogously to (50). In the coordinates (48), they are given by

$$\dot{z} = \lambda_1 z - \xi(\lambda_1)(z + \bar{z})^3 =: g(z), \quad (58a)$$

$$\dot{\bar{z}} = \lambda_2 \bar{z} - \xi(\lambda_2)(z + \bar{z})^3. \quad (58b)$$

By nature of our procedure, (58) corresponds to the graph style of parameterization. We proceed via a normal form transformation to achieve the form reported in Szaksz et al. (2025).

When  $\lambda_1, \lambda_2$  are purely imaginary, the (semisimple) normal form of (58) is given by (see e.g. Chapter 1 of Murdock 2003)

$$\dot{q} = \lambda_1 q - \beta_{21} |q|^2 q, \quad (59a)$$

$$\dot{\bar{q}} = \lambda_2 \bar{q} - \bar{\beta}_{21} |q|^2 \bar{q}. \quad (59b)$$

Classically, in normal form computations, the transformation which leads to the form (59) is sought as a change of coordinates  $z = q + p(q, \bar{q})$  for which

$$g(q + p(q, \bar{q})) = \frac{d}{dt}(q + p(q, \bar{q}))$$

holds up to third order; where  $p$  is a degree three homogeneous polynomial in  $q$  and  $\bar{q}$  over  $\mathbb{C}$  (in this example). At third order, this equation is

$$Dp(q, \bar{q})[\lambda_1 q, \lambda_2 \bar{q}] - \lambda_1 p(q, \bar{q}) = -\xi(\lambda_1)(q + \bar{q})^3 + \beta_{21} |q|^2 q. \quad (60)$$

The coefficient  $\beta_{21}$  is uniquely determined by requiring that (60) is solvable for  $p$  when  $\lambda_1, \lambda_2$  are purely imaginary. We obtain  $\beta_{21} = 3\xi(\lambda_1)$ . Moreover, the solution to (60) (unique when  $\lambda_1, \lambda_2$  are not purely imaginary) is

$$p(q, \bar{q}) = -\frac{\xi(\lambda_1)}{2\lambda_1} q^3 - \frac{3\xi(\lambda_1)}{2\lambda_2} |q|^2 \bar{q} - \frac{\xi(\lambda_1)}{3\lambda_2 - \lambda_1} \bar{q}^3.$$

We now introduce polar coordinates via  $q = re^{i\phi}$ ,  $(r, \phi) \in \mathbb{R}^{\geq 0} \times \mathbb{S}$ . For  $r > 0$ , the dynamics are hence given by

$$\dot{r} = \operatorname{Re} \lambda_1 r - 3\operatorname{Re} \xi(\lambda_1) r^3, \quad (61a)$$

$$\dot{\phi} = \operatorname{Im} \lambda_1 - 3\operatorname{Im} \xi(\lambda_1) r^2. \quad (61b)$$

To identify the associated parameterization of the manifold, we first formalize the computations done thus far. Recall that  $H^{\leq}$  denotes the original third order vector field on  $X_{\Sigma}$ ; and consider the following maps:

$$\begin{aligned} \vartheta : \mathbb{R}^{\geq 0} \times \mathbb{S} &\rightarrow \mathbb{C}^2, & \vartheta(r, \phi) &:= (re^{i\phi}, re^{-i\phi}), \\ \Phi : \mathbb{C}^2 &\rightarrow \mathbb{C}^2, & \Phi(q_1, q_2) &:= (q_1 + p(q_1, q_2), q_2 + \bar{p}(q_2, q_1)), \\ \Psi : \mathbb{C}^2 &\rightarrow (X_{\Sigma})_{\mathbb{C}}, & \Psi(z_1, z_2) &:= \theta \mapsto z_1 e^{\lambda_1 \theta} + z_2 e^{\lambda_2 \theta}, \end{aligned}$$

where  $\bar{p}$  is simply  $p$  with its coefficients conjugated; and set  $\Xi := \Psi \circ \Phi$ . Then, the vector field defining (61) (for  $r > 0$ ) is

$$\tilde{H}(r, \phi) = (D\vartheta(r, \phi))^{-1} \left( (D\Xi^{-1}) \circ \Xi[H_{\mathbb{C}}^{\leq} \circ \Xi] \right)^{\leq} \circ \vartheta(r, \phi).$$

Consequently, the associated parameterization of the manifold (to third order in  $q$ ) is

$$\tilde{K} = (K^{\leq} \circ \Xi)^{\leq} \circ \vartheta. \quad (62)$$

For  $w_1 : \theta \mapsto e^{\lambda_1 \theta}$  and  $w_2 : \theta \mapsto e^{\lambda_2 \theta}$  we write

$$K^{\leq}(zw_1 + \bar{z}w_2) = zw_1 + \bar{z}w_2 + \frac{1}{6} \left( K_{30}z^3 + 3K_{21}|z|^2z + 3K_{12}|z|^2\bar{z} + K_{03}\bar{z}^3 \right),$$

where  $K_{30} = D^3K(0)[w_1, w_1, w_1]$  and similarly for the rest. An explicit computation via (42) yields

$$\begin{aligned} K_{30} &= -6 \left[ \Delta(3\lambda_1)^{-1} e^{3\lambda_1[\cdot]} - \frac{\xi(\lambda_1)}{2\lambda_1} e^{\lambda_1[\cdot]} - \frac{\xi(\lambda_2)}{3\lambda_1 - \lambda_2} e^{\lambda_2[\cdot]} \right], \\ K_{21} &= -6 \left[ \Delta(2\lambda_1 + \lambda_2)^{-1} e^{(2\lambda_1 + \lambda_2)[\cdot]} - \frac{\xi(\lambda_1)}{\lambda_1 + \lambda_2} e^{\lambda_1[\cdot]} - \frac{\xi(\lambda_2)}{2\lambda_1} e^{\lambda_2[\cdot]} \right]; \end{aligned}$$

$K_{03}$  and  $K_{12}$  are their complex conjugates. Substituting into (62),

$$\tilde{K}(r, \phi) = r \left( e^{\lambda_1[\cdot]} e^{i\phi} + e^{\lambda_2[\cdot]} e^{-i\phi} \right) + r^3 \left( \tilde{K}_{30} e^{3i\phi} + \tilde{K}_{21} e^{i\phi} + \tilde{K}_{12} e^{-i\phi} + \tilde{K}_{03} e^{-3i\phi} \right),$$

where

$$\begin{aligned} \tilde{K}_{30} &= -\Delta(3\lambda_1)^{-1} e^{3\lambda_1[\cdot]}, \\ \tilde{K}_{21} &= -3 \left[ \Delta(2\lambda_1 + \lambda_2)^{-1} e^{(2\lambda_1 + \lambda_2)[\cdot]} - \frac{\xi(\lambda_1)}{\lambda_1 + \lambda_2} e^{\lambda_1[\cdot]} \right]; \end{aligned}$$

the rest of the coefficients are given by their complex conjugates once more. This agrees with the parameterization of Szaksz et al. (2025) (with a sign difference).

The reduced dynamics (61) predict the existence of a limit cycle whenever  $\operatorname{Re} \xi(\lambda_1) \neq 0$  and  $\operatorname{Re} \lambda_1 / \operatorname{Re} \xi(\lambda_1) > 0$ . Both of these are certainly satisfied for  $\operatorname{Re} \lambda_1 > 0$  (i.e.,  $h > \pi/2$ ), in which case the predicted limit cycle is stable (this corresponds to the 'supercritical' part of the Hopf bifurcation), with amplitude  $\hat{r} = \sqrt{\operatorname{Re} \lambda_1 / (3 \operatorname{Re} \xi(\lambda_1))}$  and angular frequency  $\omega$  determined by the right hand side of (61b) evaluated at  $\hat{r}$ . Its trajectory in physical space can be estimated – to third order – via  $x_{per}(t) \approx \tilde{K}(\hat{r}, e^{i\omega t})$  (c.f. (112) of Szaksz et al. 2025).

In examples where the nonlinearity is no longer globally Lipschitz but the existence of a compact global attractor is known, we may still conclude the existence (but no longer uniqueness) of an inertial manifold by use of cutoff functions. This still requires a moderately large spectral gap (or small delay), at least in comparison to the Lipschitz constant  $\operatorname{Lip}(R_\rho)$  of the cut-off nonlinearity (as in Section 6.1), where  $\rho$  is selected such that  $B_\rho^X(0)$  contains the global attractor  $\mathcal{A}$ . Due to the cutoff function, the foliation is no longer global either, but the exponential attraction rate in Definition 19 can still be achieved so long as  $\varphi_t(u)$  enters (and stays within)  $B_\rho^X(0)$  for all times  $t \geq t^*$  for some finite  $t^*$  depending only on  $|u|_X$  (this absorbing property can also be used to prove the existence of a global attractor via Proposition 4(iv) and the procedure described in Chapter 10 of Robinson (2001)).

In the simple example (55) considered here, we may exhibit the permissible size of the attractor  $\mathcal{A}$  explicitly, by means of (53) (or perhaps more precisely, via (76)). With  $\alpha_1, \alpha_2, \omega, \nu$  defined as prior to (53), the Lipschitz constant of the cut-off nonlinearity,  $R_\rho$  (see Sect 6.1 for a precise definition), must satisfy

$$\operatorname{Lip}(R_\rho) < \frac{\alpha_1 - \alpha_2}{\frac{8}{\omega} (e^{\omega/\nu} - 1) e^{\omega/\nu}} \quad (63)$$

for the manifold  $W_\rho^\Sigma$  to exist. By Lemma 28 and its footnote, we have

$$\operatorname{Lip}(R_\rho) \leq 2(1 + 2\|P_\Sigma\|)\operatorname{Lip}_{4\rho}(R) \leq 2(1 + 2\|P_\Sigma\|)48\rho^2.$$

Computing  $\|P_\Sigma\|$  via (41) now shows that the critical  $\rho$  for which (63) is met is bounded from below by  $\rho_{crit} \approx 0.008$  in the delay range  $h \in (\pi/2, \pi/2 + 0.1)$ . Certainly then, if  $h$  is only marginally above its critical value, the stable limit cycle can be confined

to be within  $B_{\rho_{crit}}^X(0)$ . Provided no other attractors exist, this shows the existence of an inertial manifold (this is using Remark 21 and the numerical observation that the smaller solution will be attained below 0). The conclusions obtained in the context of this example are rather weak and seemingly insignificant; our main purpose was to demonstrate the general procedure of obtaining inertial manifolds in the absence of globally Lipschitz nonlinearities.

## 6 Proof of Theorem 12

In this section, we provide a proof of Theorem 12, which will largely be based on Theorem 1.1 of Chen et al. (1997).

### 6.1 Cutoff Functions

In order to apply this result, one needs *global* control over the Lipschitz constant of the nonlinearity. This is usually achieved by modifying the nonlinearity  $R$  away from the fixed point at 0 by means of cutoff functions. Since  $X$  is not a Hilbert space, nor is it finite dimensional, the existence of *smooth* cutoff functions is a nontrivial question. In fact, it is known that  $X = C([-h, 0]; \mathbb{R}^n)$  does *not* admit a Fréchet differentiable norm or cutoff function (this follows from Theorem 24, (Fry and McManus 2002), noting that  $X$  is separable but  $X^*$  is not).

To retain smoothness on some (strictly smaller) subset of  $O$ , we proceed via two cutoff functions and define

$$R_\rho(u) := \chi_\rho(|P_\Sigma u|_X) \chi_\rho(|P_{\Sigma'} u|_X) R(u), \quad u \in O, \quad (64)$$

where  $\chi : \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  is a  $C^\infty$  function satisfying

- (i)  $\chi(y) = 1$  for  $y \in [0, 1]$ ,
- (ii)  $\chi(y) \in [0, 1]$  for  $y \in (1, 2)$ ,
- (iii)  $\chi(y) = 0$  for  $y \geq 2$ ,
- (iv)  $|D\chi(y)| \leq 2$  for all  $y \in \mathbb{R}^{\geq 0}$ ,

and  $\chi_\rho(u) := \chi(u/\rho)$ . Note that  $\text{Lip}(\chi_\rho) \leq 2/\rho$  by point (iv). Suppose  $\rho$  is small enough so that the support of the two cutoff functions is contained in  $O$ . Then, we may extend the domain of  $R_\rho$  from  $O$  to the whole of  $X$  by setting  $R_\rho|_{X \setminus O} \equiv 0$  (this modification does not alter the smoothness properties of (64)). Since  $\text{im}(P_\Sigma) = X_\Sigma$  is finite dimensional,  $R_\rho$  preserves the smoothness of  $R$  on the open set

$$S = \{u \in X \mid |P_{\Sigma'} u|_X < \max\{\rho, 2|P_\Sigma u|_X\}\}. \quad (65)$$

Note also that  $\text{supp}(R_\rho) \subset \overline{B_{4\rho}(0)}^X$ .

We have the following result.

**Lemma 28** *Let  $R_\rho : X \rightarrow X^{\odot*}$  be as in (64), extended to the whole of  $X$  as above. Denote by  $\text{Lip}_\delta(R)$  the Lipschitz constant of  $R$  on the (closed) ball  $\overline{B_\delta(0)}^X$  of radius  $\delta$*

centered at the origin, and by  $\text{Lip}(R_\rho)$  the global Lipschitz constant of  $R_\rho : X \rightarrow X^{\odot*}$ . Then, there exist constants  $C, c > 0$  such that  $\text{Lip}(R_\rho) \leq C \text{Lip}_{c\rho}(R) = o(1)$  as  $\rho \rightarrow 0$ .<sup>18</sup>

**Proof** This is a consequence of Lemma IX.4.1, Diekmann et al. (2012), and the properties of  $R$  discussed below (12). For completeness, we replicate the full proof here.

Set  $\xi_\rho(u) := \chi_\rho(|P_\Sigma u|_X) \chi_\rho(|P_{\Sigma^\perp} u|_X)$ , and note that  $\xi_\rho(u) = 0$  for  $|u|_X \geq c\rho$  for some  $c > 0$ ; moreover,  $\xi_\rho$  is globally Lipschitz with  $\text{Lip}(\xi_\rho) = C/\rho$  for some  $C > 0$  (depending on the norm of  $P_\Sigma$ ). For  $u, v \in X$ , we have

$$\begin{aligned} & |\xi_\rho(u)R(u) - \xi_\rho(v)R(v)|_{X^{\odot*}} \\ & \leq |R(u) - R(v)|_{X^{\odot*}} \xi_\rho(v) + |\xi_\rho(u) - \xi_\rho(v)| |R(u)|_{X^{\odot*}} \\ & \leq \begin{cases} \text{Lip}_{c\rho}(R)|u - v|_X + \frac{C}{\rho}|u - v|_X \text{Lip}_{c\rho}(R)c\rho & \text{if } |u|_X, |v|_X \leq c\rho \\ 0 & \text{if } |u|_X, |v|_X \geq c\rho \\ \frac{C}{\rho}|u - v|_X \text{Lip}_{c\rho}(R)c\rho & \text{if } |u|_X \leq c\rho, |v|_X \geq c\rho \end{cases} \\ & \leq \text{Lip}_{c\rho}(R)(cC + 1)|u - v|_X. \end{aligned}$$

Finally, note that  $\text{Lip}_{c\rho}(R) \leq \sup\{\|DR(u)\| \mid u \in \overline{B_{c\rho}(0)}^X\}$ , which goes to 0 as  $\rho \rightarrow 0$  by  $DR(0) = 0$  and continuity of  $u \mapsto DR(u)$ .  $\square$

## 6.2 Global Existence of Forward Solutions For the Cut-Off System

Denote by  $\varphi_t^\rho$  the semiflow produced by replacing  $R$  with  $R_\rho$  in (15), i.e.,

$$\varphi_t^\rho(u) = T(t)u + \iota^{-1} \int_0^t T^{\odot*}(t-s)R_\rho \circ \varphi_s^\rho(u) ds, \quad (t, u) \in \mathcal{D}^{\varphi^\rho}. \quad (66)$$

The statement of Theorem 1.1, Chen et al. (1997), assumes global existence of forward solutions, i.e.,  $\mathcal{D}^{\varphi^\rho} = \mathbb{R}^{\geq 0} \times X$ . This follows from arguments entirely analogous to the ODE case; we include a proof here for completeness.

**Lemma 29** (Global existence for  $\varphi^\rho$ ) Equation (66) defines a maximal semiflow  $\varphi^\rho$  with domain  $\mathcal{D}^{\varphi^\rho} = \mathbb{R}^{\geq 0} \times X$ .

**Proof** It follows directly from Proposition 4 (or Section VII.3, Diekmann et al. 2012) that (66) determines a unique maximal semiflow  $\varphi^\rho$  on some domain  $\mathcal{D}^{\varphi^\rho} \subset \mathbb{R}^{\geq 0} \times X$ . We show global existence. Assume, for contradiction, the existence of  $v \in X$  for which there exists  $t^* \notin \mathcal{D}_v^{\varphi^\rho}$ ,  $t^* \in (0, \infty)$ .

Since  $T(t)$  is strongly continuous on  $X$ , there exists  $M > 0$  and  $\omega > \omega(A)$ ,  $\omega \neq 0$ , (c.f. (18)) for which  $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ . Note that  $T^{\odot*}(t)$  satisfies the same bounds,

<sup>18</sup> A straightforward computation shows that we may take  $c = 4$  and  $C = 2(1 + 2\|P_\Sigma\|)$ .

since it is obtained from  $T(t)$  by taking the dual twice and a restriction inbetween (neither of which increase its norm). Choose  $T > 0$  such that

$$\frac{M}{\omega} (e^{\omega T} - 1) \text{Lip}(R_\rho) < 1. \quad (67)$$

Choose  $m \in \mathbb{N}$  such that  $Tm < t^* < T(m+1)$ , and set  $u = \varphi_{Tm}^\rho(v)$ .<sup>19</sup> The map  $\mathcal{T}_u : C([0, T]; X) \rightarrow C([0, T]; X)$  defined by

$$[\mathcal{T}_u(x)](t) = T(t)u + t^{-1} \int_0^t T^{\odot*}(t-s)R_\rho(x(s))ds, \quad t \in [0, T],$$

has image in  $C([0, T], X)$  (Lemma III.2.1, Diekmann et al. 2012); and moreover is a uniform contraction by (67):

$$\|\mathcal{T}_u(x) - \mathcal{T}_u(y)\|_{C([0, T]; X)} \leq \frac{M}{\omega} (e^{\omega T} - 1) \text{Lip}(R_\rho) \|x - y\|_{C([0, T]; X)},$$

where we have used also that  $\|t^{-1}\|_{\mathcal{L}(\iota(X), X)} = 1$ , with  $\iota(X) \subset X^{\odot*}$  equipped with the subspace topology. Hence  $\mathcal{T}_u$  has a unique fixed point in  $C([0, T]; X)$ , which implies  $T \in \mathcal{D}_u^{\varphi^\rho}$  (by uniqueness of  $\varphi^\rho$ ). This contradicts the maximality of  $\varphi^\rho$ .  $\square$

### 6.3 Global Lipschitz Constant For the Cut-Off Semiflow $\varphi^\rho$

Next, we translate the bounds on the Lipschitz constant of  $R_\rho$  to the nonlinearity of the semiflow  $\varphi^\rho$ . Let us define  $N_\rho : \mathbb{R}^{\geq 0} \times X \rightarrow X$  by

$$N_\rho(t, u) := t^{-1} \int_0^t T^{\odot*}(t-s)R_\rho \circ \varphi_s^\rho(u)ds.$$

**Lemma 30** *Let  $\omega > \omega(A)$ ,  $\omega \neq 0$ , and  $M > 0$  be as in the proof of Lemma 29, and fix  $t \geq 0$ . Then  $\varphi_t^\rho : X \rightarrow X$  is globally Lipschitz for  $\rho > 0$  sufficiently small (depending on  $t$ ), with*

$$\sup_{s \in [0, t]} \text{Lip}(\varphi_s^\rho) \leq 2Me^{t\omega}.$$

Moreover,

$$\text{Lip}(N_\rho(t, \cdot)) \leq \frac{2M^2}{\omega} (e^{\omega t} - 1) e^{\omega t} \text{Lip}(R_\rho). \quad (68)$$

**Proof** The case  $t = 0$  is trivial, so assume  $t > 0$ . Then, for  $u, v \in X$ ,

$$|\varphi_t^\rho(u) - \varphi_t^\rho(v)|_X \leq Me^{\omega t} |u - v|_X + \frac{M}{\omega} (e^{\omega t} - 1) \text{Lip}(R_\rho) \sup_{s \in [0, t]} |\varphi_s^\rho(u) - \varphi_s^\rho(v)|_X.$$

<sup>19</sup> The crux of the argument is that  $T$ , as defined by (67), is independent of the initial condition  $u$  due to the uniformity of the Lipschitz constant.

Taking the supremum over  $t \in [0, T]$  and choosing  $\rho$  small enough such that (c.f. Lemma 28)

$$\text{Lip}(R_\rho) < \frac{\omega}{2M(e^{\omega T} - 1)}, \quad (69)$$

we obtain

$$\sup_{t \in [0, T]} |\varphi_t^\rho(u) - \varphi_t^\rho(v)|_X \leq 2Me^{\omega T} |u - v|_X.$$

Moreover,

$$|N_\rho(t, u) - N_\rho(t, v)|_X \leq \frac{M}{\omega} (e^{\omega t} - 1) \text{Lip}(R_\rho) 2Me^{\omega t} |u - v|_X.$$

□

#### 6.4 Theorem 1.1 of Chen et al. (1997)

In this section, we quote Theorem 1.1 of Chen et al. (1997), and verify its assumptions. Theorem 12 is simply deduced as a corollary of this result.

**Theorem 31** (Theorem 1.1, Chen et al. 1997) *Assume (A.1) and (A.3). Let  $\varphi_t^\rho$  denote the cut-off semiflow (66). Suppose  $\gamma_1$  and  $\gamma_2$  are real numbers such that  $\beta < \gamma_2 < \gamma_1 < \alpha$ . For  $\rho > 0$  sufficiently small, there is a globally Lipschitz map  $\phi_\rho : X_\Sigma \rightarrow X_{\Sigma'}$  with  $\phi_\rho(0) = 0$ , such that the submanifold*

$$W_\rho^\Sigma := \text{graph}(\phi_\rho) = \{v_1 + \phi_\rho(v_1) \mid v_1 \in X_\Sigma\} \quad (70)$$

of  $X$  satisfies the following properties:

- (i) *The restriction of the semiflow  $\varphi_t^\rho$  to  $W_\rho^\Sigma$  leaves the submanifold invariant and can be uniquely extended to a Lipschitz flow  $\{\varphi_t^\rho|_{W_\rho^\Sigma}\}_{t \in \mathbb{R}}$  defined for all times. If  $u \in W_\rho^\Sigma$ , then the negative semiorbit  $\{\varphi_t^\rho(u)\}_{t \leq 0}$  satisfies*

$$\limsup_{t \rightarrow -\infty} \frac{1}{|t|} \ln |\varphi_t^\rho(u)|_X \leq -\gamma_1 \quad (71)$$

*Conversely, if  $\{w(t)\}_{t \leq 0} \subset X$  is a negative semiorbit of  $\varphi^\rho$ <sup>20</sup> that satisfies*

$$\limsup_{t \rightarrow -\infty} \frac{1}{|t|} \ln |w(t)|_X \leq -\gamma_2, \quad (72)$$

*then  $\{w(t)\}_{t \leq 0} \subset W_\rho^\Sigma$ .*

<sup>20</sup> A function  $w : \mathbb{R}^{\leq 0} \rightarrow X$  is called a negative semiorbit of  $\varphi^\rho$  if  $\varphi_t^\rho(w(s)) = w(t+s)$  for any  $t \geq 0$  and  $s \leq -t$ .

- (ii) There exists a continuous map  $\psi_\rho : X \times X_{\Sigma'} \rightarrow X_\Sigma$  such that for each  $u \in W_\rho^\Sigma$ ,  $\psi_\rho(u, P_{\Sigma'}u) = P_\Sigma u$  and the manifold  $M_\rho^{\Sigma'}(u) = \{ \psi_\rho(u, v_2) + v_2 \mid v_2 \in X_{\Sigma'} \}$  passing through  $u$  satisfies

$$\varphi_t^\rho(M_\rho^{\Sigma'}(u)) \subset M_\rho^{\Sigma'}(\varphi_t^\rho(u)), \quad t \geq 0, \quad (73a)$$

$$M_\rho^{\Sigma'}(u) = \left\{ v \in X \mid \limsup_{t \rightarrow \infty} \frac{1}{t} \ln |\varphi_t^\rho(u) - \varphi_t^\rho(v)|_X \leq \gamma_2 \right\}. \quad (73b)$$

Moreover,  $\psi_\rho : X \times X_{\Sigma'} \rightarrow X_\Sigma$  is uniformly Lipschitz with respect to its second argument. For every  $u \in X$ ,  $M_\rho^{\Sigma'}(u) \cap W_\rho^\Sigma$  is a single point. In particular,

$$M_\rho^{\Sigma'}(u) \cap M_\rho^{\Sigma'}(v) = \emptyset \text{ for } u, v \in W_\rho^\Sigma, u \neq v; \quad \bigcup_{u \in W_\rho^\Sigma} M_\rho^{\Sigma'}(u) = X;$$

that is,  $\{M_\rho^{\Sigma'}(u)\}_{u \in W_\rho^\Sigma}$  form a foliation of  $X$  over  $W_\rho^\Sigma$ .

- (iii) The Lipschitz constants  $\text{Lip}(\phi_\rho)$  and  $\text{Lip}(\psi_\rho(u, \cdot))$  tend to 0 as  $\rho \rightarrow 0$ , the latter independently of  $u \in X$ .

**Proof** We verify hypotheses (H.1)–(H.4) of Chen et al. (1997) for the case of  $\varphi_t^\rho$ , which we recall below for ease of use.

- (H.1)  $\varphi^\rho$  is continuous as a map  $\mathbb{R}^{\geq 0} \times X \rightarrow X$  and there exists a constant  $t > 0$  such that

$$\sup_{s \in [0, t]} \text{Lip}(\varphi_s^\rho) < \infty.$$

- (H.2) The map  $\varphi_t^\rho$ , with  $t$  as in (H.1), can be decomposed as  $\varphi_t^\rho = L + N$  where  $L \in \mathcal{L}(X)$  and  $N : X \rightarrow X$  is globally Lipschitz.

- (H.3) There are subspaces  $X_i \subset X$ ,  $i = 1, 2$  and continuous projections  $P_i : X \rightarrow X_i$  such that  $P_1 + P_2 = \text{id}_X$ ,  $X = X_1 \oplus X_2$ ,  $L$  leaves  $X_i$  invariant and  $L$  commutes with  $P_i$ ,  $i = 1, 2$ . Denoting by  $L_i : X_i \rightarrow X_i$  the restrictions of  $L$ ,  $L_1$  has bounded inverse and there exist constants  $\alpha_1 > \alpha_2 \geq 0$ ,  $C_1 \geq 1$ , and  $C_2 \geq 1$  such that

$$\|L_1^{-j} P_1\|_{\mathcal{L}(X)} \leq C_1 \alpha_1^{-j}, \quad j \in \mathbb{N}_0, \quad (74a)$$

$$\|L_2^j P_2\|_{\mathcal{L}(X)} \leq C_2 \alpha_2^j, \quad j \in \mathbb{N}_0. \quad (74b)$$

- (H.4) Let

$$\varrho(s) := \frac{C_1}{\alpha_1 - s} + \frac{C_2}{s - \alpha_2}, \quad s \in (\alpha_2, \alpha_1). \quad (75)$$

The numbers  $\gamma_1, \gamma_2$  from the statement satisfy  $\alpha_2 < e^{\gamma_2 t} < e^{\gamma_1 t} < \alpha_1$  and  $\varrho(s) \text{Lip}(N) < 1$  for all  $s \in (e^{\gamma_2 t}, e^{\gamma_1 t})$ .

Hypotheses (H.1) and (H.2) follow from Lemma 30, with  $t = 1$ ,  $N = N_\rho(1, \cdot)$  and  $L = T(1)$ . The dichotomy in (H.3) is supplied by Lemma 8, with  $X_1 = X_\Sigma$  and

$X_2 = X_{\Sigma'}$ . In particular, (20) implies (74) for any  $\alpha_1 < e^\alpha$  and  $\alpha_2 > e^\beta$  – we choose these so that  $\alpha_1 > e^{\gamma_1}$  and  $\alpha_2 < e^{\gamma_2}$ . The condition (H.4) can be achieved through control of  $\text{Lip}(R_\rho)$  via (68) and Lemma 28, taking  $\rho$  sufficiently small.

Statement 31(iii) is not part of the main statement in Chen et al. (1997), but is a direct consequence of equations (3.16) and (3.18) therein (see also (107)).  $\square$

**Remark 32** As remarked also in Chen et al. (1997), if the operators  $L$  and  $N$  satisfy

$$\frac{(\sqrt{C_1} + \sqrt{C_2})^2}{\alpha_1 - \alpha_2} \text{Lip}(N) < 1, \quad (76)$$

then one can find  $\gamma_1, \gamma_2$  satisfying (H.4). (Equation (76) is obtained simply by evaluating  $\varrho$  at its minimum within  $(\alpha_2, \alpha_1)$ .)

We may now deduce Theorem 12 as a corollary of Theorem 31.

**Proof of Theorem 12** Suppose  $\rho > 0$  is small enough so that Theorem 12 applies and such that  $\text{Lip}(\phi_\rho) < 1$  and  $\sup_{u \in X} \text{Lip}(\psi_\rho(u, \cdot)) < 1$ . Then, an application of the parametric Banach fixed point theorem (see, e.g., Theorem 21, Irwin 1972) to the map

$$\begin{aligned} g : X \times X_\Sigma &\rightarrow X_\Sigma \\ (u, v_1) &\mapsto \psi_\rho(u, \phi_\rho(v_1)) \end{aligned}$$

shows that, for each  $u \in X$ , the map  $g(u, \cdot) : X_\Sigma \rightarrow X_\Sigma$  has a unique fixed point. Moreover, if we denote this fixed point by  $f(u)$ , then the map  $f : X \rightarrow X_\Sigma$  is continuous. Now, defining  $\pi_\rho : X \rightarrow W_\rho^{\Sigma_1}$  as

$$\pi_\rho(u) := f(u) + \phi_\rho \circ f(u)$$

yields a continuous map, which satisfies  $M_\rho^{\Sigma'}(u) = \pi_\rho^{-1}(\pi_\rho(u))$  for each  $u \in X$ . Hence, (73a) is equivalent to

$$\pi_\rho \circ \varphi_t^\rho(u) = \varphi_t^\rho|_{W_\rho^\Sigma} \circ \pi_\rho(u), \quad \text{for all } (t, u) \in \mathbb{R}^{\geq 0} \times X. \quad (77)$$

It follows from (73b), with  $u \in X$  fixed, that for any  $\varepsilon > 0$ , there exists  $C > 0$  such that

$$|\varphi_t^\rho(u) - \varphi_t^\rho \circ \pi_\rho(u)|_X \leq C e^{(\gamma_2 + \varepsilon)t}, \quad \text{for all } t \geq 0. \quad (78)$$

Here, we have the freedom to choose  $\gamma_2 < \gamma$ , since  $\gamma_2 \in (\beta, \alpha)$  was arbitrary in Theorem 31 and  $\beta < \gamma$ . Note also that  $\pi_\rho|_{W_\rho^\Sigma} = \text{id}_{W_\rho^\Sigma}$ .

Take an open neighbourhood  $V$  of 0 such that

$$V \subset \{u \in X \mid |P_\Sigma u| < \rho, |P_{\Sigma'} u| < \rho\}.$$

Then  $\varphi \equiv \varphi^\rho$  on  $\mathcal{U}(V)$ . Set  $U := (\pi_\rho^{-1}(W_\rho^\Sigma \cap V)) \cap V$ . This is an open neighbourhood of 0.

We may now declare  $W^\Sigma := W_\rho^\Sigma \cap U (= W_\rho^\Sigma \cap V)$  and  $\pi := \pi_\rho|_U$ . Statement 12(i) is immediate from the fact that  $\varphi \equiv \varphi^\rho$  on  $\mathcal{U}(U)$ . Statement 12(iii) follows from the choice of  $U$  and (73). More precisely, by construction of  $U$ , we have  $\pi(U) \subset W^\Sigma \subset U$ . Given  $(t, u) \in \mathcal{U}(U)$ , (77) combined with the above observation implies that  $(t, \pi(u)) \in \mathcal{U}(U)$  and  $\pi \circ \varphi_t(u) = \varphi_t|_{W^\Sigma} \circ \pi(u)$ . This shows (28). If  $\mathcal{U}_u(U) \neq \mathbb{R}^{\geq 0}$ , (29) is trivially satisfied. Otherwise, (28) holds for all times on a domain on which  $\pi_\rho = \pi$  and  $\varphi^\rho = \varphi$ ; hence (78) implies (29), being that  $\gamma_2 < \gamma$ .

The first part of statement 12(iv) was covered in Remark 15. The second part follows from the argument coined by Burchard et al. (1992) for the case of center manifolds. It is written out in the specific context of the present result in Buza (2024), Lemma 4.8 (note that its proof is independent of the equation at hand, and relies only on the existence of the foliation map  $\pi$ ).

For the proof of statement 12(ii) and the  $C^1$  part of 12(i), see Appendix C. Once a certain degree of smoothness, say  $C^\ell$ ,  $\ell \leq k$ , of the manifold  $W^\Sigma$  has been obtained, Corollary (8A.9) of Marsden and McCracken (1976) immediately implies that the restricted semiflow extends to a jointly  $C^\ell$  flow. The tangency of the manifold  $W^\Sigma$  to  $X_\Sigma$  at 0 is obtained in Corollary 49.  $\square$

## 7 Proof of Theorem 16

We start by reducing the problem to the case when 0 is a (spectrally) stable fixed point, by restricting the semiflow  $\varphi$  to the stable manifold  $W^s$ . Explicitly, we consider the setting of Appendix C (or that of Chapter VIII, Diekmann et al. 2012). Choose  $\sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A; X), \operatorname{Re} \lambda < 0\} < \gamma < 0$ . The stable manifold can be characterized via a  $C^k$  map  $\mathcal{W}^s : U_s \rightarrow BC^{-\gamma}(\mathbb{R}^{\geq 0}; X)$ <sup>21</sup>, which maps points of an open neighbourhood  $U_s$  of 0 in  $X_s$  to forward orbits on the submanifold  $W^s$ . Here  $X_s$  denotes the stable subspace characterized by the half-plane of the spectrum strictly to the left of the imaginary axis; let us also denote by  $\tilde{P}_s : X \rightarrow X_s$  the associated projection with restricted range. We define

$$\psi_t := \tilde{P}_s \circ \operatorname{ev}_t \circ \mathcal{W}^s, \quad t \geq 0, \quad (79)$$

on  $U_s$ , where  $\operatorname{ev}_t : BC^{-\gamma}(\mathbb{R}^{\geq 0}; X) \rightarrow X$  is the continuous linear map given by  $f \mapsto f(t)$ .

**Lemma 33** *The maps  $\{\psi_t\}_{t \geq 0}$  from (79) constitute a semiflow with domain  $\mathcal{D}^\psi \subset \mathbb{R}^{\geq 0} \times U_s$ , which satisfies the same smoothness properties as the original semiflow  $\varphi$ .*

**Proof** Denoting by  $K : U_s \rightarrow X$  the  $C^k$  map  $\operatorname{ev}_0 \circ \mathcal{W}^s$  (whose image is  $W^s$ ), we have

$$\psi_t = \tilde{P}_s \circ \varphi_t \circ K, \quad t \geq 0. \quad (80)$$

Therefore, smoothness properties are inherited.

<sup>21</sup> The Banach space  $BC^{-\gamma}(\mathbb{R}^{\geq 0}; X)$  consists of continuous functions  $\mathbb{R}^{\geq 0} \rightarrow X$  for which the norm given by  $\|f\|_{-\gamma} = \sup_{t \geq 0} e^{-\gamma t} \|f(t)\|_X$  is finite. This is analogous to Definition 42.  $\gamma$  being negative means that forward orbits on  $W^s$  decay exponentially, at least as fast as  $e^{\gamma t}$ .

Invariance of  $W^s$  implies that  $\varphi_t \circ K(u) \in \text{im}(K)$  for all  $t \geq 0$  such that  $\psi_\tau(u) \in U_s$  for  $\tau \in [0, t]$ . By construction, we have  $K \circ \tilde{P}_s|_{\text{im}(K)} = \text{id}_{\text{im}(K)}$  and  $\tilde{P}_s \circ K = \text{id}_{U_s}$  – for a detailed derivation see Appendix C, specifically the paragraph above (106). The semiflow property thus follows from (80) on the domain of  $\psi$ , which can be identified as

$$\mathcal{D}^\psi = \{(t, u) \in \mathbb{R}^{\geq 0} \times U_s \mid \psi_\tau(u) \in U_s \text{ for all } \tau \in [0, t]\}. \quad (81)$$

□

**Lemma 34** *The fixed point 0 of  $\psi_t$  is Lyapunov and exponentially stable (with respect to the induced norm  $|\cdot|_{X_s}$ ). Moreover, there exists an open neighbourhood of the origin  $U \subset U_s$  such that  $\mathbb{R}^{\geq 0} \times U \subset \mathcal{D}^\psi$  and  $\psi_t \in C_b^k(U)$  for all  $t \geq 0$ .*

**Proof** Choose  $\varepsilon > 0$  such that  $B_\varepsilon^{X_s}(0) \subset U_s$ . Since  $\mathcal{W}^s : U_s \rightarrow BC^{-\gamma}(\mathbb{R}^{\geq 0}; X)$  is continuous and  $\mathcal{W}^s(0) = 0$ , we may choose a small enough neighbourhood  $U$  of 0 in  $U_s$  such that

$$\|P_s\|_{\mathcal{L}(X)} \sup_{t \geq 0} e^{-\gamma t} |\text{ev}_t \circ \mathcal{W}^s(u)|_X < \varepsilon, \quad u \in U.$$

Since  $\gamma < 0$ , this implies Lyapunov and exponential stability of 0 under  $\psi_t$ ; and moreover that

$$\|P_s\|_{\mathcal{L}(X)} |\text{ev}_t \circ \mathcal{W}^s(u)|_X < \varepsilon \quad \text{for } (t, u) \in \mathbb{R}^{\geq 0} \times U.$$

In particular,  $\mathbb{R}^{\geq 0} \times U \subset \mathcal{D}^\psi$ , c.f. (81), and  $\psi_t \in C_b^0(U)$  for all  $t \geq 0$ .

Since  $\mathcal{W}^s$  is of class  $C^k$ , we may shrink  $U$  such that<sup>22</sup>

$$\left\| D^j \mathcal{W}^s(u) - D^j \mathcal{W}^s(0) \right\|_{M_j(X_s; BC^{-\gamma}(\mathbb{R}^{\geq 0}; X))} < \varepsilon, \quad \text{for all } u \in U, \ 1 \leq j \leq k.$$

We have

$$D^j \psi_t(u) = \text{ev}_t \left( \tilde{P}_s D^j \mathcal{W}^s(u) \right).$$

Hence  $\psi_t \in C_b^k(U)$  follows for all  $t \geq 0$ , noting that  $\|\text{ev}_t\| \leq 1$  due to  $\gamma < 0$ . □

**Lemma 35** *For each  $0 \leq j \leq k$ ,  $(t, u) \mapsto D^j(\psi_t \circ \iota_{\tilde{\Sigma}})(u)$  is jointly continuous over the line  $\mathbb{R}^{\geq 0} \times \{0\}$ , where  $\iota_{\tilde{\Sigma}} : U_s \cap X_{\tilde{\Sigma}} \hookrightarrow U_s$  is the inclusion.*

**Proof** Apply Remark 40 with  $W = K(X_{\tilde{\Sigma}} \cap U_s)$ ; noting that  $K \circ \iota_{\tilde{\Sigma}} = \widetilde{\iota_W \circ K \circ \iota_{\tilde{\Sigma}}}$ , where  $\widetilde{K \circ \iota_{\tilde{\Sigma}}}$  is considered with range restricted to  $W$  (it retains the smoothness of  $K$  by Theorem 18.2, Munkres 2000). □

We are now ready to prove the main result.

**Proof of Theorem 16** Let

$$\Omega := \{\lambda \in \sigma(A; X) \mid \text{Re } \lambda \geq \inf \text{Re } \Sigma\}.$$

<sup>22</sup> For the introduction of  $M_j$  spaces of multilinear maps, see Appendix C.

We may apply Theorem 12 with  $\Omega$  as the spectral subset to obtain a locally invariant  $C^\ell$  submanifold  $W^\Omega$  of  $X$  (contained in a small enough neighbourhood of the origin). The inclusion  $W^\Omega \hookrightarrow X$  is transversal over  $W^s$  (locally, about 0) in the sense of Lang (2012), page 29. We hence have, up to perhaps shrinking the manifolds, that  $W^{\Omega,s} := W^\Omega \pitchfork W^s$  is a locally invariant  $C^\ell$  submanifold of  $X$ . It is tangent to  $X_\Omega \cap X_s$  at 0. If  $\tilde{\Sigma}$  is empty, we are done – hence we suppose otherwise in what follows.

Next, we apply Theorem 2.6, Buza (2025), to the semiflow  $\psi$  with respect to the splitting  $X_s = X_{\tilde{\Sigma}} \oplus X_{\tilde{\Sigma}'}.$  We verify assumptions (A.1) and (A.2) of Buza (2025). The contents of (A.1) were verified by Lemmas 33 and 34. The splitting for (A.2) therein is provided above,  $X_{\tilde{\Sigma}}$  taking the role of  $X_0$ . That  $t \mapsto D\psi_t(0)$  is strongly continuous is a consequence of the tangency of  $W^s = \text{im}(K)$  to  $X_s$  at 0:

$$\begin{aligned} D\psi_t(0) &= \tilde{P}_s D\varphi_t(0) DK(0) \\ &= T(t)|_{X_s} \tilde{P}_s DK(0) \\ &= T(t)|_{X_s}. \end{aligned}$$

This leaves  $X_{\tilde{\Sigma}'}$  invariant; its generator,  $A|_{X_s}$ , satisfies  $X_{\tilde{\Sigma}} \subset \text{dom}(A|_{X_s})$ . The joint continuity assumption was verified by Lemma 35. The remainder of (A.2) therein concerns spectral assumptions. Noting that spectral mapping holds for the semigroup  $T(t)|_{X_s}$  (Theorem 5) and that the spectrum of  $A|_{X_s}$  is confined to the region  $|z| \leq C_1 e^{-h\text{Re } z}$  for large enough  $|z|$  (Theorem I.4.1, Diekmann et al. 2012), we may verify the spectral assumptions directly on the generator, as in Remark 2.8, Buza (2025) (simply replace its sectoriality assumption with the above). (A.2) therein is hence fulfilled by (A.4) herein.

This yields a  $C^k$  submersion  $\pi : U \rightarrow X_{\tilde{\Sigma}}$  on some small enough neighbourhood  $U$  of the origin in  $X_s$  that satisfies  $\pi(0) = 0$ ,

$$D\pi(0)|_{X_{\tilde{\Sigma}}} \in \text{Aut}(X_{\tilde{\Sigma}}) \quad (82)$$

and

$$\pi \circ \psi_t = \vartheta_t \circ \pi \quad \text{for all } t \geq 0 \quad (83)$$

on some open  $V \subset U$  containing the origin; where  $\{\vartheta_t\}_{t \geq 0}$  is a semiflow on  $X_{\tilde{\Sigma}}$ , with a fixed point at 0. Up to perhaps confining  $W^{\Omega,s}$  to a smaller neighbourhood of the origin, we may assume  $\tilde{P}_s(W^{\Omega,s}) \subset V$ . Precomposing (83) with  $\tilde{P}_s|_{W^{\Omega,s}}$ , we obtain (c.f. the proof of Lemma 33)

$$\pi \circ \tilde{P}_s \circ \varphi_t|_{W^{\Omega,s}} = \vartheta_t \circ \pi \circ \tilde{P}_s|_{W^{\Omega,s}}, \quad t \geq 0.$$

By Lyapunov stability of 0 with respect to  $\varphi_t|_{W^{\Omega,s}}$  and local invariance of  $W^{\Omega,s}$ , we may choose a neighbourhood  $\tilde{V} \subset W^{\Omega,s}$  of the origin such that  $\varphi_t(\tilde{V}) \subset W^{\Omega,s}$  for all  $t \geq 0$ . We hence arrive at

$$\pi \circ \tilde{P}_s|_{W^{\Omega,s}} \circ \varphi_t = \vartheta_t \circ \pi \circ \tilde{P}_s|_{W^{\Omega,s}}, \quad t \geq 0, \quad (84)$$

which holds on  $\tilde{V}$ .

Due to the restriction map,  $\pi \circ \tilde{P}_s|_{\tilde{V}}$  is only of class  $C^\ell$ . We check that it is a submersion at 0. We have

$$D(\pi \circ \tilde{P}_s|_{\tilde{V}})(0) = D\pi(0)\tilde{P}_s D\iota_{W^{\Omega,s}}(0), \quad (85)$$

where  $\iota_{W^{\Omega,s}} : W^{\Omega,s} \hookrightarrow X$  is the inclusion. Noting the tangency of  $W^{\Omega,s}$  to  $X_\Omega \cap X_s$  at 0, the map (85) is

$$X_\Omega \cap X_s \hookrightarrow X_s \xrightarrow{D\pi(0)} X_{\tilde{\Sigma}}. \quad (86)$$

Considering that  $X_{\tilde{\Sigma}} \subset X_\Omega \cap X_s$ , (82) indeed confirms that  $\pi \circ \tilde{P}_s|_{\tilde{V}}$  is a  $C^\ell$  submersion on a neighbourhood of 0. The kernel of (86) can be identified as  $X_{\tilde{\Sigma}}$ .

Hence,  $(\pi \circ \tilde{P}_s|_{\tilde{V}})^{-1}(0)$  is a locally invariant (by (84))  $C^\ell$  submanifold of  $\tilde{V}$ , and hence of  $X$ , tangent to  $X_{\tilde{\Sigma}}$  at 0. The joint smoothness of the restricted semiflow follows from the same argument as in the proof of Theorem 12.  $\square$

## 8 Proof of Theorem 20 and Corollary 24

The proof of Theorem 20 proceeds analogously to the proof of Theorem 12 in Section 6. We simply verify the assumptions of Theorem 1.1, Chen et al. (1997), now for the full, unaltered semiflow.

**Proof of Theorem 20** Under the global Lipschitz assumption, the solutions (15) exist for all times (just as in Lemma 29), hence the domain of  $\varphi$  is  $\mathcal{D}^\varphi = \mathbb{R}^{\geq 0} \times X$ . Repeating the proof of Lemma 30 with  $N : \mathbb{R}^{\geq 0} \times X \rightarrow X$ ,

$$N(t, u) := t^{-1} \int_0^t T^{\odot*}(t-s)R \circ \varphi_s(u) ds,$$

we obtain

$$\text{Lip}(N(1/\nu, \cdot)) \leq \frac{2M^2}{\omega} (e^{\omega/\nu} - 1) e^{\omega/\nu} \text{Lip}(R), \quad (87)$$

having used (31) in place of (69).

The assumptions (H.1)-(H.3) are now easily verified, just as in the proof of Theorem 31. In place of (H.4), we shall verify (76) of Remark 32. This follows from (32) and the condition  $\nu < (\alpha - \beta - \varepsilon_1 - \varepsilon_2)/\ln 2$ . Indeed, the latter implies

$$\frac{1}{2} e^{(\alpha-\varepsilon_1)/\nu} < e^{(\alpha-\varepsilon_1)/\nu} - e^{(\beta+\varepsilon_2)/\nu},$$

whereas (32) is simply

$$(\sqrt{K_1} + \sqrt{K_2})^2 \text{Lip}(N(1/\nu, \cdot)) < \frac{1}{2} e^{(\alpha-\varepsilon_1)/\nu}.$$

This shows (76); and hence all conclusions of Theorem 20, by an application of Theorem 31 to the unaltered semiflow (i.e., without the  $\rho$  sub/superscripts). Since

Theorem 31 provides a rate of attraction strictly less than  $\alpha \leq 0$ , we must have that the resulting manifold is exponentially attracting.  $\square$

**Proof of Corollary 24** We shall produce a sufficiently large spectral gap contained in the region  $\operatorname{Re} z < 0$ . In this region, from (34), we have

$$|\det(\Delta(z))| \geq |z|^n \left( 1 - \sum_{j=1}^n \left( \frac{e^{-h\operatorname{Re} z}}{|z|} \right)^j \prod_{i=1}^j \operatorname{TV}(\eta_{ji}) \right).$$

We apply Young's inequality to each term in the summation. Explicitly, for  $j = 1, \dots, n-1$ ,

$$\left( \frac{e^{-h\operatorname{Re} z}}{|z|} \right)^j \prod_{i=1}^j \operatorname{TV}(\eta_{ji}) \leq \frac{j}{n} (\varepsilon_j)^{-\frac{n}{j}} \left( \frac{e^{-h\operatorname{Re} z}}{|z|} \right)^n + \frac{n-j}{n} \left( \varepsilon_j \prod_{i=1}^j \operatorname{TV}(\eta_{ji}) \right)^{\frac{n}{n-j}},$$

for some  $\varepsilon_j > 0$ . We choose  $\varepsilon_j$  such that

$$\frac{n-j}{n} \left( \varepsilon_j \prod_{i=1}^j \operatorname{TV}(\eta_{ji}) \right)^{\frac{n}{n-j}} = \frac{1}{2n}.$$

Therefore, we arrive at

$$|\det(\Delta(z))| \geq |z|^n \left( \frac{1}{2} - \left( \frac{e^{-h\operatorname{Re} z}}{|z|} \right)^n Q \right),$$

with  $Q$  as in (35). Hence, if

$$h < \frac{1}{-\operatorname{Re} z} \ln \left( (2Q)^{-\frac{1}{n}} |z| \right), \quad (88)$$

then  $|\det(\Delta(z))| > 0$ .

The condition (36) on  $h$  implies that (88) holds on the vertical strip

$$\{z \in \mathbb{C} \mid \operatorname{Re} z \in [-r, \gamma]\}.$$

This shows the existence of a spectral gap of size  $r + \gamma$ , containing  $\gamma$ . We hence must have  $\alpha - \beta > r + \gamma$ . The existence of  $\nu > 0$  satisfying the assumptions of Theorem 20 now follows from the choice of  $r$  in (37) and Remark 23. The rate of attraction claim follows from the fact that we could have equivalently defined  $\gamma = -r$  (since both choices are within the same spectral gap).  $\square$

## 9 Conclusions

In this paper, we have derived existence and regularity results for spectral submanifolds (SSMs) in time delay systems. The main purpose in doing so was to provide rigorous mathematical support to existing works on the subject (Szaksz et al. 2024, 2025) and those to come. The statements of all main results are gathered in Section 4. Fleshing out the theoretical aspects is perhaps more practically pertinent here than in the case of other infinite-dimensional dynamical systems (e.g., PDEs), since the embedding defining the invariant manifold can be explicitly constructed in a way that preserves the infinite-dimensional nature of the problem (Szaksz et al. 2025). This observation provided the main theme of examples explored herein. In particular, the procedure described in the examples (Section 5) is approximate only in the sense of truncating the Taylor expansion of the embedding, no other form of discretization was performed.

Explicit formulas for the Taylor coefficients of the embedding and the reduced dynamics in the graph style of parameterization are given in Lemma 26. These formulas are completely rigorous under spectral gap assumptions ensuring the necessary smoothness of the manifold, but the coefficients themselves could, in practice, be computed under less strict non-resonance conditions. We demonstrated, through the course of Examples 5.1 and 5.2, how these formulas are utilized in explicit calculations. Moreover, based on the theory of Section 4.1, we derived conditions on the nonlinearity (Example 5.1) or delay (Example 5.2) under which the SSM extends to a globally defined inertial manifold, which is then uniquely characterized by Lyapunov exponents of its trajectories. In the final example (Section 5.2.1), we demonstrated how the existence (but no longer uniqueness) of inertial manifolds may be concluded if the nonlinearity is not globally Lipschitz but the existence and size of the global attractor is known.

The examples highlight that the dimension of an inertial manifold need not be equal to the dimension of the physical configuration of the system, which is perhaps a novelty that has not been explicitly pointed out before (to the authors' knowledge).

The conditions for the existence of SSMs locally about a fixed point are quite permissive. Indeed,  $C^1$  pseudo-unstable manifolds exist whenever  $f$  (from (1)) is of class  $C^1$ , without any additional assumptions. Even the non-resonance conditions for Theorem 16, spelled out in (A.4), hold generically. In contrast, global theory, i.e., the existence of inertial manifolds, hinges on stringent conditions which may be difficult to verify in practice. Even in the academic example of Section 5.2.1, the permissible size of the attractor is rather small. Of course, if one has full control over the delay or the nonlinearity, the conditions can be easily verified, as already observed in Driver (1968).

The local theory described herein should extend to the case when the fixed point is replaced by a periodic orbit in a straightforward fashion, by applying similar invariant manifold results to the Poincaré map. More generally, neighborhoods of normally hyperbolic invariant manifolds could also be considered, but this necessarily hinges on a uniform spectral gap condition (in the Sacker-Sell sense), which would be difficult to verify in practice. Perhaps a more interesting question is whether the picture described in the work of Haller et al. (2023) holds near a fixed point of a DDE. Namely, if there exists a family of invariant manifolds of fractional smoothness tangent to the slow

subspace that tessellate the complement of the strong stable manifold in phase space. The practical significance of this question is intrinsically tied to data-driven techniques, whereas in the theoretical realm, it poses an interesting problem due to the failure of Sternberg-type linearization results in the DDE setting, which the original proof in Haller et al. (2023) relies on.

## Appendix A Preliminaries

In this appendix, we give a brief overview of some basic concepts appearing in the work that might be unfamiliar to some readers. Section A.1 is largely based on Appendix I of Diekmann et al. (2012); the definitions in Section A.2 were taken from Bühler and Salamon (2018) and Marsden and McCracken (1976); whereas Section A.3 follows III.1 and A.II.3.13 of Diekmann et al. (2012).

### A.1 Functions of Bounded Variation

A function  $\eta : [a, b] \rightarrow \mathbb{R}$  is said to be of *bounded variation*,  $\eta \in \text{BV}([a, b]; \mathbb{R})$ , if its total variation,

$$\text{TV}(\eta) = \sup_{P \in \mathcal{P}(a,b)} \sum_{j=1}^{m_P} |\eta(\sigma_j) - \eta(\sigma_{j-1})|, \quad (89)$$

is bounded; here the supremum is taken over all partitions  $P = \{\sigma_0, \dots, \sigma_{m_P}\} \in \mathcal{P}(a, b)$  of  $[a, b]$  with  $a = \sigma_0 < \sigma_1 < \dots < \sigma_{m_P} = b$ . The width of a partition is

$$w(P) = \max_{1 \leq j \leq m_P} (\sigma_j - \sigma_{j-1}).$$

A function  $\eta \in \text{BV}([a, b]; \mathbb{R})$  is called *normalized* if  $\eta(a) = 0$  and  $\eta$  is continuous from the right on  $(a, b)$ . We write  $\eta \in \text{NBV}([a, b]; \mathbb{R})$  to express that  $\eta$  is of normalized bounded variation. These definitions extend to the complex- and multidimensional case in the obvious fashion (through real/complex parts and component-wise).

Let  $\zeta : [a, b] \rightarrow \mathbb{C}^{n \times n}$  and  $u : [a, b] \rightarrow \mathbb{C}^n$  be two given maps. Then for any partition  $P \in \mathcal{P}(a, b)$  and any choice of  $\tau_j \in [\sigma_{j-1}, \sigma_j]$  we denote by

$$S(P, \zeta, u) := \sum_{j=1}^{m_P} [\zeta(\sigma_j) - \zeta(\sigma_{j-1})] u(\tau_j).$$

If there exists  $c \in \mathbb{C}^n$  such that for all  $\varepsilon > 0$  there exists  $\delta > 0$  for which  $|c - S(P, \zeta, u)| < \varepsilon$  holds for all  $P \in \mathcal{P}(a, b)$  with  $w(P) < \delta$  and all choices of  $\tau_j$ , we say that  $u$  is Riemann-Stieltjes integrable with respect to  $\zeta$  over  $[a, b]$  and write

$$c = \int_a^b d\zeta(t) u(t).$$

The (original form of the) Riesz representation theorem (Riesz 1909) asserts that any continuous linear functional on  $C([a, b]; \mathbb{C})$  is represented by a unique element of  $\text{NBV}([a, b]; \mathbb{C})$ . This is used in establishing (21) and (6). For the reader more familiar with the modern version of Riesz's theorem (e.g., Theorem 3.15, Salamon 2016), we remark that this is equivalent to the statement that the dual space of  $\mathbb{C}$ -valued continuous functions on  $[a, b]$  is (isometrically) isomorphic to the space of finite complex Borel measures on  $[a, b]$ ; with the equivalence given by setting  $\zeta : t \mapsto \mu([a, t]) \in \text{NBV}$  for  $\mu$  a Borel measure.

A function  $\eta \in \text{NBV}([a, b]; \mathbb{C})$  belongs to  $\text{SBV}([a, b]; \mathbb{C})$  if there exists  $t \in (0, b]$  such that  $\lim_{s \searrow t} \eta(s) \neq \eta(t)$  and  $\eta(s) = \eta(t)$  for all  $s \in [t, b]$ . Given a kernel  $\zeta \in \text{NBV}([a, b]; \mathbb{C}^{n \times n})$  determining  $\Delta$  through (22), we say  $\zeta \in \text{SBV}$  if the characteristic function, which can be written as (c.f. Section V.8, Diekmann et al. 2012)

$$\det \Delta(z) = z^n + \sum_{j=1}^n \int_0^{\tau_j} e^{-z\theta} d\eta_j(\theta) z^{n-j}$$

for some  $\tau_j \leq jh$  and  $\eta_j \in \text{NBV}([0, \tau_j]; \mathbb{C})$ , satisfies in addition the requirement  $\eta_j \in \text{SBV}([0, \tau_j]; \mathbb{C})$ . This is a fairly general concept; e.g.,  $\zeta \in \text{SBV}$  if it is purely atomic (that is, (1a) consists of discrete delays only), or if  $\det_* \zeta \in \text{SBV}$ , where  $\det_*$  denotes the determinant with respect to the convolution product (Definition V.5.2, Diekmann et al. 2012).

## A.2 Semiflows and Semigroups

Let  $M$  be a Banach manifold. A family of maps  $\{\varphi_t\}_{t \geq 0}$ ,  $\varphi_t : M \rightarrow M$ , is called a *semiflow* on  $M$  if

$$\varphi_0 = \text{id}_M, \quad (90a)$$

$$\varphi_t \circ \varphi_s = \varphi_{t+s}, \quad \text{for all } t, s \geq 0. \quad (90b)$$

If (90b) holds for all  $t, s \in \mathbb{R}$ ,  $\varphi$  is called a *flow*. A *local semiflow* is a map  $\varphi : \mathcal{D}^\varphi \rightarrow M$  satisfying (90) wherever defined, where  $\mathcal{D}^\varphi \subset [0, \infty) \times M$  is an open subset such that  $(0, p) \in \mathcal{D}^\varphi$  for all  $p \in M$ . Let  $\mathcal{D}_t^\varphi := \{p \in M \mid (t, p) \in \mathcal{D}^\varphi\}$  and  $\mathcal{D}_p^\varphi := \{t \in [0, \infty) \mid (t, p) \in \mathcal{D}^\varphi\}$ . The semiflow is called *maximal* if  $t \in \mathcal{D}_p^\varphi$  and  $s \in \mathcal{D}_{\varphi_t(p)}^\varphi$  imply that  $s + t \in \mathcal{D}_p^\varphi$ .

A *semigroup* (of operators on a Banach space  $(X, \|\cdot\|)$ ) is a map  $T : [0, \infty) \rightarrow \mathcal{L}(X)$  satisfying the semiflow property (90). A semigroup is *strongly continuous* if it moreover satisfies

$$\lim_{t \searrow 0} \|T(t)x - x\| = 0, \quad \text{for all } x \in X.$$

The *infinitesimal generator* of  $T$  is the linear operator  $A : \text{dom}(A) \rightarrow X$  given by

$$Ax = \lim_{t \searrow 0} \frac{1}{t} (T(t)x - x) \quad (91)$$

on its domain

$$\operatorname{dom}(A) = \left\{ x \in X \mid \text{the limit } \lim_{t \searrow 0} \frac{1}{t} (T(t)x - x) \text{ exists} \right\}.$$

If  $T$  is strongly continuous,  $A$  is necessarily a closed operator and its domain  $\operatorname{dom}(A)$  is dense in  $X$ .

Given an unbounded operator  $A : \operatorname{dom}(A) \rightarrow X$  with a dense domain  $\operatorname{dom}(A) \subset X$ , its dual operator,

$$A^* : \operatorname{dom}(A^*) \rightarrow X^*, \quad \operatorname{dom}(A^*) \subset X^*,$$

is defined via its domain, the linear subspace

$$\operatorname{dom}(A^*) = \left\{ x^* \in X^* \mid \text{there exists a constant } c \geq 0 \text{ such that } |\langle x^*, Ax \rangle| \leq c|x|_X \text{ for all } x \in \operatorname{dom}(A) \right\}, \quad (92)$$

on which its action is determined by the relation

$$\langle A^*x^*, x \rangle = \langle x^*, Ax \rangle, \quad \text{for all } x \in \operatorname{dom}(A).$$

### A.3 Weak\* Integration

Suppose  $X$  is a Banach space, denote its dual by  $X^*$ . Let  $I \subset \mathbb{R}$  be an interval and let  $f : I \rightarrow X^*$  be such that

$$\langle f(\cdot), x \rangle_{X^*, X} \in L^1(I, \mathbb{C}), \quad \text{for all } x \in X. \quad (93)$$

Then,

$$\int_I f(t) dt$$

is defined as the unique element of  $X^*$  satisfying

$$\left\langle \int_I f(t) dt, x \right\rangle_{X^*, X} = \int_I \langle f(t), x \rangle_{X^*, X} dt \quad \text{for all } x \in X.$$

See A.II.3.13 of Diekmann et al. (2012), for instance, for an argument substantiating this claim using the Closed Graph Theorem.

In the main text, weak\* integration is typically used in the context when (using the notation of Section 2)

$$f(t, s) = T_0^{\odot*}(t - s)F \circ \varphi_s(u), \quad \text{for some } u \in O.$$

Since  $f(t, \cdot) : [0, t] \rightarrow X^{\odot*}$  is norm continuous for any  $t \in \mathcal{D}_u^\varphi$ , (93) is clearly met. Moreover,  $t \mapsto \int_0^t f(t, s) ds$  is norm continuous and takes values in  $\iota(X)$ , see Lemma III.2.1, Diekmann et al. (2012).

## A.4 Notions of Smoothness

Let us gather the notions of smoothness used throughout this work.

Let  $(X, |\cdot|_X)$  and  $(Y, |\cdot|_Y)$  be Banach spaces. Let  $f : O \rightarrow Y$  denote a map, with  $O \subset X$  open. We say  $f$  is *locally Lipschitz* if for every  $u \in O$ , there exists an open neighbourhood  $U$  of  $u$  and a constant  $L \geq 0$  for which  $|f(v) - f(w)|_Y \leq L|v - w|_X$  for all  $v, w \in U$ . We say  $f$  is *globally Lipschitz* if  $U = O$  in the above. In this case, we denote by  $\text{Lip}(f)$  the global Lipschitz constant.

$f$  is called *Fréchet differentiable* at  $u \in O$  if there exists a bounded linear operator  $A \in \mathcal{L}(X; Y)$  such that

$$\lim_{|\delta|_X \rightarrow 0} \frac{|f(u + \delta) - f(u) - A\delta|_Y}{|\delta|_X} = 0.$$

In this case, we denote  $Df(u) = A$ . If  $f$  is Fréchet differentiable for all points in  $O$ , it is said to be of class  $C^1$  if the map  $u \mapsto Df(u)$  is continuous from  $O$  to  $\mathcal{L}(X; Y)$ . Inductively,  $f$  is of class  $C^k$ ,  $k \in \mathbb{N}$  if  $Df$  is of class  $C^{k-1}$ .  $f$  is said to be of class  $C^\infty$  if it is  $C^k$  for all  $k \in \mathbb{N}$ .

$f$  belongs to the space  $C_b^k(O)$  if it is  $C^k$ , and moreover,  $u \mapsto D^j f(u)$  is bounded on  $O$  for each  $0 \leq j \leq k$ . The space  $C_b^k(O)$  is Banach with the norm  $\sup\{\|D^j f(u)\| \mid u \in O, 0 \leq j \leq k\}$ .

## Appendix B Smoothness of the Semiflow

Here, we derive, for completeness,  $C^k$ -in-space smoothness properties of the semiflow generated by a delay equation (1) with  $f \in C^k(X; \mathbb{R}^n)$ , for integers  $k \geq 2$ .

Recall the following result about the smoothness of composition maps.

**Theorem 36** (Theorem 6, Irwin 1972) *Let  $A$  be a compact topological space and let  $Y, Z$  denote Banach spaces, and  $U \subset Y$  an open subset. Suppose  $g : U \rightarrow Z$  is a  $C^k$  map,  $k \in \mathbb{N}_0$ . Then  $\hat{g} : C(A; U) \rightarrow C(A; Z)$  given by  $\hat{g}(f) = g \circ f$  is  $C^k$ .*

We remark that the first derivative  $D\hat{g} : C(A; U) \rightarrow \mathcal{L}(C(A; Y); C(A; Z))$  is given by

$$(D\hat{g}(f)[\eta])(x) = Dg(f(x))[\eta(x)], \quad f \in C(A; U), \quad \eta \in C(A; Y).$$

Thus,

$$\|D\hat{g}(f)\| \leq \sup_{x \in A} \|Dg(f(x))\|. \quad (94)$$

Recall also that for an open subset  $U$  of a Banach space  $Y$ , we have that  $C(A; U)$  is an open subset of  $C(A; Y)$ .

**Proposition 37** *Suppose  $f$  in (1) is  $C^k$ , with  $k \in \mathbb{N}$ . Let  $\varphi : \mathcal{D}^\varphi \rightarrow O$  denote the semiflow from Proposition 4. Given  $u \in O$ , there exists  $t > 0$  small enough and a neighbourhood  $U \subset \mathcal{D}_t^\varphi$  of  $u$  such that  $\varphi_t|_U$  is  $C^k$ . Moreover,  $(s, u) \mapsto$*

$D^j \varphi_s(u)[v_1, \dots, v_j]$  is jointly continuous on  $[0, t] \times U$  for each integer  $j \leq k$ , and fixed  $v_i \in X$ ,  $1 \leq i \leq j$ .

**Proof** Let  $M, \omega$  be as in the proof of Lemma 29. By continuity of  $\varphi$  (Proposition 4(i)) and continuity of  $v \mapsto DR(v)$ , we may choose  $t > 0$  small enough so that

$$\frac{M}{\omega} (e^{\omega t} - 1) \sup_{s \in [0, t]} \|DR(\varphi_s(u))\| < 1. \quad (95)$$

Consider the map

$$\mathcal{F} : \mathcal{D}_t^\varphi \times C([0, t]; O) \rightarrow C([0, t]; X),$$

$$(u, y) \mapsto \left( s \mapsto y(s) - T(s)u - \iota^{-1} \int_0^s T^{\odot*}(s - \tau)R(y(\tau)) d\tau \right).$$

Note that  $\mathcal{F}$  is well defined and indeed has image in  $C([0, t]; X)$ , by Lemma III.2.1, Diekmann et al. (2012). Then, by Proposition 4, since  $u \in \mathcal{D}_t^\varphi$ ,

$$\mathcal{F}(u, s \mapsto \varphi_s(u)) = 0.$$

Moreover, the map  $\mathcal{F}$  can be written as

$$\mathcal{F} = L_1 - L_3 \circ \widehat{R} \circ L_2,$$

where

$$L_1 : \mathcal{D}_t^\varphi \times C([0, t]; O) \rightarrow C([0, t]; X), \quad L_1(u, y)(s) = y(s) - T(s)u,$$

$$L_2 : \mathcal{D}_t^\varphi \times C([0, t]; O) \rightarrow C([0, t]; O), \quad L_2(u, y) = y,$$

$$L_3 : C([0, t]; X^{\odot*}) \rightarrow C([0, t]; X), \quad L_3(z)(s) = \iota^{-1} \int_0^s T^{\odot*}(s - \tau)z(\tau) d\tau.$$

All of these are continuous linear maps or restrictions thereof. Hence, by Theorem 36,  $\mathcal{F}$  is of class  $C^k$ . Moreover,

$$D_2 \mathcal{F}(u, s \mapsto \varphi_s(u)) = \text{id}_{C([0, t]; X)} - L_3 \circ D\widehat{R}(s \mapsto \varphi_s(u)).$$

Since, by (94),

$$\|L_3 \circ D\widehat{R}(s \mapsto \varphi_s(u))\| \leq \frac{M}{\omega} (e^{\omega t} - 1) \sup_{s \in [0, t]} \|DR(\varphi_s(u))\|,$$

(95) implies that  $D_2 \mathcal{F}(u, s \mapsto \varphi_s(u))$  is a linear isomorphism.

We may hence apply the Implicit Function Theorem (Theorem I.5.9, Lang 2012) to obtain a neighbourhood  $U \subset \mathcal{D}_t^\varphi$  of  $u$  and a unique  $C^k$  map  $\mathcal{G} : U \rightarrow C([0, t]; O)$  such that

$$\mathcal{F}(w, \mathcal{G}(w)) = 0 \quad \text{for all } w \in U.$$

By uniqueness, and since  $U \subset \mathcal{D}_t^\varphi$ , we necessarily have  $\mathcal{G}(w) = s \mapsto \varphi_s(w)$ . Composing  $\mathcal{G}$  with the (continuous, linear) evaluation map  $\text{ev}_t : y \mapsto y(t)$ , we have

$$\varphi_t = \text{ev}_t \circ \mathcal{G} \quad \text{on } U.$$

This yields the first claim immediately. For the second, observe that, for  $u \in U$  and  $v_i \in X$ ,  $1 \leq i \leq j$ ,  $j \leq k$ ,

$$D^j \varphi_s(u)[v_1, \dots, v_j] = \text{ev}_s(D^j \mathcal{G}(u)[v_1, \dots, v_j]), \quad s \in [0, t].$$

Note, in particular, that  $D^j \mathcal{G}(u)[v_1, \dots, v_j] \in C([0, t]; X)$ . Hence, for  $(s, u), (\tau, w) \in [0, t] \times U$ ,

$$\begin{aligned} & \left| D^j \varphi_s(u)[v_1, \dots, v_j] - D^j \varphi_\tau(w)[v_1, \dots, v_j] \right|_X \\ & \leq \left| D^j \mathcal{G}(u)[v_1, \dots, v_j] - D^j \mathcal{G}(w)[v_1, \dots, v_j] \right|_{C([0, t]; X)} \\ & \quad + \left| \left( D^j \mathcal{G}(u)[v_1, \dots, v_j] \right)(s) - \left( D^j \mathcal{G}(u)[v_1, \dots, v_j] \right)(\tau) \right|_X, \end{aligned}$$

from which the second assertion follows.  $\square$

**Proposition 38** *Let  $t > 0$  be fixed, and assume the conditions of Proposition 37. Then  $\varphi_t$  is  $C^k$  on  $\mathcal{D}_t^\varphi$ .*

**Proof** Fix  $u \in \mathcal{D}_t^\varphi$  and set  $T_1 = 0$ . For each  $j \in \mathbb{N}$ , we may select  $t_j > 0$  inductively such that

$$\frac{1}{2} < \frac{M}{\omega} (e^{\omega t_j} - 1) \sup_{s \in [T_j, T_j + t_j]} \|DR(\varphi_s(u))\| < 1, \quad (96)$$

where  $T_j = \sum_{i < j} t_i$  for  $j \geq 2$  (using the axiom of dependent choice).

Since  $(T_j)_{j \in \mathbb{N}}$  is a (strictly) increasing sequence, it either tends to  $\infty$  or converges to a limit  $T$ . We claim  $T > t$ . For if, on the contrary,  $T \leq t$ , we have by (96) and the fact that necessarily  $t_j \rightarrow 0$ ,

$$\lim_{s \rightarrow T} \|DR(\varphi_s(u))\| = \infty,$$

which would contradict that  $R$  is  $C^1$  at  $\varphi_T(u) \in O$  (since  $T \in \mathcal{D}_u^\varphi$ ).

Hence,  $T > t$ , and we may choose  $m \in \mathbb{N}$  large enough so that  $T_{m+1} \geq t > T_m$ . Up to perhaps shrinking  $t_m$ , we may assume  $T_{m+1} = t$ . For each  $1 \leq j \leq m$ , Proposition 37 provides a neighbourhood  $U_j \subset \mathcal{D}_{t_j}^\varphi$  of  $\varphi_{T_j}(u)$  on which  $\varphi_{t_j}$  is  $C^k$ . Then,

$$V := \bigcap_{j=1}^m \varphi_{T_j}^{-1}(U_j)$$

is a nonempty open neighbourhood of  $u$  in  $\mathcal{D}_t^\varphi$  (by maximality of  $\varphi$ ) on which

$$\varphi_t = \varphi_{t_m} \circ \dots \circ \varphi_{t_1}$$

is  $C^k$ .

□

**Remark 39** (*Smoothness of  $\varphi_1^\rho$  about  $W_\rho^\Sigma$* ) While  $R_\rho$  is not smooth in general, it retains the  $C^k$  smoothness of  $R$  on the open set  $S$  defined by (65). Moreover,  $C([0, t]; S)$  is an open subset of  $C([0, t]; X)$ ;  $W_\rho^\Sigma$  is an invariant set for  $\varphi_t^\rho$  contained within  $S$  for  $\rho$  sufficiently small<sup>23</sup>, and  $s \mapsto \varphi_s^\rho(u) \in C([0, t]; S)$  for all  $u \in W_\rho^\Sigma$ . Therefore, the proof of Proposition 37 goes through with  $\mathcal{F}_\rho : X \times C([0, t]; S) \rightarrow C([0, t]; X)$  in place of  $\mathcal{F}$  (with the obvious modification  $R \mapsto R_\rho$ ). In particular,  $\mathcal{F}_\rho$  is  $C^k$  on this restricted space, and the above discussion shows that each  $u \in W_\rho^\Sigma$  produces a solution within the domain of  $\mathcal{F}_\rho$  of the form  $\mathcal{F}_\rho(u, s \mapsto \varphi_s^\rho(u)) = 0$ . As in the proof of Proposition 37, the Implicit Function Theorem then produces  $C^k$  smoothness of  $\varphi_1^\rho$  on neighbourhoods of  $u \in W_\rho^\Sigma$ , where  $\rho > 0$  is small enough so that (95) holds with  $t = 1$  and  $R = R_\rho$ , the supremum now being taken over the whole of  $W_\rho^\Sigma$ .

**Remark 40** (*Joint continuity property for Theorem 16*) In the setting of Section 3, we have that  $\varphi_t(0) = 0$  for all  $t \geq 0$ ; and  $DR(0) = 0$ . Hence, (95) is satisfied for all  $t \geq 0$  if  $u$  is taken to be 0. The last assertion of Proposition 37 then implies that, for any  $t > 0$ , there exists  $U \subset \mathcal{D}_t^\rho$  containing 0 such that  $(s, u) \mapsto D^j \varphi_s(u)[v_1, \dots, v_j]$  is jointly continuous on  $[0, t] \times U$ . Hence,  $(s, u) \mapsto D^j \varphi_s(u)[v_1, \dots, v_j]$  is jointly continuous over  $\mathbb{R}^{\geq 0} \times \{0\}$ . Suppose  $W$  is a finite dimensional submanifold embedded in  $X$ ; denote the embedding by  $\iota_W : W \hookrightarrow X$ . Recall that 'strong' continuity in the above sense implies 'norm' continuity in the finite dimensional case, hence  $(s, u_1) \mapsto D^j(\varphi_s \circ \iota_W)(u_1) = D^j \varphi_s(u_1)[\iota_W(\cdot), \dots, \iota_W(\cdot)]$  is jointly continuous over  $\mathbb{R}^{\geq 0} \times \{0\}$ .

## Appendix C Smoothness of $W_\rho^\Sigma$

Here, we outline the proof of the smoothness claim of Theorem 12(ii), following the 'scale of Banach spaces' approach of Vanderbauwhede and Van Gils (1987), designed originally for center manifolds. The discrete-time version, known as Irwin's method (Irwin 1980; de la Llave and Wayne 1995), would produce the same smoothness conclusions. The latter, specifically, would be directly applicable to the setting of Section 6 if  $R_\rho$  was  $C^k$  on its whole domain (using uniqueness of  $W_\rho^\Sigma$  once a cutoff function has been fixed). If one traces through the proof of Theorem 2.1 in de la Llave and Wayne (1995), it is apparent (see Lemma 3.4 specifically) that  $C^k$  smoothness of the map  $\varphi_1^\rho$  is only used on a neighbourhood of negative semiorbits contained in  $W_\rho^\Sigma$  – which we do indeed have (c.f. Remark 39).

We include a proof anyway so as to avoid concealing any details. It will moreover serve to provide proofs of the assertions made in Remark 15. We assume throughout this section that  $f$  is  $C^k$ ,  $k \geq 1$ .

First, we remark that the projections  $P_\Sigma$  of Lemma 6 can be extended to the larger space  $X^{\odot*}$  via  $P_\Sigma^{\odot*} := (P_\Sigma^*|_{X^{\odot*}})^*$  – these will have the same range and satisfy  $\iota P_\Sigma = P_\Sigma^{\odot*} \iota$ . (They moreover coincide with spectral projections of the operator  $A^{\odot*}$ .) We

<sup>23</sup> This can be ensured via Theorem 31(iii), which gives control over  $\text{Lip}(\phi_\rho)$ . If this is chosen to be less than 2,  $W_\rho^\Sigma \subset S$  according to (65).

shall only make use of  $P_{\Sigma}^{\odot*}$  in the special case when  $\Sigma$  corresponds to a half space of the spectrum, as in (A.3), for which we can directly quote the following result.

**Lemma 41** (Theorem 2.12, Diekmann et al. 2012) *Let  $A$  be as in (13), let  $T(t)$  denote the semigroup it generates. If*

$$\Sigma = \{\lambda \in \sigma(A; X) \mid \operatorname{Re} \lambda \geq \gamma\},$$

*then  $\Sigma = \{\lambda \in \sigma(A^{\odot*}; X) \mid \operatorname{Re} \lambda \geq \gamma\}$ ,*

$$X^{\odot*} = \operatorname{im}(P_{\Sigma}^{\odot*}) \oplus \ker(P_{\Sigma}^{\odot*})$$

*and, if  $\alpha := \inf_{\lambda \in \Sigma} \operatorname{Re} \lambda$ , and  $\beta := \sup_{\lambda \in \Sigma'} \operatorname{Re} \lambda$ , for any  $\varepsilon > 0$  there exists  $K > 0$  such that*

$$\|T^{\odot*}(t)P_{\Sigma}^{\odot*}\|_{\mathcal{L}(X^{\odot*})} \leq Ke^{(\alpha-\varepsilon)t} \|P_{\Sigma}^{\odot*}\|_{\mathcal{L}(X^{\odot*})}, \quad t \leq 0, \quad (97a)$$

$$\|T^{\odot*}(t)P_{\Sigma'}^{\odot*}\|_{\mathcal{L}(X^{\odot*})} \leq Ke^{(\beta+\varepsilon)t} \|P_{\Sigma'}^{\odot*}\|_{\mathcal{L}(X^{\odot*})}, \quad t \geq 0. \quad (97b)$$

To characterize backwards orbits, we introduce the following Banach spaces (see Vanderbauwhede and Van Gils 1987).

**Definition 42** For each  $\eta \in \mathbb{R}$ ,

$$BC^{\eta}(\mathbb{R}^{\leq 0}; X) = \left\{ f \in C(\mathbb{R}^{\leq 0}; X) \mid \sup_{t \in \mathbb{R}^{\leq 0}} e^{\eta t} |f(t)|_X < \infty \right\}$$

is a Banach space when equipped with the norms

$$|f|_{\eta} = \sup_{t \in \mathbb{R}^{\leq 0}} e^{\eta t} |f(t)|_X.$$

Note that for  $\tilde{\eta} \leq \eta$  there exists a continuous embedding  $J_{\tilde{\eta}}^{\eta} : BC^{\tilde{\eta}}(\mathbb{R}^{\leq 0}; X) \hookrightarrow BC^{\eta}(\mathbb{R}^{\leq 0}; X)$  with norm one. We shall occasionally use the shorthand notation  $X_{\eta} := BC^{\eta}(\mathbb{R}^{\leq 0}; X)$  in sub/superscripts.

**Lemma 43** *The operator defined by*

$$\begin{aligned} [\mathcal{K}F](t) &= -t^{-1} \int_t^0 T^{\odot*}(t-s)P_{\Sigma}^{\odot*}F(s)ds \\ &\quad + t^{-1} \int_{-\infty}^t T^{\odot*}(t-s)P_{\Sigma'}^{\odot*}F(s)ds, \quad t \leq 0, \end{aligned} \quad (98)$$

*is bounded as a map  $\mathcal{K} : BC^{-\eta}(\mathbb{R}^{\leq 0}; X^{\odot*}) \rightarrow BC^{-\eta}(\mathbb{R}^{\leq 0}; X)$  for each  $\eta \in (\beta, \alpha)$ . Moreover,  $\|\mathcal{K}\|_{\mathcal{L}(X_{-\eta}^{\odot*}, X_{-\eta})}$  can be bounded uniformly with respect to  $\eta$  on compact subsets of  $(\beta, \alpha)$ .*

**Proof** Fix  $\eta \in (\beta, \alpha)$  and  $F \in BC^\eta(\mathbb{R}^{\leq 0}; X^{\odot*})$ . Choose  $0 < \varepsilon < \min\{\eta - \beta, \alpha - \eta\}$ . For  $t \leq 0$ , we have, by (97),

$$\begin{aligned} e^{-\eta t} |[\mathcal{K}F](t)|_X &\leq C|F|_{-\eta} \left( \int_t^0 e^{(\alpha-\varepsilon-\eta)(t-s)} ds + \int_{-\infty}^t e^{(\beta+\varepsilon-\eta)(t-s)} ds \right) \\ &\leq C|F|_{-\eta} \left( \frac{1}{\alpha-\varepsilon-\eta} - \frac{1}{\beta+\varepsilon-\eta} \right), \end{aligned}$$

for some  $C > 0$ . The uniformity claim, for a compact subset  $[\tilde{\eta}, \eta] \subset (\beta, \alpha)$ , follows by choosing  $\varepsilon < \min\{\tilde{\eta} - \beta, \alpha - \eta\}/2$ .  $\square$

We need the following result on composition operators acting on  $BC^\eta$  spaces.

**Lemma 44** (Lemma A.IV.1.1, Diekmann et al. 2012) *Let  $Y$  and  $Z$  be two Banach spaces. Let  $g : Y \rightarrow Z$  be a  $C_b^k$  mapping with  $g(0) = 0$ ,  $k \geq 1$ . Then, the map  $\hat{g} : f \mapsto g \circ f$  satisfies the following.*

- (i) *For  $\eta > 0$ , the map  $\hat{g} : BC^{-\eta}(\mathbb{R}^{\leq 0}; Y) \rightarrow BC^{-\eta}(\mathbb{R}^{\leq 0}; Z)$  is  $C^k$ .*
- (ii) *For  $\eta, \tilde{\eta} < 0$  such that  $\tilde{\eta} < k\eta$ , the map  $\hat{g} : BC^{-\eta}(\mathbb{R}^{\leq 0}; Y) \rightarrow BC^{-\tilde{\eta}}(\mathbb{R}^{\leq 0}; Z)$  is  $C^k$ .*

In both cases, for  $f, h \in BC^{-\eta}(\mathbb{R}^{\leq 0}; Y)$ ,

$$(D\hat{g}(f)[h])(s) = Dg(f(s))[h(s)]. \quad (99)$$

**Proof** The proof is given in Appendix IV of Diekmann et al. (2012) for the spaces  $BC^\eta(\mathbb{R}; Y)$  (equipped with norms  $\sup_{t \in \mathbb{R}} e^{-\eta|t|} |f(t)|_Y$ , for a Banach space  $Y$ ). The proof for our case would be virtually the same, but the result can also be deduced directly, by considering the embedding  $\iota_Y : BC^\eta(\mathbb{R}^{\leq 0}; Y) \hookrightarrow BC^\eta(\mathbb{R}; Y)$  given by mirroring about the origin  $(\iota_Y f)(t) := f(-|t|)$ ,  $t \in \mathbb{R}$ ) and noting that  $\hat{g}$  preserves its image; that is,  $\iota_Z \circ \hat{g} = \hat{g} \circ \iota_Y$ .  $\square$

Invariant manifolds will be extracted as fixed points of (some variant of) the map

$$\mathcal{F}(u_1, y)(t) = T(t)u_1 + [\mathcal{K} \circ \hat{R}(y)](t). \quad (100)$$

Here,  $u_1 \in X_\Sigma = \text{im}(P_\Sigma^{\odot*})$ ; but the exact  $BC$  space on which  $y$  is considered depends on the number  $\gamma$  (from Lemma 41) defining  $\Sigma$ , since  $\mathcal{K}$  needs to act boundedly on the image of  $\hat{R}$  (c.f. Lemma 43). We first deal with the simplest case, when  $\gamma > 0$ , for which we will be able to show  $C^k$  smoothness without any spectral gap assumptions (hence proving the statements of Remark 15).

### C.1 The Strong-Unstable Case

**Lemma 45** *Suppose  $\eta > 0$  and that  $R : O \rightarrow X^{\odot*}$  is as in Section 3, of class  $C^k$ ,  $k \geq 1$ . For  $\delta > 0$  sufficiently small, the map  $\hat{R} : B_\delta^{X_{-\eta}}(0) \rightarrow BC^{-\eta}(\mathbb{R}^{\leq 0}; X^{\odot*})$  is  $C^k$ . Moreover,  $\sup_{f \in B_\delta^{X_{-\eta}}(0)} \|D\hat{R}(f)\|_{\mathcal{L}(X_{-\eta}; X_{-\eta}^{\odot*})} = o(1)$  as  $\delta \rightarrow 0$ .*

**Proof** Choose  $\delta > 0$  small enough so that  $B_\delta^X(0) \subset O$  and  $R|_{B_\delta^X(0)} \in C_b^k$ . Since  $\eta > 0$ , we have  $\cup_{t \leq 0} \text{ev}_t(B_\delta^{X-\eta}(0)) \subset B_\delta^X(0)$ , and hence the assumptions of Lemma 44 are met on the open set  $B_\delta^{X-\eta}(0)$ . Noting that smoothness is a local property, we may conclude that  $\widehat{R}$  is  $C^k$  on  $B_\delta^{X-\eta}(0)$ . The bound is an immediate consequence of (99), the inclusions above, and

$$\sup_{|u| \leq \delta} \|DR(u)\| = o(1) \quad \text{as } \delta \rightarrow 0,$$

(c.f. Section 3). □

**Partial proof of Remark 15** Suppose  $\gamma > 0$  and choose  $\eta \in (\min\{0, \beta\}, \alpha)$ . Consider (100) as  $\mathcal{F} : X_\Sigma \times B_\delta^{X-\eta}(0) \rightarrow BC^{-\eta}(\mathbb{R}^{\leq 0}; X)$ . This is well defined and  $C^k$  for  $\delta$  small enough by Lemmas 43 and 45. In fact, we may ensure, by perhaps further adjusting  $\delta$ , that

$$\|\mathcal{K}\|_{\mathcal{L}(X_{-\eta}^{\odot*}, X_{-\eta})} \sup_{|u| \leq \delta} \|DR(u)\| < \frac{1}{2}, \quad (101)$$

so that  $\mathcal{F}$  is a contraction on the second factor. We may hence apply the Implicit Function Theorem to the map  $(u_1, y) \mapsto y - \mathcal{F}(u_1, y)$  about the fixed point  $\mathcal{F}(0, 0) = 0$  to obtain a neighbourhood  $U_\Sigma$  of 0 in  $X_\Sigma$  and a unique  $C^k$  map

$$\mathcal{W} : U_\Sigma \rightarrow B_\delta^{X-\eta}(0)$$

such that  $\mathcal{F}(u_1, \mathcal{W}(u_1)) = \mathcal{W}(u_1)$ . The image of the  $C^k$  map  $\text{ev}_0 \circ \mathcal{W} : U_\Sigma \rightarrow B_\delta^X(0)$  is then the desired 'strong-unstable' manifold. It is uniquely characterized by backwards Lyapunov exponents of its orbits, manifest in the choice of  $BC^{-\eta}(\mathbb{R}^{\leq 0}; X)$ ,  $\eta \in (\min\{0, \beta\}, \alpha)$ .

Next, we show that it agrees with  $W_\rho^\Sigma$  from Theorem 31 on a small enough neighbourhood of the origin. For a given selection of  $\gamma_1, \gamma_2 > 0$  in Theorem 31, we may consider  $\eta$  on the interval  $[\gamma_2 - \varepsilon, \gamma_1 + \varepsilon]$ , where  $\varepsilon$  is chosen such that this interval lies within  $(\min\{0, \beta\}, \alpha)$ . By the final claim of Lemma 43, we may ensure (101) holds uniformly for  $\eta \in [\gamma_2 - \varepsilon, \gamma_1 + \varepsilon]$ .

Since the spaces  $BC^{-\eta}(\mathbb{R}^{\leq 0}; X)$ ,  $\eta \in [\gamma_2 - \varepsilon, \gamma_1 + \varepsilon]$ , are all embedded in  $BC^{-(\gamma_2 - \varepsilon)}(\mathbb{R}^{\leq 0}; X)$ , we must have  $J_{-(\gamma_2 - \varepsilon)}^{-\eta} \circ \mathcal{W}^{-(\gamma_2 - \varepsilon)} = \mathcal{W}^{-\eta}$  by the uniqueness claim of the Implicit Function Theorem (where the superscript of  $\mathcal{W}$  refers to the space in which it was obtained).

Comparing these observations with Theorem 31(i); in particular, (71) and (72), we must have that  $\text{im}(\text{ev}_0 \circ \mathcal{W}) = W_\rho^\Sigma$  on a sufficiently small neighbourhood of the origin where both  $\mathcal{W}$  and  $W_\rho^\Sigma$  are defined and  $\varphi = \varphi^\rho$ . □

Strong-stable manifolds of Remark 15 are proven analogously, with appropriate modifications to  $\mathcal{K}$ , and considering positive semiorbits.

## C.2 The Pseudo-Unstable Case

The case  $\gamma \leq 0$  (when necessarily  $\eta < 0$  is required for the  $BC$  space) is much more complicated; firstly because Lemma 45 no longer applies (this necessitates the application of a cutoff function), and secondly due to the difference in items (i) and (ii) of Lemma 44. We first state a replacement of Lemma 45 for the modified nonlinearity  $R_\rho$  from Section 6.1.

**Lemma 46** *For all  $\eta \in \mathbb{R}$ , the substitution operators*

$$\widehat{R}_\rho : BC^\eta(\mathbb{R}^{\leq 0}; X) \rightarrow BC^\eta(\mathbb{R}^{\leq 0}; X^{\odot*})$$

*defined by  $\widehat{R}_\rho : f \mapsto R_\rho \circ f$  are globally Lipschitz continuous with  $\text{Lip}(\widehat{R}_\rho) \leq \text{Lip}(R_\rho) = o(1)$  as  $\rho \rightarrow 0$ .*

**Proof** Observe, for  $f, g \in BC^\eta(\mathbb{R}^{\leq 0}; X)$ , that

$$\begin{aligned} |\widehat{R}_\rho(f) - \widehat{R}_\rho(g)|_\eta &= \sup_{t \leq 0} e^{\eta t} |R_\rho(f(t)) - R_\rho(g(t))|_{X^{\odot*}} \\ &\leq \text{Lip}(R_\rho) |f - g|_\eta. \end{aligned}$$

This shows that  $\widehat{R}_\rho$  is well defined and  $\text{Lip}(\widehat{R}_\rho) \leq \text{Lip}(R_\rho)$ . Lemma 28 completes the proof.  $\square$

We may now produce a Lipschitz manifold via the same argument as above. We shall consider, for later purposes, a range of  $\eta \in [\tilde{\eta}, \bar{\eta}] \subset (\beta, \alpha)$  with  $k\bar{\eta} > \tilde{\eta}$ ,  $k \geq 1$ . In particular, consider  $\mathcal{F}_\rho : X_\Sigma \times BC^{-\eta}(\mathbb{R}^{\leq 0}; X) \rightarrow BC^{-\eta}(\mathbb{R}^{\leq 0}; X)$ , with  $\eta \in [\tilde{\eta}, \bar{\eta}]$ , given by

$$\mathcal{F}_\rho(u_1, y)(t) = T(t)u_1 + [\mathcal{K} \circ \widehat{R}_\rho(y)](t). \quad (102)$$

Shrink  $\rho$  so that

$$\|\mathcal{K}\|_{\mathcal{L}(X_{-\eta}^{\odot*}, X_{-\eta})} \text{Lip}(R_\rho) < \frac{1}{2}, \quad \text{for all } \eta \in [\tilde{\eta}, \bar{\eta}]. \quad (103)$$

(This can be done by the last assertion of Lemma 43.) Since  $\mathcal{F}_\rho$  is globally Lipschitz (in both entries), the (parametric) Banach fixed point theorem (Theorem 21, Irwin 1972) implies the existence of a globally Lipschitz map  $\mathcal{W}_\rho^{-\eta} : X_\Sigma \rightarrow BC^{-\eta}(\mathbb{R}^{\leq 0}; X)$  such that

$$\mathcal{F}_\rho(u_1, \mathcal{W}_\rho^{-\eta}(u_1)) = \mathcal{W}_\rho^{-\eta}(u_1). \quad (104)$$

Note that uniqueness of the solution implies  $J_{-\bar{\eta}}^{-\eta} \circ \mathcal{W}_\rho^{-\eta} = \mathcal{W}_\rho^{-\eta}$  (since  $\mathcal{K} \circ \widehat{R}_\rho$  commutes with  $J_{-\bar{\eta}}^{-\eta}$ ). Uniqueness also implies the invariance property

$$\text{ev}_t \circ \mathcal{W}_\rho^{-\eta} = \text{ev}_0 \circ \mathcal{W}_\rho^{-\eta} \circ \tilde{P}_\Sigma \circ \text{ev}_t \circ \mathcal{W}_\rho^{-\eta}, \quad t \leq 0. \quad (105)$$

We may estimate the Lipschitz constant of  $\mathcal{W}_\rho^{-\eta}$  via

$$|\mathcal{W}_\rho^{-\eta}(u_1) - \mathcal{W}_\rho^{-\eta}(v_1)|_{-\eta} \leq \|\mathcal{K}\|_{\mathcal{L}(X_{-\eta}^{\odot*}; X_{-\eta})} \text{Lip}(R_\rho) |\mathcal{W}_\rho^{-\eta}(u_1) - \mathcal{W}_\rho^{-\eta}(v_1)|_{-\eta} + K|u_1 - v_1|_X$$

to be  $\text{Lip}(\mathcal{W}_\rho^{-\eta}) \leq 2K$ , with  $K$  as in (97) (for  $\varepsilon < \alpha - \bar{\eta}$ ). Observe, from (98), that  $P_\Sigma \circ \text{ev}_0 \circ \mathcal{K} = 0$ . This shows  $\tilde{P}_\Sigma \circ \text{ev}_0 \circ \mathcal{W}_\rho^{-\eta} = \text{id}_{X_\Sigma}$ . Hence, the manifold  $\text{im}(\text{ev}_0 \circ \mathcal{W}_\rho^{-\eta})$  can be alternatively written as the graph of the map

$$\phi_\rho := \tilde{P}_{\Sigma'} \circ \text{ev}_0 \circ \mathcal{W}_\rho^{-\eta} : X_\Sigma \rightarrow X_{\Sigma'}. \quad (106)$$

From (104), we have  $\phi_\rho = \tilde{P}_{\Sigma'} \circ \text{ev}_0 \circ \mathcal{K} \circ \widehat{R}_\rho \circ \mathcal{W}_\rho^{-\eta}$  and

$$\text{Lip}(\phi_\rho) \leq 2K \|P_{\Sigma'}\|_{\mathcal{L}(X)} \|\mathcal{K}\|_{\mathcal{L}(X_{-\eta}^{\odot*}; X_{-\eta})} \text{Lip}(R_\rho), \quad \text{for all } \eta \in [\tilde{\eta}, \bar{\eta}]. \quad (107)$$

Up to potentially adjusting  $\rho$ , we may hence assume  $\text{Lip}(\phi_\rho) < 2$ . We have thus ensured that is contained in the open set  $S$  on which  $R_\rho$  is  $C^k$  (65). Since  $\text{supp}(R_\rho) \subset \overline{B_{4\rho}(0)}^X$ , we moreover have

$$\sup_{x \in X_\Sigma} |\phi_\rho(x)| < \rho \quad (108)$$

for  $\rho$  small enough. With the cutoff function fixed, by the same argument as above in the proof of Remark 15, we have  $\text{im}(\text{ev}_0 \circ \mathcal{W}_\rho) = W_\rho^\Sigma$  (which now holds globally), hence the notation  $\phi_\rho$  is no mistake, it is indeed equal to the one in (70).

For  $\eta \in [\tilde{\eta}, \bar{\eta}]$ , let

$$V^{-\eta} = \left\{ y \in BC^{-\eta}(\mathbb{R}^{\leq 0}; X) \mid |P_{\Sigma'} y|_0 < \infty \right\}$$

(recall  $|\cdot|_0$  stands for the  $BC^0$  norm).  $V^{-\eta}$  is a Banach space continuously embedded in  $BC^{-\eta}(\mathbb{R}^{\leq 0}; X)$  when equipped with the norm  $|y|_{V^{-\eta}} = |P_\Sigma y|_{-\eta} + |P_{\Sigma'} y|_0$ . Consider the open subset

$$V_\rho^{-\eta} = \left\{ y \in BC^{-\eta}(\mathbb{R}^{\leq 0}; X) \mid |P_{\Sigma'} y|_0 < \rho \right\}.$$

It follows from the invariance property (105) that

$$\text{ev}_t \circ \mathcal{W}_\rho^{-\eta} \in \text{im}(\text{ev}_0 \circ \mathcal{W}_\rho^{-\eta}) = \text{graph}(\phi_\rho), \quad t \leq 0.$$

By (108), we have

$$\text{im}(\mathcal{W}_\rho^{-\eta}) \subset V_\rho^{-\eta}. \quad (109)$$

The reason we cannot repeat the proof of the  $\eta > 0$  case above is that now, according to Lemma 44(ii), the map  $\widehat{R}_\rho$  and hence  $\mathcal{F}_\rho$  are only smooth with differing domain and range spaces – thus the implicit function theorem no longer directly applies. Roughly speaking, this is a consequence of the spaces  $BC^{-\eta}$  allowing growth (as  $t \rightarrow -\infty$ ) for  $\eta < 0$ . To show smoothness of  $\mathcal{W}_\rho^{-\eta}$ , we consider an abstract version of (102),

and provide a version of implicit mapping that allows for such differences in domains and ranges (following Vanderbauwhede and Van Gils 1987; Diekmann et al. 2012).

Let  $(Y_0, |\cdot|_{Y_0})$ ,  $(Y, |\cdot|_Y)$ ,  $(Y_1, |\cdot|_{Y_1})$  and  $(\Lambda, |\cdot|_\Lambda)$  be Banach spaces such that

$$Y_0 \xhookrightarrow{J_0} Y \xhookrightarrow{J} Y_1$$

with  $J_0, J$  continuous embeddings. Let  $O_0$  denote an open subset of  $Y_0$ . Consider the fixed point equation

$$f(\lambda, y) = y \quad (110)$$

for some  $f : \Lambda \times Y \rightarrow Y$  satisfying the following hypotheses:

(Hf.1) The function  $g : (\lambda, y_0) \mapsto J \circ f(\lambda, J_0(y_0))$  is of class  $C^1$  on  $\Lambda \times O_0$ . There exist mappings  $f^{(1)} : \Lambda \times J_0(O_0) \rightarrow \mathcal{L}(Y)$  and  $f_1^{(1)} : \Lambda \times J_0(O_0) \rightarrow \mathcal{L}(Y_1)$  such that

$$D_2 g(\lambda, y_0)[z_0] = J f^{(1)}(\lambda, J_0(y_0))[J_0(z_0)] \quad \text{for all } (\lambda, y_0, z_0) \in \Lambda \times O_0 \times Y_0$$

and

$$J f^{(1)}(\lambda, J_0(y_0))[y] = f_1^{(1)}(\lambda, J_0(y_0))[J(y)] \quad \text{for all } (\lambda, y_0, y) \in \Lambda \times O_0 \times Y.$$

(Hf.2) There exists some  $\kappa \in [0, 1)$  such that for all  $y, \tilde{y} \in Y$  and  $\lambda \in \Lambda$

$$|f(\lambda, y) - f(\lambda, \tilde{y})|_Y \leq \kappa |y - \tilde{y}|_Y, \quad (111)$$

and for all  $\bar{y} \in J_0(O_0)$

$$\|f^{(1)}(\lambda, \bar{y})\|_{\mathcal{L}(Y)} \leq \kappa, \quad \|f_1^{(1)}(\lambda, \bar{y})\|_{\mathcal{L}(Y_1)} \leq \kappa.$$

(Hf.3) Let  $\Psi : \Lambda \rightarrow Y$  denote the map associating to each  $\lambda$  a solution of (110) (this exists by (111)). Suppose  $\Psi = J_0 \circ \Phi$  for some continuous  $\Phi : \Lambda \rightarrow Y_0$  with image contained in  $O_0$ .

(Hf.4)  $f_0 : \Lambda \times Y_0 \rightarrow Y$ ,  $(\lambda, y_0) \mapsto f(\lambda, J_0(y_0))$  has continuous partial derivative

$$D_1 f_0 : \Lambda \times Y_0 \rightarrow \mathcal{L}(\Lambda; Y).$$

(Hf.5) The mapping  $(\lambda, y_0) \mapsto J \circ f^{(1)}(\lambda, J_0(y_0))$  is continuous on  $\Lambda \times O_0$ .

**Lemma 47** (Lemma IX.6.7, Diekmann et al. 2012) *Assume (Hf.1)-(Hf.5) hold. Then  $\Psi$  is locally Lipschitz continuous. Moreover,  $J \circ \Psi$  is of class  $C^1$  with  $D(J \circ \Psi)(\lambda) = J \circ \mathcal{A}(\lambda)$ , where  $\mathcal{A} : \Lambda \rightarrow \mathcal{L}(\Lambda; Y)$  is the solution map of the equation*

$$A = D_1 f_0(\lambda, \Phi(\lambda)) + f^{(1)}(\lambda, \Psi(\lambda))A$$

for  $A$  (whose existence is guaranteed by (Hf.2)-(Hf.4)).

Let  $Y_1, \dots, Y_n$  and  $Z$  be Banach spaces. By convention, we endow product spaces with the norm  $|(y_1, \dots, y_n)|_{Y_1 \times \dots \times Y_n} = \sup\{|y_1|_{Y_1}, \dots, |y_n|_{Y_n}\}$ . We denote by  $M(Y_1, \dots, Y_n; Z)$  the Banach space of bounded multilinear maps

$$Y_1 \times \dots \times Y_n \longrightarrow Z$$

equipped with the norm

$$\|N\|_{M(Y_1, \dots, Y_n; Z)} = \sup\{|N(y_1, \dots, y_n)|_Z \mid |(y_1, \dots, y_n)|_{Y_1 \times \dots \times Y_n} < 1\}.$$

We use the shorthand notation  $M_n(Y; Z)$  for the space  $M(Y, \dots, Y; Z)$ , where  $Y$  appears  $n \geq 0$  times, with the convention that  $M_0(Y; Z) = Z$ .

Recall that  $R_\rho$  is  $C^k$  on  $S$  (c.f. (65)). Let  $1 \leq j \leq k$ , and let  $\xi_i < 0$  for  $i = 1, \dots, j$ ,  $\sum \xi_i =: \xi$ , and  $\eta < 0$  denote real numbers. We may define, if  $\eta \leq \xi$ , the map  $\widehat{R}_\rho^{(j)} : V_\rho^{-\sigma} \rightarrow M_j(BC^{-\xi_1}(\mathbb{R}^{\leq 0}; X), \dots, BC^{-\xi_j}(\mathbb{R}^{\leq 0}; X); BC^{-\eta}(\mathbb{R}^{\leq 0}; X^{\odot*}))$ ,  $\sigma < 0$ , by

$$\left(\widehat{R}_\rho^{(j)}(y)[h_1, \dots, h_j]\right)(t) := D^j R_\rho(y(t))[h_1(t), \dots, h_j(t)], \quad t \leq 0. \quad (112)$$

It is immediate that this is well defined and takes values in  $M_j(X_{-\xi_1}, \dots, X_{-\xi_j}; X_\eta^{\odot*})$  by  $\eta \leq \xi$ .

**Lemma 48** *Let  $1 \leq j \leq k$  and  $\xi_i < 0$  for  $i = 1, \dots, j$ ,  $\sum \xi_i =: \xi$ . Then we have*

- (i) *If  $\eta \leq \xi$ , we have  $\widehat{R}_\rho^{(j)}(y) \in M_j(X_{-\xi_1}, \dots, X_{-\xi_j}; X_\eta^{\odot*})$  for all  $y \in V_\rho^{-\sigma}$ ,  $\sigma < 0$ . The mapping  $y \mapsto \widehat{R}_\rho^{(j)}(y)$  is continuous if  $\eta < j\xi$ .*
- (ii) *If  $\eta < j\xi$ , the map  $J_{-\xi}^{-\eta/j} \circ \widehat{R}_\rho|_{V_\rho^{-\xi}} : V_\rho^{-\xi} \rightarrow BC^{-\eta}(\mathbb{R}^{\leq 0}; X^{\odot*})$  is of class  $C^j$ .*
- (iii) *Let  $\Phi : X_\Sigma \rightarrow V_\rho^{-\sigma}$ ,  $\sigma < 0$ , be a  $C^1$  map. Then the mapping  $\widehat{R}_\rho^{(j)} \circ \Phi : X_\Sigma \rightarrow M_j(X_{-\xi_1}, \dots, X_{-\xi_j}; X_\eta^{\odot*})$  is  $C^1$ , provided that  $\eta < \sigma + \xi$ ; its derivative is given as*

$$D(\widehat{R}_\rho^{(j)} \circ \Phi)(u_1)[v_1, \dots, v_j](s) = \widehat{R}_\rho^{(j+1)}(\Phi(u_1))[v_1(s), \dots, v_j(s), D\Phi(u_1)[\cdot](s)] \quad (113)$$

**Proof** These can be inferred from the corresponding results in Section IX.7, Diekmann et al. (2012), as in the proof of Lemma 44. For clarity, we provide a direct proof of  $\square$

Note that, by its very definition (112),  $\widehat{R}_\rho^{(j)}$  is insensitive to the space  $V_\rho^{-\sigma}$ , in the sense that

$$\widehat{R}_\rho^{(j)}(y) = \widehat{R}_\rho^{(j)} \circ J_{-\sigma}^{-\tau}(y), \quad \text{for } y \in V_\rho^{-\sigma}, \text{ and } \tau < \sigma < 0. \quad (114)$$

For future reference, we also extend the definition of the  $J$  maps to  $M_j$  spaces in the obvious fashion: If  $A \in M_j(X_\Sigma; BC^{-\xi}(\mathbb{R}^{\leq 0}; X))$ , then

$$(J_{-\xi}^{-\eta} \circ A)[v_1, \dots, v_j] := J_{-\xi}^{-\eta}(A[v_1, \dots, v_j]). \quad (115)$$

If  $\widehat{R}_\rho^{(j)} : V_\rho^{-\sigma} \rightarrow M_j(X_{-\xi_1}, \dots, X_{-\xi_j}; X_\eta^{\odot*})$  and  $\zeta < \eta$ , then the map

$$\begin{aligned} J_{-\eta}^{-\zeta} \circ \widehat{R}_\rho^{(j)} : V_\rho^{-\sigma} &\rightarrow M_j(X_{-\xi_1}, \dots, X_{-\xi_j}; X_\zeta^{\odot*}) \\ J_{-\eta}^{-\zeta} \circ \widehat{R}_\rho^{(j)}(y)[v_1, \dots, v_j] &= J_{-\eta}^{-\zeta} \left( \widehat{R}_\rho^{(j)}(y)[v_1, \dots, v_j] \right) \end{aligned}$$

can be factored as

$$J_{-\eta}^{-\zeta} \circ \widehat{R}_\rho^{(j)}(y)[v_1, \dots, v_j] = \widehat{R}_\rho^{(j)}(y)[J_{-\xi_1}^{-\zeta_1} \circ v_1, \dots, J_{-\xi_j}^{-\zeta_j} \circ v_j] \quad (116)$$

for some choice of  $\zeta_i, i = 1, \dots, j$ , with  $\sum \zeta_i = \zeta$ .

The proof of smoothness will be entirely analogous to Section IX.7 of Diekmann et al. (2012) for center manifolds. The only difference in our setting is that we consider a different interval of  $\eta$  possibly bounded away from 0, and that we work with negative semiorbits only – manifest in the spaces  $BC(\mathbb{R}^{\leq 0}; X)$ . It turns out that these changes barely modify the arguments required.

**Proof of the smoothness assertion, Theorem 12(ii)** Let  $\eta \in [\tilde{\eta}, \bar{\eta})$ . To prove that  $\phi_\rho$  is of class  $C^1$ , we apply Lemma 47 with

$$\begin{aligned} O_0 &= V_\rho^{-\bar{\eta}}, \quad Y_0 = V^{-\bar{\eta}}, \quad Y = BC^{-\bar{\eta}}(\mathbb{R}^{\leq 0}; X), \quad Y_1 = BC^{-\eta}(\mathbb{R}^{\leq 0}; X), \\ \Lambda &= X_\Sigma, \quad f(u_1, y) = \mathcal{F}_\rho(u_1, y) = T(\cdot)u_1 + \mathcal{K} \circ \widehat{R}_\rho(y), \quad u_1 \in \Lambda, \quad y \in Y, \\ f^{(1)}(u_1, y) &= \mathcal{K} \circ \widehat{R}_\rho^{(1)}(y) \in \mathcal{L}(Y), \quad u_1 \in \Lambda, \quad y \in J_0(O_0), \\ f_1^{(1)}(u_1, y) &= \mathcal{K} \circ \widehat{R}_\rho^{(1)}(y) \in \mathcal{L}(Y_1), \quad u_1 \in \Lambda, \quad y \in J_0(O_0), \end{aligned}$$

with  $J_0$  being the embedding  $V^{-\bar{\eta}} \hookrightarrow BC^{-\bar{\eta}}(\mathbb{R}^{\leq 0}; X)$  and  $J = J_{-\bar{\eta}}^{-\eta}$ . Note that we have obtained  $\Phi$  from (Hf.3) as  $\mathcal{W}_\rho^{-\bar{\eta}}$  in (104); it satisfies the requirements of (Hf.3) by (109) (it is an elementary fact from point-set topology that  $\mathcal{W}_\rho^{-\bar{\eta}}$  is continuous  $X_\Sigma \rightarrow Y_0$ , Theorem 18.2, Munkres 2000). Let us verify the remaining assumptions of Lemma 47. For (Hf.1), we use Lemma 48(ii) on the term  $J_{-\bar{\eta}}^{-\eta} \circ \mathcal{K} \circ \widehat{R}_\rho \circ J_0 = \mathcal{K} \circ J_{-\bar{\eta}}^{-\eta} \circ \widehat{R}_\rho \circ J_0$ , which shows  $g$  is  $C^1$  – differentiability in the  $\Lambda = X_\Sigma$  direction is a consequence of  $X_\Sigma \subset \text{dom}(A)$  and  $\eta < \alpha$  (c.f. Lemma 41). We have

$$D_2 g(u_1, y_0)[z_0] = J_{-\bar{\eta}}^{-\eta} \circ \mathcal{K} \circ \widehat{R}_\rho^{(1)}(J_0(y_0))[J_0(z_0)] = J_{-\bar{\eta}}^{-\eta} \circ f^{(1)}(u_1, J_0(y_0))[J_0(z_0)]$$

for  $(u_1, y_0, z_0) \in \Lambda \times O_0 \times Y_0$ ; and similarly for the final requirement of (Hf.1). (Hf.2) holds by the choice of  $\rho$  in (103). (Hf.4) is once more a consequence of  $X_\Sigma \subset \text{dom}(A)$  and  $\bar{\eta} < \alpha$ . For (Hf.5), we may apply Lemma 48(i) directly.

Applying Lemma 47, we infer that  $\mathcal{W}_\rho^{-\eta} = J_{-\bar{\eta}}^{-\eta} \circ \mathcal{W}_\rho^{-\bar{\eta}}$  is of class  $C^1$ , the Fréchet derivative of which,  $D\mathcal{W}_\rho^{-\eta}(u_1) \in \mathcal{L}(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X))$ , is the unique solution to

$$\mathcal{F}_\rho^1(u_1, A^{(1)}) = A^{(1)},$$

where  $\mathcal{F}_\rho^1 : X_\Sigma \times \mathcal{L}(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X)) \rightarrow \mathcal{L}(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X))$  is given by

$$T(\cdot) + \mathcal{K} \circ \widehat{R}_\rho^{(1)}(\mathcal{W}_\rho^{-\bar{\eta}}(u_1))[A^{(1)}]. \quad (117)$$

Since  $\eta$  was arbitrary, we infer this property holds over all  $\eta \in [\tilde{\eta}, \bar{\eta})$ .

Choose  $\sigma_{k-1} < \dots < \sigma_1 < \bar{\eta}$  such that  $k\sigma_{k-1} > \tilde{\eta}$ . For higher degrees of smoothness, we proceed by induction on  $1 \leq j \leq k-1$ . We assume, as in the statement Theorem 12(ii), that the spectral gap satisfies  $\beta < k\alpha < 0$  (this is no less general than the original statement with  $\ell$ ). The induction hypotheses are:

(IH.1) For all  $1 \leq \ell \leq j$ , the map  $\mathcal{W}_\rho^{-\eta}$  is of class  $C^\ell$  for  $\eta \in [\tilde{\eta}, \ell\sigma_\ell]$ .

(IH.2) For  $\eta \in [\tilde{\eta}, j\sigma_j]$ ,  $D^j \mathcal{W}_\rho^{-\eta}(u_1)$  is the unique solution of an equation of the form

$$\mathcal{F}_\rho^j(u_1, A^{(j)}) = A^{(j)},$$

where  $\mathcal{F}_\rho^j : X_\Sigma \times M_j(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X)) \rightarrow M_j(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X))$  is given by

$$\mathcal{F}_\rho^j(u_1, A^{(j)}) = \mathcal{K} \circ \widehat{R}_\rho^{(1)}(\mathcal{W}_\rho^{-\sigma_j}(u_1))[A^{(j)}] + H_j(u_1), \quad \text{for } j \geq 1,$$

where  $H_j : X_\Sigma \rightarrow M_j(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X))$  is given by  $H_1(u_1) = T(\cdot)$  for all  $u_1 \in X_\Sigma$  if  $j = 1$ ; if  $j \geq 2$ ,  $H_j$  is given by a finite sum of terms of the form

$$J_{-j\sigma_j}^{-\eta} \circ \mathcal{K} \circ \widehat{R}_\rho^{(\ell)}(\mathcal{W}_\rho^{-\sigma_j}(u_1)) \left[ D^{r_1} \mathcal{W}_\rho^{-r_1\sigma_j}(u_1), \dots, D^{r_\ell} \mathcal{W}_\rho^{-r_\ell\sigma_j}(u_1) \right] \quad (118)$$

with  $2 \leq \ell \leq j$ ,  $1 \leq r_i < j$  for  $1 \leq i \leq \ell$ , and  $r_1 + \dots + r_\ell = j$ .

(IH.3)  $\mathcal{F}_\rho^j$  is a contraction on the second factor, uniformly in  $\eta \in [\tilde{\eta}, j\sigma_j]$ .

That (IH.1)-(IH.3) hold for  $j = 1$  was shown above; noting that  $\mathcal{W}_\rho^{-\bar{\eta}}(u_1)$  in the argument of  $\widehat{R}_\rho^{(1)}$  may be replaced by  $\mathcal{W}_\rho^{-\sigma_1}(u_1)$ , using (114). (Uniformity of the contraction in (IH.3) is satisfied by the choice of  $\rho$ , as specified in (103).) We may hence suppose (IH.1)-(IH.3) hold for  $2 \leq j < k-1$ ; we show they are then satisfied for  $j+1$ . For this, fix  $\eta \in [\tilde{\eta}, (j+1)\sigma_j]$  and  $\xi \in (\eta, (j+1)\sigma_j)$ .

We shall apply Lemma 47 with

$$\begin{aligned} O_0 &= Y_0 = M_j(X_\Sigma; BC^{-j\sigma_j}(\mathbb{R}^{\leq 0}; X)), \quad Y = M_j(X_\Sigma; BC^{-\xi}(\mathbb{R}^{\leq 0}; X)), \\ Y_1 &= M_j(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X)), \quad \Lambda = X_\Sigma, \quad f = \mathcal{F}_\rho^j, \\ f^{(1)}(u_1, \cdot) &= \mathcal{K} \circ \widehat{R}_\rho^{(1)}(\mathcal{W}_\rho^{-\sigma_j}(u_1)) \in \mathcal{L}(Y), \quad u_1 \in X_\Sigma, \\ f_1^{(1)}(u_1, \cdot) &= \mathcal{K} \circ \widehat{R}_\rho^{(1)}(\mathcal{W}_\rho^{-\sigma_j}(u_1)) \in \mathcal{L}(Y_1), \quad u_1 \in X_\Sigma, \end{aligned}$$

with  $J_0 = J_{-j\sigma_j}^{-\xi}$  and  $J = J_{-\xi}^{-\eta}$  defined as in (115).

We have, for  $g$  as in (Hf.1),

$$g(u_1, A^{(j)}) = J_{-\xi}^{-\eta} \circ \mathcal{K} \circ \widehat{R}_\rho^{(1)}(\mathcal{W}_\rho^{-\sigma_j}(u_1))[J_{-j\sigma_j}^{-\xi} \circ A^{(j)}] + H_j(u_1).$$

Since  $g$  is bounded linear in  $A^{(j)}$ , it is  $C^1$  in its second argument if  $u_1 \mapsto J_{-\xi}^{-\eta} \circ \mathcal{K} \circ \widehat{R}_\rho^{(1)}(\mathcal{W}_\rho^{-\sigma_j}(u_1)) \circ J_{-\sigma_j}^{-\xi}$  is continuous. The latter assertion follows from Lemma 48(i),  $\eta < j\sigma_j$ , and the continuity of  $\mathcal{W}_\rho^{-\sigma_j}$ , as  $\sigma_j < \bar{\eta}$ . With respect to its first argument, we may apply Lemma 48(iii) (to both terms) to conclude it is  $C^1$ , since  $\eta < (j+1)\sigma_j$  and  $\mathcal{W}_\rho^{-\sigma_j} : X_\Sigma \rightarrow V_\rho^{-\sigma_j}$  is  $C^1$ . The rest of (Hf.1) is obvious – noting that  $f^{(1)}$  and  $f_1^{(1)}$  indeed take values in  $\mathcal{L}(Y)$  and  $\mathcal{L}(Y_1)$  by Lemma 48(i). The same argument also shows (Hf.4), given that  $\xi < (j+1)\sigma_j$ . (Hf.2) follows from induction hypothesis (IH.3). (Hf.3) is a consequence of induction hypotheses (IH.1) and (IH.2). (Hf.5) is once more an application of Lemma 48(i), noting that  $\eta < \xi$ .

We may hence apply Lemma 47 to obtain that  $u_1 \mapsto D^j \mathcal{W}_\rho^{-\eta}(u_1)$  is  $C^1$ ; its derivative  $D^{j+1} \mathcal{W}_\rho^{-\eta}(u_1) \in \mathcal{L}(X_\Sigma; M_j(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X))) \cong M_{j+1}(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X))$  satisfies

$$A^{(j+1)} = \mathcal{K} \circ \widehat{R}_\rho^{(1)}(\mathcal{W}_\rho^{-\sigma_j}(u_1))[A^{(j+1)}] + H_{j+1}(u_1), \quad (119)$$

where, by (113),

$$H_{j+1}(u_1) = J_{-(j+1)\sigma_j}^{-\eta} \circ \mathcal{K} \circ \widehat{R}_\rho^{(2)}(\mathcal{W}_\rho^{-\sigma_j}(u_1))[D^j \mathcal{W}_\rho^{-j\sigma_j}(u_1), D\mathcal{W}_\rho^{-\sigma_j}(u_1)] + DH_j(u_1). \quad (120)$$

Note that a sample term of  $H_j(u_1)$  from (118) corresponds to a term

$$\begin{aligned} & J_{-(j+1)\sigma_j}^{-\eta} \circ \mathcal{K} \circ \widehat{R}_\rho^{(\ell+1)}(\mathcal{W}_\rho^{-\sigma_j}(u_1)) \left[ D^{r_1} \mathcal{W}_\rho^{-r_1\sigma_j}(u_1), \dots, D^{r_\ell} \mathcal{W}_\rho^{-r_\ell\sigma_j}(u_1), D\mathcal{W}_\rho^{-\sigma_j}(u_1) \right] \\ & + \sum_{l=1}^{\ell} J_{-(j+1)\sigma_j}^{-\eta} \circ \mathcal{K} \circ \widehat{R}_\rho^{(\ell)}(\mathcal{W}_\rho^{-\sigma_j}(u_1)) \left[ D^{r_1} \mathcal{W}_\rho^{-r_1\sigma_j}(u_1), \dots, D^{r_{l+1}} \mathcal{W}_\rho^{-r_{l+1}\sigma_j}(u_1), \dots, D^{r_\ell} \mathcal{W}_\rho^{-r_\ell\sigma_j}(u_1) \right] \end{aligned} \quad (121)$$

of  $DH_j(u_1)$ , using (113) once more. (The  $j$  inclusion terms appear as a consequence of considering (119) on  $M_{j+1}(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X))$ .) This implies  $\mathcal{W}_\rho^{-\eta}$  is of class  $C^{j+1}$  for  $\eta \in [\tilde{\eta}, (j+1)\sigma_j]$ .

Thus, in particular,  $\mathcal{W}_\rho^{-\eta}$  is of class  $C^{j+1}$  for  $\eta \in [\tilde{\eta}, (j+1)\sigma_{j+1}]$ , showing (IH.1) for  $j+1$ . On this interval of  $\eta$ , (IH.2) will be shown if we can modify all  $\sigma_j$ 's in (119), (120) and (121) to  $\sigma_{j+1}$ . For the  $\sigma_j$ 's appearing in the first argument of  $\widehat{R}_\rho^{(j)}$ , we may simply use (114) – this already treats (119). For the rest, we separate the  $J_{-(j+1)\sigma_j}^{-\eta}$  terms into  $J_{-(j+1)\sigma_{j+1}}^{-\eta} \circ J_{-(j+1)\sigma_j}^{-(j+1)\sigma_{j+1}}$  and move the  $J_{-(j+1)\sigma_j}^{-(j+1)\sigma_{j+1}}$  term to the terms within square brackets via (116). Note that for each term, there is a unique choice of  $(\zeta_i)$  – in the notation of (116) – that transforms each term to their  $\sigma_{j+1}$  counterpart. For instance, the last term(s) in (121) have  $\zeta = (r_1\sigma_{j+1}, \dots, (r_l+1)\sigma_{j+1}, \dots, r_\ell\sigma_{j+1})$ . The rest of the terms can be treated similarly. Noting also that  $J_{-r_l\sigma_j}^{-r_l\sigma_{j+1}} D^{r_l} \mathcal{W}_\rho^{-r_l\sigma_j}(u_1) = D^{r_l} \mathcal{W}_\rho^{-r_l\sigma_{j+1}}(u_1)$ , we arrive at the form (IH.2) for  $j+1$ . Finally, (IH.3) for  $j+1$  follows from its version for  $j$ , noting the form of (119).  $\square$

Tangency at the origin can be deduced in the exact same way as for center manifolds. (Note that this has no assumptions on the spectral gap, since neither does  $C^1$  smoothness.)

**Corollary 49** (Corollary IX.7.10, Diekmann et al. 2012) *The map  $\phi_\rho$  from (106),*

$$\phi_\rho = \tilde{P}_{\Sigma'} \circ \text{ev}_0 \circ \mathcal{W}_\rho^{-\eta} : X_\Sigma \rightarrow X_{\Sigma'}$$

*satisfies  $D\phi_\rho(0) = 0$ .*

**Proof** Fix  $\eta \in [\tilde{\eta}, \bar{\eta})$ . Then,  $D\mathcal{W}_\rho^{-\eta}(0)$  must satisfy  $\mathcal{F}_\rho^1(0, D\mathcal{W}_\rho^{-\eta}(0)) = D\mathcal{W}_\rho^{-\eta}(0)$ , with  $\mathcal{F}_\rho^1$  as in (117). Hence, noting that  $\hat{R}_\rho^{(1)}(0) = 0$ ,

$$D\mathcal{W}_\rho^{-\eta}(0) = T(\cdot)\iota_\Sigma \in \mathcal{L}(X_\Sigma; BC^{-\eta}(\mathbb{R}^{\leq 0}; X)), \quad (122)$$

where  $\iota_\Sigma : X_\Sigma \hookrightarrow X$  is the inclusion. Writing out the definition of  $\phi_\rho$ , we have

$$D\phi_\rho(0) = \tilde{P}_{\Sigma'} \circ \text{ev}_0 \circ D\mathcal{W}_\rho^{-\eta}(0) = \tilde{P}_{\Sigma'} \circ \iota_\Sigma = 0.$$

□

## Appendix D Proof of Lemma 26

Recall that the manifold  $W^\Sigma$  was obtained upon restricting  $W_\rho^\Sigma$  to a small enough neighbourhood of the origin (Section 6.4). Since the conclusions of the lemma are local, we may as well consider  $W_\rho^\Sigma$ . Recall also that  $W_\rho^\Sigma$  can be obtained as the fixed point of the contraction mapping (102); explicitly (c.f. (104) – we dispose of the  $\rho$  and  $\eta$  sub/superscripts),

$$\begin{aligned} [\mathcal{W}(u_1)](t) &= T(t)u_1 - \iota^{-1} \int_t^0 T^{\odot*}(t-s)P_{\Sigma'}^{\odot*}R(\mathcal{W}(u_1)(s))ds \\ &\quad + \iota^{-1} \int_{-\infty}^t T^{\odot*}(t-s)P_{\Sigma'}^{\odot*}R(\mathcal{W}(u_1)(s))ds, \end{aligned} \quad (123)$$

where  $\mathcal{W} : X_\Sigma \rightarrow BC^{-\eta}(\mathbb{R}^{\leq 0}; X)$  maps points of  $X_\Sigma$  to negative semiorbits in  $W_\rho^\Sigma$ , and where  $\eta \in (\beta, \alpha)$  is chosen such that  $\mathcal{W}$  is  $C^\ell$  (as in Appendix C).

The desired embedding  $K$  is simply (a restriction of)  $\text{ev}_0 \circ \mathcal{W}$ ,

$$K(u_1) = u_1 + \iota^{-1} \int_0^\infty T^{\odot*}(s)P_{\Sigma'}^{\odot*}R(\mathcal{W}(u_1)(-s))ds.$$

Considering  $\mathcal{W}(0) = 0$  and  $DR(0) = 0$ , we obtain  $DK(0) = \iota_\Sigma$  (as in Corollary 49). Differentiating once more,

$$D^2K(0) = \iota^{-1} \int_0^\infty T^{\odot*}(s)P_{\Sigma'}^{\odot*}D^2R(0)[D\mathcal{W}(0)(-s), D\mathcal{W}(0)(-s)]ds.$$

Considering that  $DW(0)(s) = T(s)_t \Sigma$  (see (122)), we arrive at the desired formula.

Similarly to Section 7, the reduced dynamics can be written as

$$\psi_t(u_1) = \tilde{P}_\Sigma \circ \text{ev}_t \circ \mathcal{W}(u_1) \quad (124)$$

where defined. To obtain the vector field  $H$ , we differentiate (124) in time (we may do this since  $\psi$  is jointly  $C^\ell$ , as per Theorem 12). Explicitly, via (123), we have

$$\frac{d}{dt} \Big|_{t=0} \psi_t(u_1) = A_\Sigma u_1 + \tilde{P}_\Sigma^{-1} P_\Sigma^{\odot*} R(\mathcal{W}(u_1)(0)),$$

hence

$$H = A_\Sigma + \tilde{P}_\Sigma^{\odot*} R \circ K.$$

The expansion formula follows immediately, considering  $DR(0) = 0$ .

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