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INFINITE DIMENSIONAL GEOMETRIC SINGULAR PERTURBATION THEORY FOR THE MAXWELL–BLOCH EQUATIONS*

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Abstract. We study the Maxwell–Bloch equations governing a two-level laser in a ring cavity. For Class A lasers, these equations have two widely separated time scales and form a singularly perturbed, semilinear hyperbolic system with two distinct characteristics. We extend Fenichel's geometric singular perturbation theory [N. Fenichel, *J. Differential Equations*, 31 (1979), pp. 53–98] to the Maxwell–Bloch equations by proving the persistence of a $C^k, 0 < k < \infty$, slow manifold under an unbounded perturbation. The proof is obtained by a modified graph transform method. We use uniform decay estimates of Constantin, Foias, and Gibbon [*Nonlinearity*, 2 (1989), pp. 241–269] to obtain a cone condition. These estimates rely on the energy preserving nature of the nonlinearity and the existence of two distinct characteristics. The cone condition and the fact that the unbounded perturbation generates a continuous group are used to define the graph transform. The slow manifold is a globally attracting, positively invariant manifold, with infinite dimension and codimension, that contains the attractor of the system. The slow manifold depends only continuously on ε and converges uniformly on (strongly) compact sets to the critical manifold. This enables us to rigorously decouple the slow and fast time scales and obtain a reduced (but still infinite-dimensional) dynamical system described by a functional differential equation.

Key words. Maxwell–Bloch equations, invariant manifolds, geometric singular perturbation theory

AMS subject classifications. 58F30, 35Q, 78A60, 37L, 35B

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1. Introduction.

1.1. The Maxwell–Bloch equations. We shall study the asymptotic dynamics of the laser equations proposed by Risken and Nummedal [25]. These are amplitude equations describing a two-level laser derived by a semiclassical approximation. The electric field obeys the classical Maxwell equations, and the light-matter interaction is modeled by the quantum mechanical Bloch equations. There are numerous simplifications in this model, several of which are pointed out in [25]. Nevertheless, these equations are quite faithful to the underlying physics and are also mathematically tractable in certain limits. The equations we will study are

$$E_\tau + E_x = \kappa(P - E), \tag{1.1}$$

$$P_\tau = \gamma_\perp[ED - (1 + i\delta)P], \tag{1.2}$$

$$D_\tau = \gamma_\| \left[\lambda + 1 - D - \frac{\lambda}{2}(E^*P + EP^*)\right]. \tag{1.3}$$

$E, P \in \mathbb{C}$, and $D \in \mathbb{R}$ are periodic on the domain $[0, 1]$; E is the electric field, P is the polarization of the gain medium, and D is a measure of the population inversion;

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$\kappa, \gamma_\perp, \gamma_\parallel > 0$ are phenomenological damping constants; $\lambda > 0$ is a pumping term; $\delta \in \mathbb{R}$ is a detuning parameter. All the variables are dimensionless and have been scaled to the continuous wave (cw) solutions. By these we mean spatially homogeneous steady states of (1.1)–(1.3) that correspond to a steady output from the laser. In another scaling these equations are also known as the Lorenz PDE. The Lorenz ODEs are contained in the system (1.1)–(1.3) when $\delta = 0$, and attention is restricted to real valued, spatially independent solutions.

Constantin, Foias, and Gibbon were the first to study these equations rigorously [9]. They proved the existence of global weak solutions and a C^∞ global attractor in L^2 with finite Hausdorff dimension. Recently Xin and Moloney studied the equations in three dimensions with the addition of a transverse dispersive term [28]. They proved the existence and uniqueness of weak solutions in L^p, $2 \le p < \infty$, and an attractor with finite regularity. Naturally, one must expect the dynamics to depend strongly on the parameter values, and in some parameter ranges the analysis will be easier than in others. Kovacic and Wettergren studied the Maxwell–Bloch equations (in a different scaling) near an integrable limit in [20, 27], respectively. The motivation there is to use the knowledge of the geometry of the integrable limit to understand the dynamics when the damping and forcing are small.

1.2. Adiabatic elimination for Class A lasers. Different types of lasers have vastly different dynamics because of the wide variation in parameters $\kappa, \gamma_\perp$, and $\gamma_\parallel$. Arecchi proposed a characterization of lasers based on the range of damping parameters, and we shall consider what he terms Class A lasers [1, p. 17]. For this class of lasers, we have $\gamma_\perp \approx \gamma_\parallel \gg \kappa$. This scaling has also been called the good cavity limit [10], but strictly speaking, the good cavity limit refers to the case where $\gamma_\perp + \gamma_\parallel \ge \kappa$ and includes a much broader range of dynamics than we consider. Nevertheless, the range of Class A lasers is sufficiently wide to be physically and mathematically interesting. For Class B lasers, $\gamma_\perp \gg \kappa \tilde{>} \gamma_\parallel$, and for Class C lasers, all three damping constants are comparable. For Class A and B lasers one may hope to simplify the dynamics by separating the evolution on fast and slow time scales. Such adiabatic eliminations are common in the physics literature (see the expository article [1] and the references therein). Our aim is to examine such a reduction from a more mathematical viewpoint. The simplest case is of Class A lasers, and we consider only this scaling henceforth.

Let $\gamma_\perp \to \frac{1}{\varepsilon}, \gamma_\parallel \to \frac{\gamma_\parallel}{\varepsilon}$, with $0 < \varepsilon \ll 1$. The laser equations are rewritten as

$$E_\tau + E_x = \kappa(P - E), \tag{1.4}$$

$$\varepsilon P_\tau = ED - (1 + i\delta)P, \tag{1.5}$$

$$\varepsilon D_\tau = \gamma_\parallel \left[\lambda + 1 - D - \frac{\lambda}{2}(E^* P + EP^*)\right]. \tag{1.6}$$

This suggests we formally eliminate the "fast" variables P and D, i.e., we set the left-hand sides of (1.5) and (1.6) to zero, solve for P, D as functions of E, and substitute in (1.4). This adiabatic approximation is often used in the physics literature [1, 10, 15] although it is typically used with finite-dimensional modal truncations [16, pp. 156, 290]. It is not apparent that this formal reduction is valid or if the solutions to the singular limit are similar to those of the full system. Indeed, we will show that this reduction leads to false predictions about the asymptotic behavior.

The failure of the formal reduction should not be unexpected. The laser equations are a semilinear hyperbolic system with two characteristics: $x - t$ =constant and

x =constant. The formal reduction procedure eliminates one of these characteristics and thus neglects essential information. Nevertheless, there is some merit in studying the reduced system since it provides some insight into the range of possible asymptotic behavior. We may then attempt to verify if similar behavior persists when $\gamma_\perp$ and $\gamma_\|$ are sufficiently large but finite.

1.3. Geometric singular perturbation theory. A rigorous geometric theory for singularly perturbed ODE was developed by Fenichel [12]. To apply his methods to this problem, one would proceed as follows. First, one regularizes the problem by rescaling time, $t = \frac{\tau}{\varepsilon}$. The new time scale, t, is referred to as fast time. In this variable, the laser equations are

$$E_t = \varepsilon[-E_x + \kappa(P - E)], \tag{1.7}$$

$$P_t = ED - (1 + i\delta)P, \tag{1.8}$$

$$D_t = \gamma_\| \left[\lambda + 1 - D - \frac{\lambda}{2}(E^*P + EP^*)\right]. \tag{1.9}$$

Here E changes slowly with time ($O(\varepsilon)$), and P and D have a time rate of change that is $O(1)$. In the limit $\varepsilon = 0$ the slow variable E is constant. The fast variables still change rapidly except at the equilibria of (1.7)–(1.9). Solving for these equilibria we see that they form a manifold, $\mathcal{M}_0$, given as a graph over the slow variable E. Thus the singularities of the slow time system (1.4)–(1.6) are equilibria of the fast time system (1.7)–(1.9). The formal reduction is equivalent to the assumption that $\mathcal{M}_0$ remains invariant and there is a well-defined flow in slow time restricted to it. How good is this assumption? Some intuition is provided by considering ODE.

The underlying geometry is essentially the same for singularly perturbed ODE. We are given a manifold of equilibria and we want to justify a reduction of the flow to this manifold. Under the crucial hypothesis of *normal hyperbolicity*, Fenichel proved that a compact manifold of equilibria, $\mathcal{M}_0$, continues smoothly to a family of *slow manifolds*, $\mathcal{M}_\varepsilon$, for sufficiently small $\varepsilon > 0$. Furthermore, if we consider the augmented system obtained by appending the equation $\varepsilon_t = 0$ to the ODE, then these manifolds are contained in a global center manifold given as a graph over the slow variable and ε. The singular perturbation problem is then reduced to a regular perturbation problem restricted to this center manifold, and asymptotic expansions in ε are reduced to Taylor series calculations. Fenichel's methods are powerful, and several problems that lie outside the reach of conventional (and typically heuristic) asymptotic methods are easily studied within his framework. There has been much progress in this area; see [19] for a readable introduction.

For PDE the situation is not so simple. There are several obstructions, some of which are technical; for instance, the phase space is no longer locally compact. But another obstruction is essential. For $\varepsilon > 0$ the perturbed flow is not close to the unperturbed flow in the C^1 topology because of the unbounded operator $\varepsilon\partial_x$. Hence there are no general persistence theorems that one can invoke to prove the existence of the slow manifolds $\mathcal{M}_\varepsilon$ (the definitive results in this direction are due to Bates, Lu, and Zeng and may be found in a set of articles beginning with [4]). Furthermore, even if these manifolds exist, one should not expect them to fit together smoothly in ε or to be contained in a smooth global center manifold restricted to which we obtain a regular perturbation problem. Thus there are difficulties in justifying the existence of asymptotic expansions. The addition of an unbounded perturbation also leads to some unexpected phenomenon. For example, one finds new instabilities that are hidden in the adiabatic elimination. Risken and Nummedal [25] showed that the cw solutions of (1.1)–(1.3) are linearly unstable for sufficiently large λ. On the other hand, the adiabatic elimination predicts that these solutions are always stable. We comment on this point again in section 6.

1.4. Main results. The goals of this paper are to understand rigorously the relation between the adiabatic elimination and the full Maxwell–Bloch system and to develop in the process geometric singular perturbation theory for PDE in the setting of a concrete example. Our main theorem, Theorem 4.1, establishes the persistence of a globally attracting, positively invariant manifold diffeomorphic to the manifold of equilibria. This manifold contains the attractor of the system. In infinite dimensions, the persistence of a global invariant manifold under unbounded perturbations is itself a significant fact, and Theorem 4.1 lies considerably outside the scope of general theorems in this field (see, e.g., [2, 3, 8, 4]). This being said, we must in fairness note that the proof relies strongly on the structure of the Maxwell–Bloch equations and is special to this system. There are two facts that play a key role in the analysis. The first is that the addition of the unbounded perturbation for $\varepsilon > 0$ corresponds to the splitting of characteristics that are parallel in the limit. The second is that the nonlinearities of the Maxwell–Bloch equations satisfy strong energy estimates that follow from their physical origin. In particular, the nonlinearities in (1.1)–(1.3) appear only as skew terms and ensure the uniform decay of the polarization and inversion (see 2.5). These estimates were derived by Constantin, Foias, and Gibbon [9]. We utilize these estimates to establish a cone condition of the flow similar to that in [4, 13]. The cone condition, and the fact that the unbounded perturbation generates a continuous group, are crucial ingredients of the proof.

The convergence of the slow manifold to the critical manifold is subtly altered by the unbounded perturbation. We are only able to prove that the convergence is uniform on strongly compact sets (Theorem 6.1). However, one should keep in mind that the phase space is not locally compact.

As a sidelight, we note that the persistence theorem provides an example of an inertial manifold (albeit infinite dimensional) in a problem with no diffusion. Infinite-dimensional inertial manifolds for reaction diffusion equations coupled to ODE (e.g., the Hodgkin–Huxley equations) have been studied by Marion [22]. Marion's methods are a natural complement to methods used for reaction diffusion equations [13] and depend on the control over high wave numbers provided by diffusion. Our methods are quite different and depend strongly on the absence of diffusion.

The rest of this paper is organized as follows. Section 2 contains a priori estimates and results on well-posedness. Section 3 studies the peculiarities of the singular limit. Sections 4 and 5 are dedicated to a proof of the main theorem. The existence of the invariant manifold provides a basis for rigorously decoupling the slow and fast time scales in the system. This is considered in section 6. We also remark on the relation between the formal limit and the slow dynamics there.

2. Existence and uniqueness. In this section we will prove that the laser equations define a smooth (C^∞) dynamical system in the space of continuous functions. Constantin, Foias, and Gibbon [9] proved that the laser equations define a Lipschitz dynamical system in L^2. The reason for choosing a more restrictive phase space is that smoothness of the flow is essential for invariant manifold techniques. The obstruction to smoothness in L^2 is the quadratic nonlinearity in (1.8) and (1.9). The product of two L^2 functions does not lie in L^2 in general. For continuous functions, however, multiplication is a smooth map. The motivation for choosing L^2 as a phase space is that the laser rises out of noise and the initial data cannot be prepared to be smooth. In view of this, choosing C^0 as the phase space is obviously a restriction in our study. Nevertheless, Theorem 4.2 of [9] states that asymptotically all solutions approach an attractor composed of C^∞ functions. Thus, in order to study asymptotic behavior it is sufficient to restrict our attention to continuous functions.

Our work relies strongly on the a priori estimates proved by Constantin, Foias, and Gibbon and the estimates of this section are largely rescaled versions of their work [9]. A derivation of these estimates, motivated by the underlying physics, may be found in their work. We make no claim to originality for these estimates, and they are included for completeness and to prove well-posedness of the laser equations in a form appropriate for this paper.

To better illustrate the structure of the equations, we rescale the dependent variables. Define $\mu = \sqrt{\lambda\gamma_\|}$ and set

$$u = E\ , \quad v = \mu P,\ w = D. \tag{2.1}$$

Thus, (1.7)–(1.9) are transformed to

$$u_t = \varepsilon\left[-u_x + \kappa\left(\frac{v}{\mu} - u\right)\right], \tag{2.2}$$

$$v_t = \mu u w - (1 + i\delta)v, \tag{2.3}$$

$$w_t = \gamma_\|(\lambda + 1 - w) - \frac{\mu}{2}(u^* v + u v^*). \tag{2.4}$$

2.1. Notation. The space of continuous functions from the circle into a Euclidean space $\mathbb{E}$ is denoted by $C(S^1;\mathbb{E})$. The phase space for our dynamical system is $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$, where $\mathbb{X}_1 = C(S^1;\mathbb{C})$ and $\mathbb{X}_2 = C(S^1;\mathbb{C}\times\mathbb{R})$. A typical element of $\mathbb{X}$ is denoted by the triplet (u, v, w). The norm in $\mathbb{X}_1$ is $\|u\| = \sup_{x\in S^1}|u(x)|$, and the norm of $(v, w) \in \mathbb{X}_2$ is $\|(v, w)\| = \sup_{x\in S^1}(|v(x)|^2 + |w(x)|^2)^{1/2}$. The norm of $(u, v, w) \in \mathbb{X}$ is $(\|u\|^2 + \|(v, w)\|^2)^{1/2}$. The projections from $\mathbb{X}$ into $\mathbb{X}_i$ are denoted by Π_i. The space of k-linear maps between two Banach spaces $\mathbb{Y}_1$ and $\mathbb{Y}_2$ will be denoted as $L^k(\mathbb{Y}_1, \mathbb{Y}_2)$. For $k = 1$, we drop the superscript.

2.2. A priori estimates. Notice that if $\varepsilon > 0$, it is sufficient to obtain a priori estimates for either the slow or fast system since they are equivalent. In the rest of this section $\varepsilon > 0$ is fixed.

We first derive a pointwise decay estimate. For all $x \in S^1$, we have

$$\begin{aligned}\partial_t(|v(t,x)|^2 + |w(t,x)|^2) &= -2|v|^2 - 2\gamma_\||w|^2 + 2(\lambda+1)\gamma_\| w\\ &\le -2\beta(|v|^2 + |w|^2) + \gamma_\|(\lambda+1)^2,\end{aligned}$$

where $\beta = \min(1, \gamma_\|/2)$. Integrating the resulting inequality and taking the sup over $x \in S^1$ we obtain

$$\begin{aligned}\|(v,w)(t)\|^2 &\le e^{-2\beta t}(\|(v,w)(0)\|^2) + (\lambda+1)^2\frac{\gamma_\|}{2\beta}|1 - e^{-2\beta t}|\\ &=: e^{-2\beta t}(\|(v,w)(0)\|^2) + \rho_v^2|1 - e^{-2\beta t}|,\end{aligned} \tag{2.5}$$

where we have defined the constant $\rho_v^2 = \gamma_\|(\lambda+1)^2/2\beta$. The miraculous cancellation in the nonlinear terms that leads to this strong energy estimate is actually a consequence of the underlying physics; see [9] for details. Since the nonlinear terms uv^* and uw in (2.3)–(2.4) do not influence the change in energy, we say that the nonlinearity is energy preserving. Equation (2.2) admits an equally strong estimate. A smooth solution satisfies

$$(\partial_t + \varepsilon\partial_x)(|u(t,x)|^2) = -2\kappa\varepsilon|u(t,x)|^2 + 2\frac{\varepsilon\kappa}{\mu}\,\mathrm{Re}\,(u^*v) \\ \le -\varepsilon\kappa|u(t,x)|^2 + \frac{\varepsilon\kappa}{\mu^2}|v(t,x)|^2.$$

Integrating this inequality along the characteristic $x - \varepsilon t$ =constant, we have

$$|u(t,x)|^2 \le e^{-\varepsilon\kappa t}|u(0,x-\varepsilon t)|^2 + \frac{\varepsilon\kappa}{\mu^2}\int_0^t e^{-\varepsilon\kappa(t-s)}|v(s,x-\varepsilon(t-s))|^2 ds. \tag{2.6}$$

Taking the sup over $x \in S^1$, and using the energy estimate (2.5), we obtain

$$\|u(t)\|^2 \le e^{-\varepsilon\kappa t}\|u(0)\|^2 + \frac{\varepsilon\kappa}{\mu^2}e_{2\beta}(t)\|(v,w)(0)\|^2 + \frac{\rho_v^2}{\mu^2}(1-e^{-\varepsilon\kappa t}), \tag{2.7}$$

where we have defined the exponentially decaying function

$$e_\alpha(t) = \frac{e^{-\varepsilon\kappa t} - e^{-\alpha t}}{\alpha - \varepsilon\kappa}, \tag{2.8}$$

assuming that $\varepsilon\kappa < \alpha$.

These energy estimates will be used to establish the existence of global mild solutions. They also immediately establish the existence of positively invariant regions in $\mathbb{X}$. Trajectories will satisfy $\|(v,w)(t)\| < \|(v,w)(0)\|$, for all $t > 0$, provided

$$\|(v,w)(0)\| > \rho_v. \tag{2.9}$$

Let $c(\varepsilon) = \sup_{t\ge 0} \varepsilon\kappa e_{2\beta}(t)/(1-e^{-\varepsilon\kappa t})$. Since $e_{2\beta}(t) \le te^{-\varepsilon\kappa t}$ we find that

$$c(\varepsilon) \le \sup_{y>0}\frac{y}{e^y - 1} = 1.$$

Suppose that the initial conditions satisfy (2.9). The energy estimate (2.7) shows that a sufficient condition for $\|u(t)\| < \|u(0)\|$ for all $t > 0$ is

$$\|u(0)\|^2 > \frac{\|(v,w)(0)\|^2}{\mu^2} + \frac{\rho_v^2}{\mu^2}. \tag{2.10}$$

Conditions (2.9) and (2.10) show that the region

$$\mathcal{D}_0 = \{\|u\|^2 \le 4\rho_v^2/\mu^2, \|(v,w)\|^2 \le 2\rho_v^2\} \tag{2.11}$$

is strictly positively invariant. $\mathcal{D}_0$ is also an absorbing region for the flow. The energy estimates (2.5) and (2.7) show that all trajectories enter $\mathcal{D}_0$ at the slow exponential rate $e^{-\varepsilon\kappa t}$ and that the time taken to enter $\mathcal{D}_0$ is uniform on bounded sets.

Remark 2.1. It is important to note that the size of the absorbing region is uniform for $0 < \varepsilon\kappa < 2\beta$. We will use this in our construction of slow manifolds.

Let $(u_i, v_i, w_i), i = 1, 2$, be two smooth solutions. We will estimate the growth of their difference. Define $(\xi,\eta,\zeta) = (u_1,v_1,w_1) - (u_2,v_2,w_2)$ and $(\bar{u},\bar{v},\bar{w}) = ((u_1,v_1,w_1) + (u_2,v_2,w_2))/2$. The differences satisfy

$$\xi_t = \varepsilon\left[-\xi_x + \kappa\left(\frac{\eta}{\mu} - \xi\right)\right], \tag{2.12}$$

$$\eta_t = -(1+i\delta)\eta + \mu(\bar{u}\zeta + \bar{w}\xi), \tag{2.13}$$

$$\zeta_t = -\gamma_{\|}\zeta - \mu\,\mathrm{Re}\,(\bar{u}^*\eta + \bar{v}^*\xi). \tag{2.14}$$

Equations (2.13) and (2.14) give the pointwise error estimate

$$\partial_t(|\eta|^2+|\zeta|^2) = -2|\eta|^2 - 2\gamma_{\|}|\zeta|^2 + 2\mu \operatorname{Re}(\xi\eta^*\bar{w} - \xi\zeta\bar{v}^*)$$
$$\le -\beta(|\eta|^2+|\zeta|^2) + \frac{\mu^2|\xi|^2}{\beta}(|\bar{v}|^2+|\bar{w}|^2).$$

In the second step we have used the elementary inequality $2pq \le \beta p^2 + q^2/\beta$. Integrating and taking the sup over $x \in S^1$, we obtain

$$\|(\eta,\zeta)(t)\|^2 \le e^{-\beta t}\|(\eta,\zeta)(0)\|^2 + \frac{\mu^2}{\beta}\int_0^t e^{-\beta(t-s)}\|\xi(s)\|^2\|(\bar{v},\bar{w})(s)\|^2 ds. \tag{2.15}$$

The definition of $(\bar{v},\bar{w})$, combined with the energy estimate (2.5), gives

$$\|(\bar{v},\bar{w})(t)\|^2 \le C \quad \forall t \ge 0, \tag{2.16}$$

where C is a constant that is uniform for initial conditions in any fixed ball.

An energy estimate for ξ can be obtained from (2.12). For two smooth solutions we have

$$(\partial_t + \varepsilon\partial_x)(|\xi|^2) = 2\varepsilon\kappa\left(-|\xi|^2 + \frac{1}{\mu}\operatorname{Re}(\xi^*\eta)\right)$$
$$\le 2\varepsilon\kappa\left(-|\xi|^2 + \frac{|\xi|^2}{2} + \frac{|\eta|^2}{2\mu^2}\right) = -\varepsilon\kappa|\xi|^2 + \frac{\varepsilon\kappa}{\mu^2}|\eta|^2.$$

Integrating this inequality along the characteristic $x - \varepsilon t =$constant, we have

$$|\xi(t,x)|^2 \le e^{-\varepsilon\kappa t}|\xi(0,x-\varepsilon t)|^2 + \frac{\varepsilon\kappa}{\mu^2}\int_0^t e^{-\varepsilon\kappa(t-s)}|\eta(s,x-\varepsilon(t-s))|^2 ds,$$

and taking the sup over $x \in S^1$ we obtain

$$\|\xi(t)\|^2 \le e^{-\varepsilon\kappa t}\|\xi(0)\|^2 + \frac{\varepsilon\kappa}{\mu^2}\int_0^t e^{-\varepsilon\kappa(t-s)}\|(\eta,\zeta)(s)\|^2 ds. \tag{2.17}$$

Since the expressions $\|\xi(t)\|^2$ and $\|(\eta(t),\zeta(t))\|^2$ occur often below, we now introduce separate notation for them. Let

$$a(t) = \|\xi(t)\|^2, \quad b(t) = \|(\eta,\zeta)(t)\|^2. \tag{2.18}$$

Combining the inequalities (2.15), (2.16), and (2.17) with the notation of (2.18), we obtain

$$b(t) \le b(0)e^{-\beta t} + \frac{C\mu^2}{\beta}\int_0^t e^{-\beta(t-s)}\left(e^{-\varepsilon\kappa s}a(0) + \frac{\varepsilon\kappa}{\mu^2}\int_0^s e^{-\varepsilon\kappa(s-\tau)}b(\tau)d\tau\right)ds$$
$$= b(0)e^{-\beta t} + \frac{C\mu^2}{\beta}a(0)e_\beta(t) + \frac{C\varepsilon\kappa}{\beta}\int_0^t\int_0^s e^{-\beta(t-s)-\varepsilon\kappa(s-\tau)}b(\tau)d\tau ds$$
$$= b(0)e^{-\beta t} + \frac{C\mu^2}{\beta}a(0)e_\beta(t) + \frac{C\varepsilon\kappa}{\beta}\int_0^t b(\tau)e^{-\beta t+\varepsilon\kappa\tau}\int_\tau^t e^{(\beta-\varepsilon\kappa)s}ds d\tau.$$

Computing the inner integral we obtain the estimate

$$b(t) \le b(0)e^{-\beta t} + \frac{C\mu^2}{\beta}a(0)e_\beta(t) + \frac{C\varepsilon\kappa}{\beta}\int_0^t e_\beta(t-\tau)b(\tau)d\tau. \tag{2.19}$$

Here $e_\beta(t)$ is defined as in (2.8). Thus, we have $e_\beta(t) \le te^{-\varepsilon\kappa t}$ for positive times (we suppose that $0 < \varepsilon\kappa < \beta$). As in [9], we apply Gronwall's inequality to (2.19), and use the resulting estimate in (2.17) to obtain

$$\sup_{t\in[0,T]} (a(t) + b(t)) \le C(T, \|(u_i, v_i, w_i)(0)\|)(a(0) + b(0)).$$

We have made the assumption that $t \ge 0$ for simplicity. One may work through the estimates again to find that for any fixed $T > 0$,

$$\sup_{t\in[-T,T]} (a(t) + b(t)) \le C(T, \|(u_i, v_i, w_i)(0)\|)(a(0) + b(0)). \tag{2.20}$$

2.3. Existence of a smooth flow. We now define precisely the dynamical system we will be studying and then prove the existence of a smooth global flow.

DEFINITION 2.2. $(u(t), v(t), w(t)) \in \mathbb{X}$ *is a* mild solution *to the laser equations* (2.2)–(2.4) *if it satisfies the integral equations*

$$u(t) = e^{-\varepsilon\kappa t}e^{-\varepsilon t\partial_x}u(0) + \frac{\varepsilon\kappa}{\mu}\int_0^t e^{-\varepsilon\kappa(t-s)}e^{-\varepsilon(t-s)\partial_x}v(s)ds, \tag{2.21}$$

$$v(t) = e^{-(1+i\delta)t}v(0) + \mu\int_0^t e^{-(1+i\delta)(t-s)}u(s)w(s)ds, \tag{2.22}$$

$$w(t) = e^{-\gamma_\| t}w(0) + (\lambda+1)(1-e^{-\gamma_\| t}) - \mu\int_0^t e^{-\gamma_\|(t-s)}\,\mathrm{Re}\,(u(s)^*v(s))ds. \tag{2.23}$$

The notation $e^{-\varepsilon t\partial_x}$, *with* $t \in \mathbb{R}$, *refers to the one parameter linear group generated by the wave equation* $u_t + \varepsilon u_x = 0$ *in* $C(S^1;\mathbb{C})$. *It is defined by the shift map* $(e^{-\varepsilon t\partial_x}u)(x) = u(x - \varepsilon t)$.

Remark 2.3. The integrals in (2.21)–(2.23) are interpreted as elements in $\mathbb{X}$. Since we are considering continuous functions, the integrals are well defined if and only if they are defined at each point $x \in S^1$. Notice that the product $u(s)w(s)$ is a well-defined continuous function. In [9] the laser equations do not admit a variation of constants formula in L^2. In our work a variation of constants formula is essential.

Remark 2.4. The integral equations (2.22) and (2.23) are equivalent to the differential equations (2.3) and (2.4) since the right-hand side of the differential equations contains no unbounded operator. Thus the a priori estimates (2.5) and (2.15) apply to *all* mild solutions, not just smooth solutions. The a priori estimates on u and ξ, (2.7) and (2.17), are extended to all mild solutions by approximating continuous functions with C^1 functions.

THEOREM 2.5 (C^∞ flow). *The laser equations define a* C^∞ *global flow in the sense of mild solutions. That is, there exists a* C^∞ *map* $\Phi : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ *with the following properties:*

(a) $\Phi(t, u_0, v_0, w_0)$ *is the unique solution to* (2.21)–(2.23) *with initial conditions* $\Phi(0, u_0, v_0, w_0) = (u_0, v_0, w_0)$.

(b) *The set of maps* $\varphi_t : \mathbb{X} \to \mathbb{X}, t \in \mathbb{R}$ *defined by* $\varphi_t(u_0, v_0, w_0) = \Phi(t, u_0, v_0, w_0)$ *is a one parameter group of* C^∞ *diffeomorphisms of* $\mathbb{X}$.

Sketch of the proof. A contraction mapping argument shows that for every point in $\mathbb{X}$ within the ball of radius ρ there is a unique mild solution defined for a time interval $[-T(\rho), T(\rho)]$. A well-known theorem of Segal [26] asserts that solutions fail to exist after a finite time, T_{crit}, if and only if they blow up, i.e., $\|(u(t), v(t), w(t))\| \to \infty$ as $t \to T_{crit}$. The a priori estimates (2.5) and (2.7) show that this is impossible.

Thus through every point (u_0, v_0, w_0) there is a unique solution for all $t \in \mathbb{R}$ denoted by $\Phi(t, u_0, v_0, w_0)$ with $\Phi(0, u_0, v_0, w_0) = (u_0, v_0, w_0)$. Let φ_t be defined as in (b). Clearly, $\varphi_0 = \text{Id}$. Then (2.20) shows that each φ_t is continuous (in fact, locally Lipschitz). The group property follows from uniqueness of solutions.

The proof that the flow is C^∞ is by induction on the order of the derivative. Each step of the argument follows. For any positive integer r, formal differentiation of the equations for the $(r-1)$th derivative yields a linear integral equation that the rth derivative must satisfy (for $r = 1$, we differentiate (2.21)–(2.23)). The existence of a unique solution to this integral equation on a time interval $[-T(\rho, r), T(\rho, r)]$ is proven by a contraction mapping argument. Gronwall estimates show that the derivative grows at worst exponentially in time. Thus, the derivatives of the flow are defined for all $t \in \mathbb{R}$. This is a standard calculation (see, e.g., [6, 18]) and we omit the details. The heart of the matter is that the nonlinear terms on the right-hand side of (2.22)–(2.23) are smooth, and thus all derivatives exist. □

Remark 2.6. We did not need the full strength of the estimates for differences (2.15)–(2.17) in this proof. The estimates will be used in section 4 to prove the cone property.

2.4. Asymptotic dynamics. The laser equations are dissipative. All trajectories must enter the trapping region $\mathcal{D}_0$ in finite time. To capture the asymptotic behavior of the system, we define the global attractor

$$\mathcal{A} = \bigcap_{t \geq 0} \varphi_t(\mathcal{D}_0).$$

Since $\mathcal{D}_0$ is absorbing and closed, this agrees with the definition of the attractor as the ω-limit set of the absorbing ball

$$\omega(\mathcal{D}_0) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi_t(\mathcal{D}_0)}.$$

Although the flow is dissipative, it is not smoothing, and it is not obvious that this definition of the attractor is meaningful. However, this follows from the asymptotic smoothing property of the laser equations proved by Constantin, Foias, and Gibbon [9]; see also [23]. Let $\mathcal{B}$ denote the attractor in L^2. The main result of Constantin, Foias, and Gibbon is that $\mathcal{B}$ is composed of C^∞ functions and that it has finite Hausdorff dimension. Thus it also has finite topological dimension. In [23] the regularity result was improved: The attractor $\mathcal{B}$ is in every Gevrey class G^s for $s > 1$, i.e., the attractor is "almost analytic." Furthermore, the estimates in [9, 23] show that the attractor is compact by the Arzela–Ascoli theorem. Since the inclusion $\iota : \mathbb{X} \to L^2$ is continuous, these results apply immediately to the flow in $\mathbb{X}$. Applying the regularity result we see that $\iota(\mathcal{A}) = \mathcal{B}$. Furthermore, since $\mathcal{B}$ is compact, the inverse map restricted to $\mathcal{B}$ is continuous. Hence $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic and have the same topological dimension.

These theorems are independent of the scaling assumptions of our paper. We assert that under suitable scaling hypotheses, one can simplify the geometry further by constructing a normally hyperbolic invariant manifold that contains the attractor.

3. Geometry in the limit $\varepsilon = 0$. If $\varepsilon = 0$, then $u_t = 0$ in (2.2). By inspection one sees the existence of a manifold of equilibria, $\mathcal{M}_0$, given as the graph of a map $h : \mathbb{X}_1 \to \mathbb{X}_2$. We denote its components by $h(u) = (h_v(u), h_w(u))$. These maps are defined pointwise for $x \in S^1$ by

$$h_v(u)(x) = \mu(1-i\delta)\frac{(\lambda+1)u(x)}{1+\delta^2+\lambda|u(x)|^2}, \quad h_w(u)(x) = \frac{(1+\delta^2)(\lambda+1)}{1+\delta^2+\lambda|u(x)|^2}. \tag{3.1}$$

For large $|u(x)|$ the denominator dominates; therefore, $\mathcal{M}_0$ is uniformly bounded. The pointwise maps $h_v(u)(x)$ and $h_w(u)(x)$ are C^∞ as functions of $u(x)$. Since pointwise operations extend naturally in $C(S^1)$, we find that h is C^∞ as a map between $\mathbb{X}_1 \to \mathbb{X}_2$. Thus, $\mathcal{M}_0$ is a C^∞ manifold.

In this limit we can solve the laser equations explicitly. We split (u, v) into their real and imaginary parts, i.e., $(u, v) = (\mathrm{Re}(u), \mathrm{Re}(v)) + i(\mathrm{Im}(u), \mathrm{Im}(v))$, and then rewrite (2.3)–(2.4) as

$$\partial_t \begin{pmatrix} \mathrm{Re}(v) \\ \mathrm{Im}(v) \\ w \end{pmatrix} = A(u) \begin{pmatrix} \mathrm{Re}(v) \\ \mathrm{Im}(v) \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \gamma_{\|}(\lambda+1) \end{pmatrix}, \tag{3.2}$$

where $A(u)$ is the bounded multiplication operator defined by

$$A(u) = \begin{pmatrix} -1 & \delta & \mu\mathrm{Re}(u) \\ -\delta & -1 & \mu\mathrm{Im}(u) \\ -\mu\mathrm{Re}(u) & -\mu\mathrm{Im}(u) & -\gamma_{\|} \end{pmatrix}. \tag{3.3}$$

Thus, the solution to the laser equations (2.3)–(2.4) in this limit is $u = u(x)$, and

$$\begin{pmatrix} \mathrm{Re}(v)(t) \\ \mathrm{Im}(v)(t) \\ w(t) \end{pmatrix} = e^{tA(u)} \begin{pmatrix} \mathrm{Re}(v)(0) \\ \mathrm{Im}(v)(0) \\ w(0) \end{pmatrix} + \int_0^t e^{(t-s)A(u)} \begin{pmatrix} 0 \\ 0 \\ \gamma_{\|}(\lambda+1) \end{pmatrix} ds. \tag{3.4}$$

Here u is treated as a parameter and the fibers of constant u are invariant under the flow. Within each fiber, trajectories decay to the equilibrium $(u, h(u))$. The next lemma states that the decay rate is uniform over $\mathcal{M}_0$.

LEMMA 3.1. $\|e^{tA(u)}\| \le e^{-\beta t}$ *for all* $u \in \mathbb{X}_1$.

Proof. This follows from an estimate similar to (2.5). The operator $A(u)$ is broken into two parts: a diagonal matrix that is independent of u and a skew matrix that depends on u. The skew matrix does not influence the growth or decay of energy, and hence u cannot influence the decay in $\|(v, w)(t)\|$. ◻

Clearly, Lemma 3.1 reflects a strong stability of $\mathcal{M}_0$ that depends on the skew nonlinearity. As we have emphasized earlier, this is actually a consequence of the underlying physics. Figure 3.1 describes the geometry of the flow with two key geometric objects. The first is the critical manifold $\mathcal{M}_0$, the second is the smooth invariant family $\mathcal{F}_{u_0} := \{(u, v, w)| u = u_0\}$ parametrized by $u_0 \in \mathbb{X}_1$. There is a purely metric characterization of $\mathcal{F}_{u_0}$: For any $0 < \gamma < \beta$ these manifolds are γ-stable manifolds in the sense of Chow, Lin, and Lu [7]; i.e., for $t \in \mathbf{R}_+$ and fixed $(u_0, h(u_0))$ the set of points $\{(u, v, w) : \|\varphi_t(u, v, w) - \varphi_t(u_0, h(u_0))\| = O(e^{-\gamma t})\}$ is identical to $\mathcal{F}_{u_0}$. For $\varepsilon > 0$ the system is dissipative in the $\mathbb{X}_1$ direction as well, and all trajectories are sucked into the absorbing region $\mathcal{D}_0$. Thus, it is sufficient to show that $\mathcal{M}_0$ and $\mathcal{F}_{u_0}$ persist within $\mathcal{D}_0$. Roughly speaking, we shall show that there is an $\varepsilon_* > 0$ so that for all $0 \le \varepsilon \le \varepsilon_*$, there is a smooth (but not C^∞) invariant manifold $\mathcal{M}_\varepsilon$ given as a graph $(u, h_\varepsilon(u))$ over $\Pi_1(\mathcal{D}_0)$ that contains the asymptotic dynamics (in particular the attractor $\mathcal{A}$) and is exponentially attracting.

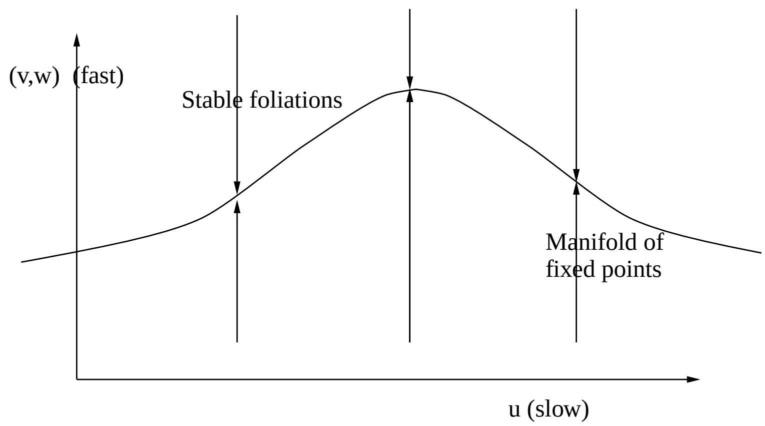

Fig. 3.1. *Geometry in the singular limit* $\varepsilon = 0$.

4. Existence of the invariant manifold.

4.1. The main theorem.

Theorem 4.1. *For any integer* r*, there is an* $\varepsilon_*(r) > 0$ *so that for each* $\varepsilon \in [0, \varepsilon_*(r)]$ *there is a positively invariant* C^r *manifold,* $\mathcal{M}_\varepsilon$*, given as a graph over* $\Pi_1(\mathcal{D}_0)$*. This manifold attracts all initial conditions exponentially fast and contains the attractor* $\mathcal{A}$ *of the Maxwell–Bloch equations.*

Sections 4 and 5 are devoted to a proof of the main theorem. The consequences of this theorem are explored in section 6.

4.2. The modified equations. We will use Hadamard's graph transform method to prove the existence of a persisting manifold [14]. We will restrict our attention to the flow within an absorbing ball and modify the nonlinearity outside this ball. This approach has been used to prove the existence of finite-dimensional attracting manifolds for dissipative dynamical systems (e.g., reaction diffusion equations) [13].

Let $R_1 = 2\rho_v/\mu$ and $R_2 = \sqrt{2}\rho_v$. Then R_1 and R_2 are sufficiently large that the region

$$\mathcal{D} = \{\|u\|_{\mathbb{X}_1} \leq 2R_1, \|(v,w)\|_{\mathbb{X}_2} \leq 2R_2\} \tag{4.1}$$

is absorbing and positively invariant (see (2.11) and the discussion preceding it). We denote this region by $\mathcal{D}$ and note that $\mathcal{D} = 2\mathcal{D}_0$.

Let $\chi_1 : \mathbb{C} \to [0,1]$ be a C^∞ function with compact support that takes the values $\chi_1(s) = 1,\ 0 \leq |s| \leq 1,\ \chi_1(s) = 0$ for $2 \leq |s| < \infty$ and has uniformly bounded derivative $|D\chi_1(s)| \leq 2$. Let $\chi_2 : \mathbb{C} \times \mathbb{R} \to [0,1]$ be a cut-off function with analogous properties. Define the cut-off functions $\chi_{R_i} : \mathbb{X}_i \to [0,1]$ by $\chi_{R_1}(u)(x) = \chi_1(u(x)/R_1)$ and $\chi_{R_2}(v,w)(x) = \chi_2((v(x), w(x))/R_2)$. One may prove that χ_{R_i}, $i = 1, 2$, are C^∞. As is common in invariant manifold theory, we will modify the laser equations so as to obtain global estimates. Let

$$\kappa\left(\frac{v}{\mu} - u\right)\chi_{R_1}(u)\chi_{R_2}(v,w) = g(u,v,w) \quad \text{and} \quad u\chi_{R_1}(u) = f(u). \tag{4.2}$$

Consider the modified laser equations

$$u_t = -\varepsilon u_x + \varepsilon g(u, v, w), \tag{4.3}$$

$$v_t = -(1 + i\delta)v + \mu f(u)w, \tag{4.4}$$

$$w_t = \gamma_{\|}(\lambda + 1 - w) - \mu \mathrm{Re}(f(u)v^*). \tag{4.5}$$

We modify only the u term in the nonlinearity in (2.3)–(2.4). This allows us to retain an estimate similar to the energy estimate (2.15).

LEMMA 4.1. *For* $(u_i, v_i, w_i) \in \mathbb{X}$, $i = 1, 2$, *we have*

(a) $\|f(u_1, v_1, w_1) - f(u_2, v_2, w_2)\| \leq 5\|u_1 - u_2\|$,

(b) $\|g(u_1, v_1, w_1)-g(u_2, v_2, w_2)\| \leq \kappa(5+4\sqrt{2})(\|u_1-u_2\|+\mu^{-1}\|(v_1, w_1)-(v_2, w_2)\|)$.

Proof. Without loss of generality suppose $\max(|u_i(x)|) = |u_2(x)|$. If $|u_2(x)| \leq 2R_1$,

$$\begin{aligned}
&|u_1\chi_{R_1}(u_1)(x) - u_2\chi_{R_1}(u_2)(x)| \\
&\leq |u_1(x) - u_2(x)||\chi_{R_1}(u_1)(x)| + |u_2(x)||\chi_{R_1}(u_1)(x) - \chi_{R_1}(u_2)(x)| \\
&\leq |u_1(x) - u_2(x)| + 2R_1\frac{2}{R_1}|u_1(x) - u_2(x)| \leq 5\|u_1 - u_2\|.
\end{aligned}$$

If $\min(|u_1(x)|, |u_2(x)|) > 2R_1$, the above inequality is trivial since the left-hand side is zero. Finally, if $|u_2(x)| > 2R_1$ and $|u_1(x)| \leq 2R_1$, we have

$$|u_1\chi_{R_1}(u_1)(x) - u_2\chi_{R_1}(u_2)(x)| = |u_1\chi_{R_1}(u_1)(x) - u_1\chi_{R_1}(u_2)(x)| \leq 4\|u_1 - u_2\|. \tag{4.6}$$

Taking the sup over x we obtain (a). Similar calculations show that the difference in g is bounded by

$$\kappa\left(\left(5 + \frac{4R_2}{\mu R_1}\right)\|u_1 - u_2\| + \left(\frac{5}{\mu} + \frac{4R_1}{R_2}\right)\|(v_1, w_1) - (v_2, w_2)\|\right).$$

But $R_2/R_1 = \mu/\sqrt{2}$. Simplifying the above estimate, we obtain (b). ∎

Remark 4.2. We make the following important observation regarding the modified flow. Suppose $\|u(0)\| > 2R_1$. Then there exists an open interval I in S^1 so that $|u(0)(x)| > 2R_1$ for all $x \in I$, and hence $g((u, v, w)(0)) = 0$, on this interval. Integrating (4.3) along the characteristic $x - \varepsilon t =$ constant, we find that $u(t, x)$ is constant on the characteristics through $I \times \{t = 0\}$. Thus, $\|u(t)\| > 2R_1$ for all $t \in \mathbb{R}$, and the region $\{\|u\| > 2R_1\}$ in phase space is invariant for the modified flow. This implies its complement is also invariant. Hence the phase space splits into two *invariant* regions, the closed cylinder $\{\|u\| \leq 2R_1\}$ and its exterior.

Remark 4.3. Within the region $\{\|u\| \leq R_1, \|(v, w)\| \leq R_2\}$ the modified and unmodified equations agree on a dense set, and hence their flows agree locally in time. But by the choice of R_i, this region is positively invariant, and thus the flows agree for all positive time. As a result they have identical asymptotic dynamics within this region. We will prove the following invariant manifold theorem for the mild formulation of the modified equations (4.3)–(4.5). The mild formulation is

$$u(t) = e^{-\varepsilon t\partial_x}u(0) + \varepsilon\int_0^t e^{-\varepsilon(t-s)\partial_x}g(u(s), v(s), w(s))ds, \tag{4.7}$$

$$v(t) = e^{-(1+i\delta)t}v(0) + \mu\int_0^t e^{-(1+i\delta)(t-s)}f(u(s))w(s)ds, \tag{4.8}$$

$$\begin{aligned}
w(t) = {}& e^{-\gamma_{\|}t}w(0) + (\lambda + 1)(1 - e^{-\gamma_{\|}t}) \\
&-\mu\int_0^t e^{-(t-s)\gamma_{\|}} \,\mathrm{Re}\,(f(u(s))^*v(s))ds.
\end{aligned} \tag{4.9}$$

THEOREM 4.4. *For any integer r, there exists an* $\varepsilon_*(r) > 0$ *so that for each* $\varepsilon \in [0, \varepsilon_*(r)]$ *there is a* C^r *manifold,* $\mathcal{M}_\varepsilon$*, invariant under the flow of the modified Maxwell–Bloch equations* (4.7)–(4.9). $\mathcal{M}_\varepsilon$ *is given as a graph over* $\Pi_1(\mathcal{D})$. *This manifold attracts all points in the absorbing region exponentially fast and contains the attractor* $\mathcal{A}$ *of the Maxwell–Bloch equations* (2.21)–(2.23).

Theorem 4.4 implies Theorem 4.1 because, by Remark 4.3, the asymptotic dynamics of modified and unmodified systems agree within $\mathcal{D}_0$. Since $\mathcal{D}_0$ is only positively invariant, the invariance of the manifold in Theorem 4.4 is weakened to positive invariance in Theorem 4.1.

4.3. A priori estimates. We reconsider the a priori estimates of section 2 in light of the above modifications. Henceforth, in sections 4 and 5, φ_t denotes the flow of the modified equations (4.7)–(4.9). In all that follows, we will only consider trajectories that start within the positively invariant region $\mathcal{D}$. Thus the constants, C_j, that occur in inequalities will generally depend on R_i and the parameters $\kappa, \lambda, \gamma_\|$, and μ. We also assume that the time t is positive.

Remark 4.2 implies the uniform bound

$$\|u(t)\| \leq 2R_1, \quad t \in \mathbb{R}, \tag{4.10}$$

for all trajectories starting within $\mathcal{D}$. The modification has also been chosen so that the energy estimate (2.15) is unchanged (i.e., we retain the cancellation of nonlinear terms). Thus, by the choice of R_2 trajectories starting within $\mathcal{D}$ satisfy the uniform bound

$$\|(v, w)(t)\| \leq 2R_2, \quad t \geq 0. \tag{4.11}$$

Estimates for differences between trajectories are derived as in section 2. As in (2.17) we have

$$a(t) \leq e^{C_1 \varepsilon t} a_0 + C_2 \varepsilon \int_0^t e^{C_1 \varepsilon (t-s)} b(s) ds \tag{4.12}$$

for $C_i = C_i(\kappa, \mu, R_i), i = 1, 2$. The analogue of (2.15) is derived from (4.3) and (4.4). The differences (η, ζ) now satisfy

$$\begin{aligned} \partial_t(|\eta|^2 + |\zeta|^2) &= -2|\eta|^2 - 2\gamma_\| |\zeta|^2 + 2\mathrm{Re}\,(\eta^*(f(u_1)w_1 - f(u_2)w_2)) \\ &\quad -2\mathrm{Re}\,(\zeta(f(u_1)v_1^* - f(u_2)v_2^*)) \\ &= -2|\eta|^2 - 2\gamma_\| |\zeta|^2 + \mathrm{Re}\,((f(u_1) - f(u_2))(\bar{w}\eta^* - \bar{v}^*\zeta)). \end{aligned} \tag{4.13}$$

Notice that the choice of the modification is such that the term involving $f(u_1) + f(u_2)$ cancels. This is important as it ensures that we retain the uniform decay normal to the manifold $\mathcal{M}_0$, independent of the basepoint u. One can now use Lemma 4.1 and the energy estimate (4.11) in (4.13) and integrate to find

$$b(t) \leq e^{-\beta t} b(0) + C_3 \int_0^t e^{-\beta(t-s)} a(s) ds. \tag{4.14}$$

The constant C_3 depends only on the parameters $\kappa, \mu, \lambda, \beta$ and the radii R_i. We also need lower estimates on $a(t)$ and $b(t)$ that are derived similarly. For example, (4.3) yields

$$(\partial_t + \varepsilon \partial_x)|\xi(t, x)|^2 \geq -C\varepsilon|\xi|(|\xi| + |\eta|) \geq -C_1 \varepsilon |\xi|^2 - C_2 \varepsilon |\eta|^2,$$

so that integrating between $t_1 \le t_2$ and taking the sup over x we have

$$a(t_2) \ge e^{-C_1\varepsilon(t_2-t_1)}a(t_1) - C_2\varepsilon \int_{t_1}^{t_2} e^{-C_1\varepsilon(t_2-s)}b(s)ds. \tag{4.15}$$

Finally, from (4.13) and the energy estimate (4.11) we have the pointwise inequality

$$\partial_t(|\eta(t,x)|^2 + |\zeta(t,x)|^2) \ge -3\tilde{\beta}(|\eta|^2 + |\zeta|^2) - C|\xi|^2, \tag{4.16}$$

where we have defined $\tilde{\beta} = \max(1, \gamma_\|)$. Integrating this inequality and taking the sup over $x \in S^1$ we obtain

$$b(t_2) \ge e^{-3\tilde{\beta}(t_2-t_1)}b(t_1) - C_4 \int_{t_1}^{t_2} e^{-3\tilde{\beta}(t_2-s)}a(s)ds. \tag{4.17}$$

These a priori estimates can be used to prove the existence of a C^∞ flow for the dynamical system defined by (4.7)–(4.9) as in Theorem 2.5. We will not state a separate theorem.

4.4. The cone property. The graph transform will be defined by applying the map φ_T to Lipschitz sections of the normal bundle of the critical manifold $\mathcal{M}_0$. Over sufficiently large time we expect the flow to contract strongly in the normal direction. This is made precise in the cone condition formulated by Conley, and used since then by several authors. It is an essential geometric feature in the persistence theorem of Bates, Lu, and Zeng and a comprehensive list of references may be found in their article [4].

Choose $T > 0$ so that

$$e^{-\beta T/2} = \frac{1}{32}. \tag{4.18}$$

T will be held fixed in all that follows. In the following propositions ε_* denotes an upper limit that may only decrease from one assertion to the next. This follows the convention in [4]. For $(u, v, w) \in \mathcal{D}$, we will use the cone

$$K_L(u,v,w) = \{(u_1,v_1,w_1) \in \mathcal{D} : \|(v_1,w_1) - (v,w)\|_{\mathbb{X}_2} \le L\|u_1 - u\|_{\mathbb{X}_1}\}. \tag{4.19}$$

LEMMA 4.2 (the moving cone lemma). *There exists $\varepsilon_* > 0$ and $L > 0$ such that for $\varepsilon \in [0, \varepsilon_*]$, $t \in [0, T]$, and each point $(u, v, w) \in \mathcal{D}$, the cone $K_L(u, v, w)$ is carried by the diffeomorphism φ_t into the cone $K_L(\varphi_t(u, v, w))$.*

Remark 4.5. The statement of Lemma 4.2 is uniform over all points in the absorbing region. Geometrically, this implies a squeezing property of the flow.

Proof. $\mathcal{D}$ is positively invariant: thus for any $(u, v, w) \in \mathcal{D}$, $L > 0$, and $t \ge 0$, φ_t carries the cone $K_L(u, v, w)$ into $\mathcal{D}$. It remains to prove that for suitable $L > 0$, if two trajectories start in $\mathcal{D}$ and satisfy $b_0 \le L^2 a_0$, then $b(t) \le L^2 a(t)$ for all $t \in [0, T]$. Since the initial conditions lie in $\mathcal{D}$, $a(t)$ and $b(t)$ must satisfy the a priori estimates (4.12) and (4.14). Our proof will demonstrate a technique of dealing with these coupled inequalities by exploiting the gap in the exponential rates.

For any $\gamma \in (C_1\varepsilon, \beta)$ we define $|a|_{\gamma,t} = \sup_{s\in[0,t]} a(s)e^{\gamma s}$. Similarly, we define $|b|_{\gamma,t}$. It follows that $|a|_{\gamma,t}$ is an increasing function of t. We will use $\gamma = \beta/2$, though the argument will work for any γ that satisfies the gap condition $C_1\varepsilon < \gamma < \beta$. We further assume that ε_* is so small that $C_1\varepsilon_* < \beta/2$.

We multiply (4.12) by $e^{\beta s/2}$ to obtain

$$\begin{aligned} a(s)e^{\beta s/2} &\le e^{(\beta/2+C_1\varepsilon)s}a_0 + C_2\varepsilon\int_0^s e^{(\beta/2+C_1\varepsilon)(s-\tau)}e^{\beta\tau/2}b(\tau)d\tau \\ &\le e^{(\beta/2+C_1\varepsilon)s}a_0 + \frac{C_2\varepsilon}{\beta/2+C_1\varepsilon}\left(e^{(\beta/2+C_1\varepsilon)s}-1\right)|b|_{\beta/2,s} \\ &\le e^{(\beta/2+C_1\varepsilon)s}\left(a_0+\varepsilon C_2 s|b|_{\beta/2,s}\right). \end{aligned}$$

In the last step we have used the elementary inequality $1-e^{-t}\le t$ for positive t. Taking the sup over $s\in[0,t]$, and using the fact that $|a|_{\gamma,s}$ is an increasing function of s, we obtain

$$|a|_{\beta/2,t} \le e^{(\beta/2+C_1\varepsilon)t}\left(a_0 + C_2\varepsilon t|b|_{\beta/2,t}\right). \tag{4.20}$$

We apply a similar calculation to (4.14) to obtain

$$\begin{aligned} b(s)e^{\beta s/2} &\le b_0 e^{-\beta s/2} + C_3\int_0^s e^{-\beta(s-\tau)/2}e^{\beta\tau/2}a(\tau)d\tau \\ &\le b_0 + C_3\frac{(1-e^{-\beta s/2})}{\beta/2}|a|_{\beta/2,s} \le b_0 + C_3 s|a|_{\beta/2,s}. \end{aligned}$$

Taking the sup over $s\in[0,t]$ we find

$$|b|_{\beta/2,t} \le b_0 + C_3 t|a|_{\beta/2,t}. \tag{4.21}$$

Combining the inequalities (4.20) and (4.21) we find

$$|b|_{\beta/2,t} \le b_0 + C_3 t e^{(\beta/2+C_1\varepsilon)t}\left(a_0 + C_2\varepsilon t|b|_{\beta/2,t}\right). \tag{4.22}$$

We suppose that ε_* is chosen so small that for all $\varepsilon\in[0,\varepsilon_*]$, we have

$$\varepsilon e^{(\beta/2+C_1\varepsilon)t}C_2C_3t^2 \le \frac{1}{2}. \tag{4.23}$$

Then using the hypothesis $b_0\le l^2a_0$, and (4.23) in (4.22) we find

$$|b|_{\beta/2,t} \le a_0\left[\frac{l^2 + C_3te^{(\beta/2+C_1\varepsilon)t}}{1-\varepsilon C_2C_3t^2e^{(\beta/2+C_1\varepsilon)t}}\right] =: a_0\theta(t,\varepsilon), \tag{4.24}$$

where we have defined a new function $\theta(t,\varepsilon)$ to simplify notation. Furthermore, we set $t_1=0$, and $t_2=t$ in the backward time estimate (4.15) to deduce that

$$\begin{aligned} a_0 &\le e^{C_1\varepsilon t}a(t) + C_2\varepsilon\int_0^t e^{C_1\varepsilon s}b(s)ds \\ &\le e^{C_1\varepsilon t}\left(a(t) + C_2\varepsilon t|b|_{\beta/2,t}\right). \end{aligned} \tag{4.25}$$

Thus, combining (4.24) and (4.25), we have

$$|b|_{\beta/2,t} \le \theta(t,\varepsilon)a_0 \le \theta(t,\varepsilon)e^{C_1\varepsilon t}\left(a(t) + C_2\varepsilon t|b|_{\beta/2,t}\right). \tag{4.26}$$

We reduce ε_* if necessary so that $\sup_{t\in[0,T]}\varepsilon C_2 t\theta(t,\varepsilon)e^{C_1\varepsilon t}\le 1/2$. Then we have

$$b(t) \le e^{-\beta/2t}|b|_{\beta/2,t} \le \frac{\theta(t,\varepsilon)e^{-(\beta/2-C_1\varepsilon)t}}{1-C_2\varepsilon t\theta(t,\varepsilon)e^{C_1\varepsilon t}}a(t). \tag{4.27}$$

Thus, the cone condition (i.e., $b(t) \leq L^2 a(t)$) will be satisfied if we ensure that for all $t \in [0, T]$ we have

$$\tilde{\theta}(t, \varepsilon) := \frac{\theta(t, \varepsilon) e^{-(\beta/2 - C_1 \varepsilon) t}}{1 - C_2 \varepsilon t \theta(t, \varepsilon) e^{C_1 \varepsilon t}} - L^2 \leq 0. \tag{4.28}$$

The function $\tilde{\theta}(t, \varepsilon)$ is smooth in t and ε for $0 \leq t \leq T, 0 \leq \varepsilon \leq \varepsilon_*$, since by the choice of ε_* the denominator is bounded away from zero. Notice that if we let $t = 0$ in (4.24) we have $\theta(0, \varepsilon) = L^2$; hence $\tilde{\theta}(0, \varepsilon) = 0$. If $\varepsilon = 0$, then the inequality (4.28) reduces to

$$\tilde{\theta}(t, 0) = -L^2 (1 - e^{-\beta t/2}) + C_3 t \leq 0. \tag{4.29}$$

Thus we choose

$$L^2 \geq 2C_3 \max\left(2/\beta, T(1 - e^{-\beta T/2})^{-1}\right) = 2C_3 T \frac{32}{31} \tag{4.30}$$

(see 4.18). This choice ensures that $\tilde{\theta}(t, 0)$ is a decreasing function of t in the range $[0, T]$ and the inequality (4.29) is an equality only at $t = 0$. But then to show that (4.28) is true for small positive ε, it suffices to ascertain its validity near $t = 0$. The choice of L in (4.30) ensures that the slope

$$\frac{d\tilde{\theta}(t, 0)}{dt}|_{t=0} \leq -C_3 < 0,$$

which implies that for sufficiently small ε_* the inequality $\max_{t \in [0,T]} \tilde{\theta}(t) \leq 0$ is satisfied. In other words, $b(t) \leq L^2 a(t)$ for all $t \in [0, T]$. ◻

Remark 4.6. To simplify some estimates later, we further suppose that

$$L^2 = 8 \max(C_3 T, C_9), \tag{4.31}$$

where C_9 is a constant that occurs in the proof of Lemma 5.1. This simplifies some estimates in the proof of existence and smoothness of the slow manifold $\mathcal{M}_\varepsilon$.

A point about the proof that an expert may find strange is the use of direct estimates on the flow as opposed to estimates from the linearization near the manifold. The laser equations admit strong estimates which is why this approach works. Typically, the best one can do is obtain a cone condition in a neighborhood of the manifold. Another unusual feature is the use of a Lipschitz constant L that is not small. In Fenichel's work [11] the slope of the Lipschitz sections (i.e., L) is small. The distinction is that we use a single coordinate chart for $\mathcal{M}_0$, so L is finite to account for the nonzero slope of $\mathcal{M}_0$. This is to avoid a global coordinate transformation that would lead to vexing technical difficulties.

The next three lemmas pick out special cases of estimates in the moving cone lemma that will be used in the proof that the graph transform is a contraction mapping (see Proposition 4.11).

LEMMA 4.3. *Suppose that* $a_0 = 0$. *Then there is* $\varepsilon_* > 0$ *so that for all* $\varepsilon \in [0, \varepsilon_*]$,

$$b(T) \leq b_0/16.$$

Proof. The inequality (4.20) with $t = T$, and $a_0 = 0$, reduces to

$$|a|_{\beta/2,T} \leq C_2 \varepsilon T e^{(\beta/2 + C_1 \varepsilon) T} |b|_{\beta/2,T},$$

and inserting this in (4.21) we have

$$\begin{aligned}|b|_{\beta/2,T} &\le b_0 + \varepsilon C_2 C_3 T^2 e^{(\beta/2+C_1\varepsilon)T}|b|_{\beta/2,T}\\ &\le b_0 + \frac{1}{2}|b|_{\beta/2,T}\end{aligned}$$

by the choice of ε_* in Lemma 4.2 (see (4.23)). Thus $|b|_{\beta/2,T} \le 2b_0$. But then

$$b(T) \le e^{-\beta T/2}|b|_{\beta/2,T} \le \frac{2}{32}b_0 = \frac{1}{16}b_0. \qquad \square$$

LEMMA 4.4. *Suppose $b_0 = 0$. There is $\varepsilon_* > 0$ such that for all $\varepsilon \in [0, \varepsilon_*]$,*

$$b(T) \le \left(\frac{3L}{4}\right)^2 a_0.$$

Proof. A calculation similar to that above reveals that $b(T) \le 2C_3 T e^{C_1\varepsilon T} a_0$. When $\varepsilon = 0$ this reduces to

$$b(T) \le 2C_3 T a_0 \le \frac{L^2}{4}a_0$$

by the choice of L^2 in Remark 4.6. Thus, for sufficiently small ε_* we obtain the required estimate. $\square$

We conclude with a backward time estimate.

LEMMA 4.5. *Suppose $a(T) = 0$. There is $\varepsilon_* > 0$ such that for all $\varepsilon \in [0, \varepsilon_*]$*

$$a_0 \le \frac{1}{4L^2}b(T).$$

Proof. We use (4.15) with $t_1 = t$ and $t_2 = T$ to find

$$\begin{aligned}a(t) &\le e^{C_1\varepsilon(T-t)}a(T) + C_2\varepsilon\int_t^T e^{C_1\varepsilon(s-t)}b(s)ds\\ &= C_2\varepsilon\int_t^T e^{C_1\varepsilon(s-t)}b(s)ds\end{aligned}$$

by our hypothesis. We multiply by $e^{\beta t/2}$ and take the sup over $t \in [0, T]$ to obtain

$$|a|_{\beta/2,T} \le \frac{C_2\varepsilon}{\beta/2 - C_1\varepsilon}|b|_{\beta/2,T}.$$

Similarly by (4.17), the backward time estimate for $b(t)$ is

$$b(t) \le e^{3\tilde\beta(T-t)}b(T) + C_4\int_t^T e^{3\tilde\beta(s-t)}a(s)ds.$$

We multiply by $e^{\beta t/2}$ and take the sup in t to obtain

$$\begin{aligned}|b|_{\beta/2,T} &\le e^{3\tilde\beta T}\left(b(T) + \frac{C_4 e^{-\beta T/2}}{3\tilde\beta - \beta/2}|a|_{\beta/2,T}\right)\\ &\le e^{3\tilde\beta T}\left(b(T) + \frac{C_4 e^{-\beta T/2}}{3\tilde\beta - \beta/2}\frac{C_2\varepsilon}{\beta/2 - C_1\varepsilon}|b|_{\beta/2,T}\right).\end{aligned}$$

Let ε_* be so small that for all $\varepsilon \in [0, \varepsilon_*]$,

$$\varepsilon \frac{C_4}{3\tilde{\beta} - \beta/2} \frac{C_2}{2\beta - C_1\varepsilon} e^{(3\tilde{\beta}-\beta/2)T} \leq \frac{1}{2}.$$

Then $|b|_{\beta/2,T} \leq 2e^{3\tilde{\beta}T} b(T)$, and hence

$$a_0 \leq |a|_{\beta/2,T} \leq \varepsilon \frac{2C_2 e^{3\tilde{\beta}T}}{\beta/2 - C_1\varepsilon} b(T).$$

We further reduce ε_* if necessary to obtain $a_0 \leq b(T)/4L^2$ for all $\varepsilon \in [0, \varepsilon_*]$. □

4.5. The graph transform. Define the metric space

$$\mathcal{S}_L = \left\{ h : \Pi_1(\mathcal{D}) \to \mathbb{X}_2 | \ \operatorname{Lip}(h) \leq L, \sup_{u \in \Pi_1(\mathcal{D})} \|h(u)\|_{\mathbb{X}_2} \leq 2R_2 \right\}$$

with the distance function

$$d(h_1, h_2) = \sup_{u \in \Pi_1(\mathcal{D})} \|h_1(u) - h_2(u)\|_{\mathbb{X}_2}.$$

$\mathcal{S}_L$ is complete in this metric. We show below that for any $h \in \mathcal{S}_L$, the image of graph (h) under $\varphi_t, t \in [0, T]$, is the graph of a function in $\mathcal{S}_L$. Taking $t = T$, we define the *graph transform* $\mathcal{G} : \mathcal{S}_L \to \mathcal{S}_L$ by graph $(\mathcal{G}(h)) = \varphi_T(\text{graph } (h))$. Most of this subsection is devoted to showing that this definition is unambiguous.

PROPOSITION 4.7 (uniqueness). *Fix $h \in S_L$ and a point $u \in \Pi_1(\mathcal{D})$. There is at most one preimage $u_0 \in \Pi_1(\mathcal{D})$ so that $\Pi_1(\varphi_t(u_0, h(u_0))) = u$.*

Proof. Suppose that $u_1 \neq u_2$ but $\Pi_1(\varphi_t(u_1, h(u_1)) - \varphi_t(u_2, h(u_2))) = 0$. Since $\operatorname{Lip}(h) \leq L$, the point $(u_2, h(u_2))$ lies in the cone $K_L(u_1, h(u_1))$. By the moving cone lemma $\varphi_t(u_2, h(u_2)) \in K_L(\varphi_t(u_1, h(u_1)))$. But then $\Pi_1(\varphi_t(u_1, h(u_1)) - \varphi_t(u_2, h(u_2))) \neq 0$. □

To prove the existence of at least one preimage requires more effort. If $\Pi_1(\mathcal{D})$ were finite-dimensional one could use topological arguments based on degree and the Wazewski principle to prove existence (see, e.g., [3]). This approach would fail here since the manifold to be constructed has both infinite dimension and infinite codimension. Moreover, though we know that there is a solution for $\varepsilon = 0$, we cannot use an implicit function theorem (e.g., as in Fenichel's work [11]) to establish existence for $\varepsilon > 0$ since the perturbation is not Lipschitz in ε. We resort to an explicit solution of the modified equations (4.3)–(4.5) in backward time.

Let $u_T \in \Pi_1(\mathcal{D})$ be fixed. We will show that there exists $(u_0, h(u_0)) \in \mathcal{D}$ such that $\Pi_1(\varphi_T(u_0, h(u_0))) = u_T$. We will rewrite the modified differential equations (4.3)–(4.5) as integral equations in a form different from the mild formulation (4.7)–(4.9). The motivation for this will be clear in the consequent estimates.

Let $S(t, s; u_T)$, $t, s \in \mathbb{R}$, be the two-parameter family in $L(\mathbb{X}_2, \mathbb{X}_2)$ defined as the solution operator to the following linear nonautonomous differential equation:

$$\begin{pmatrix} \operatorname{Re}(v)_t \\ \operatorname{Im}(v)_t \\ w_t \end{pmatrix} = \begin{pmatrix} -1 & \delta & \mu \operatorname{Re} f_1(t) \\ -\delta & -1 & \mu \operatorname{Im} f_1(t) \\ -\mu \operatorname{Re} f_1(t) & -\mu \operatorname{Im} f_1(t) & -\gamma_\| \end{pmatrix} \begin{pmatrix} \operatorname{Re}(v) \\ \operatorname{Im}(v) \\ w \end{pmatrix}, \tag{4.32}$$

where

$$f_1(t) = f(e^{\varepsilon(T-t)\partial_x} u_T),$$

and f is defined in (4.2). $S(t,s;u_T)$ is well defined since the right-hand side is a bounded linear operator, and we have the a priori estimate

$$|v(t,x)|^2 + |w(t,x)|^2 \le e^{-2\beta(t-s)}(|v(s,x)|^2 + |w(s,x)|^2),$$

which ensures the existence of global solutions. In fact, this a priori estimate proves the following.

LEMMA 4.6. $\|S(t_1,t_2;u_T)\| \le e^{-\beta(t_1-t_2)}$ *for each* $u_T \in \mathbb{X}_1$.

Any mild solution to (4.7)–(4.9) that passes through u_T at time T must satisfy the integral equations

$$u(t) = e^{\varepsilon(T-t)\partial_x}u_T - \varepsilon\int_t^T e^{\varepsilon(s-t)\partial_x} g(u(s),v(s),w(s))ds, \tag{4.33}$$

$$\begin{aligned}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} &= S(t,0;u_T)\begin{pmatrix} v(0) \\ w(0) \end{pmatrix} \\ &\quad + \int_0^t S(t,s;u_T)\begin{pmatrix} 0 \\ \gamma_\parallel(\lambda+1) \end{pmatrix} ds \\ &\quad + \mu\int_0^t S(t,s;u_T)F(u(s))\begin{pmatrix} v(s) \\ w(s) \end{pmatrix} ds,\end{aligned} \tag{4.34}$$

where $F(u(s))$ is a skew-symmetric multiplication operator in $L(\mathbb{X}_2,\mathbb{X}_2)$ whose only nonzero terms are

$$\begin{aligned} F_{13} = -F_{31} &= \mathrm{Re}(f(u(s))) - f(e^{\varepsilon(T-s)\partial_x}u_T), \\ F_{23} = -F_{32} &= \mathrm{Im}(f(u(s))) - f(e^{\varepsilon(T-s)\partial_x}u_T) \end{aligned} \tag{4.35}$$

(we split v into its real and imaginary parts). The virtue of rewriting the equations in this form is that the nonlinear terms are now small. More precisely, by Lemma 4.1 and (4.33)–(4.34), the norm of F is bounded by

$$\begin{aligned} \sup_{s\in[0,T]} \|F(u(s))\| &= \sup_{s\in[0,T]} \|f(u(s)) - f(e^{\varepsilon(T-s)\partial_x}u_T)\| \\ &\le 5 \sup_{s\in[0,T]} \|u(s) - e^{\varepsilon(T-s)\partial_x}u_T\| \le \varepsilon C_5 T \end{aligned} \tag{4.36}$$

for a constant $C_5 = \sup\|g(u,v,w)\| = C_5(\mu,\kappa,R_i)$.

PROPOSITION 4.8 (existence). *There is* $\varepsilon_* > 0$ *such that for each* $\varepsilon \in [0,\varepsilon_*]$ *there exists* $u_0 \in \Pi_1(\mathcal{D})$ *with* $\Pi_1(\varphi_T(u_0,h(u_0))) = u_T$.

Proof. If a preimage exists it must lie in $\Pi_1(\mathcal{D})$ by Remark 4.2. To prove the existence of such a preimage we use iteration on the integral equations (4.33) with the additional condition $(v,w)(0) = h(u(0))$.

Let $u^0(t) = 0$ and $(v,w)^0(t) = 0$ for $0 \le t \le T$. For $n \ge 0$ we define the sequence of iterates

$$u^{n+1}(t) = e^{\varepsilon(T-t)\partial_x}u_T - \varepsilon\int_t^T e^{\varepsilon(s-t)\partial_x} g(u^n(s),v^n(s),w^n(s))ds, \tag{4.37}$$

$$\begin{aligned}\begin{pmatrix} v^{n+1}(t) \\ w^{n+1}(t) \end{pmatrix} &= S(t,T;u_T)h(u^{n+1}(0)) \\ &\quad + \int_0^t S(t,s;u_T)\begin{pmatrix} 0 \\ \gamma_\parallel(\lambda+1) \end{pmatrix} ds \\ &\quad + \int_0^t S(t,s;u_T)F(u^{n+1}(s))\begin{pmatrix} v^n(s) \\ w^n(s) \end{pmatrix} ds.\end{aligned} \tag{4.38}$$

Notice that we solve (4.37) before (4.38).

The sequence defined above satisfies some uniform bounds. First, it is clear that

$$\|u^{n+1}(t) - e^{\varepsilon(T-t)\partial_x} u_T\|_{0,T} \leq \varepsilon T \sup \|g(u,v,w)\| = \varepsilon C_5 T. \tag{4.39}$$

Thus, by Lemmas 4.1, 4.6, and (4.36), we have

$$\|(v,w)^{n+1}(t)\| \leq e^{\beta(T-t)}\|h\| + \frac{(1-e^{-\beta t})}{\beta}(\lambda+1) + 5\varepsilon C_5 T \int_0^t e^{-\beta(t-s)}\|(v,w)^n(s)\|ds.$$

Now $\|h\| \leq 2R_2$ since $h \in \mathcal{S}_L$, so reducing ε_* further if necessary we have

$$\|(v,w)^{n+1}\|_{0,T} \leq \frac{1}{2}\|(v,w)^n\|_{0,T} + \frac{(\lambda+1)}{\beta} + 2R_2 e^{\beta T},$$

where $\|\cdot\|_{0,T} = \sup_{t\in[0,T]}\|\cdot\|$. This implies the uniform bound

$$\sup_{n\geq 0}\|(v,w)^n\|_{0,T} \leq C_6(\beta,\mu,\kappa,\lambda,R_i,T). \tag{4.40}$$

Next, we note that by (4.37) and Lemma 4.1, the difference between consequent iterates of u must satisfy

$$\|u^{n+1} - u^n\|_{0,T} \leq \varepsilon C_7 T(\|u^n - u^{n-1}\|_{0,T} + \|(v,w)^n - (v,w)^{n-1}\|_{0,T}). \tag{4.41}$$

We will estimate each term in the difference between $(v,w)^{n+1}$ and $(v,w)^n$ separately (see (4.38)). The first term is controlled by the uniform Lipschitz constant L.

$$\begin{aligned} &\sup_{t\in[0,T]}\|S(t,T;u_T)(h(u^{n+1}(0)) - h(u^n(0)))\| \\ &\leq \sup_{t\in[0,T]}\|S(t,T;u_T)\|\|h(u^{n+1}(0)) - h(u^n(0))\| \\ &\leq \sup_{t\in[0,T]} e^{-\beta(t-T)}L\|u^{n+1}(0) - u^n(0)\| \leq e^{\beta T}L\|u^{n+1} - u^n\|_{0,T} \\ &\leq \varepsilon C_7 T e^{\beta T}L(\|u^n - u^{n-1}\|_{0,T} + \|(v,w)^n - (v,w)^{n-1}\|_{0,T}) \end{aligned} \tag{4.42}$$

by Lemma 4.6 and inequality (4.41). The differences between the terms on the second line of (4.38) cancel, and the differences between the integrands in the third line are estimated as follows:

$$\begin{aligned} &\left\| F(u^{n+1}(s))\begin{pmatrix} v^n(s) \\ w^n(s)\end{pmatrix} - F(u^n(s))\begin{pmatrix} v^{n-1}(s) \\ w^{n-1}(s)\end{pmatrix}\right\| \\ &\leq \|F(u^n)\|\|(v,w)^n - (v,w)^{n-1}\| + \|(v,w)^n\|\|F(u^{n+1}) - F(u^n)\| \\ &\leq \varepsilon C_5 T\|(v,w)^n(s) - (v,w)^{n-1}(s)\| + 5C_6\|u^{n+1}(s) - u^n(s)\|. \end{aligned}$$

In the last step we have used Lemma 4.1 and the uniform estimates (4.36) and (4.40). These terms estimate the integrands. Take the sup over $t \in [0,T]$ and combine the resulting inequality with (4.41) and (4.42) to conclude that the difference between two iterates in (v,w) must satisfy

$$\|(v,w)^{n+1} - (v,w)^n\|_{0,T} \leq \varepsilon C_8 T(\|u^n - u^{n-1}\|_{0,T} + \|(v,w)^n - (v,w)^{n-1}\|_{0,T}). \tag{4.43}$$

We choose ε_* so small that $\max(C_7, C_8)\varepsilon_* T < 1/2$. Then the sequence of iterates is a contraction in the Banach space $C([0,T];\mathbb{X})$. The limit is a trajectory $(u,v,w)(t)$ with $u(T) = u_T$ and $(v,w)(0) = h(u(0))$. ∎

Remark 4.9. A closer look reveals that we have not used the condition that T is large anywhere in the proof. Thus we have in fact established the stronger statement that for fixed u_T and any $t \in [0,T]$, there is a preimage u_0 so that $\Pi_1(\varphi_t(u_0, h(u_0))) = u_T$. Since u_0 is obtained from a contraction mapping, a slight variant of this argument may be used to prove the existence and uniqueness simultaneously, providing another proof of Proposition 4.7 without invoking Lemma 4.2 (the moving cone lemma). However, the moving cone lemma is of independent interest, and as the proof of Proposition 4.7 shows, it directly implies uniqueness of the preimage.

In essence, the proof reduces to compensating for the unbounded perturbation by viewing the equation in a rotating frame. However, despite the direct proof, the proposition is not trivial. The perturbation is not small but the argument works since the unbounded part of the perturbation generates a unitary group.

We are now in a position to conclude that the graph transform is well defined as a map from $\mathcal{S}_L$ into itself.

COROLLARY 4.10. $\mathcal{G} : \mathcal{S}_L \to \mathcal{S}_L$.

Proof. Proposition 4.7 and Proposition 4.8 prove that the image of a Lipschitz graph in $\mathcal{S}_L$ is a graph. That the image is also Lipschitz, with Lipschitz constant L, follows from the moving cone lemma. Finally, since $\mathcal{D}$ is positively invariant, the image must satisfy $\|\mathcal{G}(h)\| \leq 2R_2$. Thus the graph transform is well defined. □

Now we establish that the graph transform is a contraction mapping on $\mathcal{S}_L$.

PROPOSITION 4.11. *For $\varepsilon \in [0, \varepsilon_*]$ the graph transform $\mathcal{G} : \mathcal{S}_L \to \mathcal{S}_L$ is a contraction.*

Proof. Let $h_i \in \mathcal{S}_L,\ i = 1, 2$. We will show that $d(\mathcal{G}(h_1), \mathcal{G}(h_2)) \leq d(h_1, h_2)/2$. Fix u_T in $\Pi_1(\mathcal{D})$. Let $(u_i, h_i(u_i))$ be the unique preimages of $(u_T, \mathcal{G}(h_i)(u_T))$. The distance

$$\begin{aligned}
&\|\mathcal{G}(h_1)(u_T) - \mathcal{G}(h_2)(u_T)\| \\
&= \|\Pi_2(\varphi_T(u_1, h_1(u_1)) - \varphi_T(u_2, h_2(u_2)))\| \\
&\leq \|\Pi_2(\varphi_T(u_1, h_1(u_1)) - \varphi_T(u_1, h_2(u_1)))\| \qquad (4.44)\\
&\qquad + \|\Pi_2(\varphi_T(u_1, h_2(u_1)) - \varphi_T(u_2, h_2(u_1)))\| \qquad (4.45)\\
&\qquad + \|\Pi_2(\varphi_T(u_2, h_2(u_1)) - \varphi_T(u_2, h_2(u_2)))\|. \qquad (4.46)
\end{aligned}$$

By Lemma 4.3, (4.44) $\leq \|h_1(u_1) - h_2(u_1)\|/4$. By Lemma 4.4, (4.45) $\leq 3L\|u_1 - u_2\|/4$. And by Lemma 4.3, (4.46) $\leq \|h_2(u_1) - h_2(u_2)\|/4 \leq L/4\|u_1 - u_2\|$. Thus,

$$\begin{aligned}
\|\mathcal{G}(h_1)(u_T) - \mathcal{G}(h_2)(u_T)\| &\leq \frac{1}{4}\|h_1(u_1) - h_2(u_1)\| + L\|u_1 - u_2\| \\
&\leq \frac{1}{4} d(h_1, h_2) + L\|u_1 - u_2\|.
\end{aligned}$$

Furthermore, by the backward time estimate in Lemma 4.5

$$\|u_1 - u_2\| \leq \frac{1}{2L}\|\mathcal{G}(h_1)(u_T) - \mathcal{G}(h_2)(u_T)\|$$

so that

$$\|\mathcal{G}(h_1)(u_T) - \mathcal{G}(h_2)(u_T)\| \leq \frac{1}{2} d(h_1, h_2).$$

Since u_T was arbitrary, the lemma is proved. □

Corollary 4.12. *There is a unique solution to $\mathcal{G}(h_\varepsilon) = h_\varepsilon$ in $\mathcal{S}_L$. The graph of h_ε, denoted by $\mathcal{M}_\varepsilon$, is invariant under φ_t.*

Proof. We have established that $\varphi_T(\mathcal{M}_\varepsilon) = \mathcal{M}_\varepsilon$. If $0 < t < T$, then $\varphi_t(\mathcal{M}_\varepsilon)$ is the graph of a Lipschitz map in $\mathcal{S}_L$. This follows from Remark 4.9 and the moving cone lemma (notice that Lemma 4.2 is true for all $t \in [0, T]$). But then $\varphi_{T+t}(\mathcal{M}_\varepsilon) = \varphi_t(\mathcal{M}_\varepsilon)$ so that $\varphi_t(\mathcal{M}_\varepsilon)$ is also a fixed point of $\mathcal{G}$. By uniqueness, it follows that $\varphi_t(\mathcal{M}_\varepsilon) = \mathcal{M}_\varepsilon$. Since φ_t is a diffeomorphism, we must have $\varphi_t(\mathcal{M}_\varepsilon) = \mathcal{M}_\varepsilon$ for all $t \in \mathbb{R}$. ∎

$\mathcal{M}_\varepsilon$ is a *slow manifold*, as it is given as a graph over the slow variable, u. It is clear that all solutions within $\mathcal{D}_0$ are attracted exponentially fast onto the slow manifold. Indeed, given any point in $\mathcal{D}_0$ that does not lie in $\mathcal{M}_\varepsilon$, we may construct a graph in $\mathcal{S}_L$ that passes through this point. Then Proposition 4.11 shows that this graph is attracted exponentially fast onto $\mathcal{M}_\varepsilon$. In particular, this means that the attractor $\mathcal{A}$ is contained within $\mathcal{M}_\varepsilon$. Hence this construction partially answers the open question in [9] on the existence of an inertial manifold in the Maxwell–Bloch equations in the sense that we significantly simplify the geometry of the flow and prove the existence of a smooth, normally hyperbolic invariant manifold attracting all initial conditions. The answer is only partial since this manifold is infinite dimensional.

5. Smoothness of the invariant manifold. The smoothness of the slow manifold, $\mathcal{M}_\varepsilon$, is established by differentiating the following functional equation that h must satisfy:

$$h(\Pi_1(\varphi_T(u, h(u)))) = \Pi_2(\varphi_T(u, h(u))), \quad u \in \Pi_1(\mathcal{D}). \tag{5.1}$$

For brevity we let $u_T = \Pi_1(\varphi_T(u, h(u)))$ so that (5.1) may be rewritten as

$$h(u_T) = \Pi_2(\varphi_T(u, h(u))), \quad u \in \Pi_1(\mathcal{D}). \tag{5.2}$$

We differentiate (5.2) to obtain a nonlinear functional equation that the derivative of h must satisfy. We prove the existence of a solution to this equation by a contraction mapping argument.

5.1. Notation. We use the notation in [12] for differentiation. Let $\mathbb{X}_i, \mathbb{Y}$ be Banach spaces. If $F : \mathbb{X} \to \mathbb{Y}$, then $DF : \mathbb{X} \to L(\mathbb{X}, \mathbb{Y})$. If F is a function of several variables, say, $F : \mathbb{X}_1 \times \cdots \times \mathbb{X}_n \to \mathbb{Y}$, then $DF = (D_1F, \ldots, D_nF)$, where $D_iF : \mathbb{X}_1 \times \cdots \times \mathbb{X}_n \to L(\mathbb{X}_i, \mathbb{Y})$. In the interest of brevity we denote

$$P_i = D_i(\Pi_1 \circ \varphi_T), \quad Q_i = D_i(\Pi_2 \circ \varphi_T), \quad i = 1, 2. \tag{5.3}$$

5.2. C^1 smoothness. Differentiating (5.2) with respect to u we obtain

$$Dh(u_T)Du_T(u) = Q_1(u, h(u)) + Q_2(u, h(u))Dh(u).$$

Since $u_T = \Pi_1 \circ \varphi_T(u, h(u))$, its derivative is

$$Du_T(u) = P_1(u, h(u)) + P_2(u, h(u))Dh(u).$$

Thus, we obtain the formal expression

$$Dh(u_T) = [Q_1 + Q_2 Dh(u)]\,[P_1 + P_2 Dh(u)]^{-1} \tag{5.4}$$

for the derivative of h. (Here and henceforth we suppress the arguments of P_i, Q_i to simplify notation.) When $\varepsilon = 0$, the derivatives satisfy $P_1 = \mathrm{Id}$, $P_2 = 0$, and

$\|Q_2\| \le e^{-\beta T}$. Furthermore, $u_T = u$. Thus, in this limit, (5.4) reduces to $Dh(u) = Q_1 + Q_2 Dh(u)$, which has a unique solution since $\|Q_2\|$ is small. This suggests that we use iteration to solve (5.4) for $\varepsilon > 0$.

We now define the function space in which we wish to construct the derivative Dh. Let

$$\mathcal{T}_L = \left\{ A : \Pi_1(\mathcal{D}) \to L(\mathbb{X}_1, \mathbb{X}_2) | \sup_{u \in \Pi_1(\mathcal{D})} \|A(u)\|_{L(\mathbb{X}_1, \mathbb{X}_2)} \le L \right\}$$

be the metric space of continuous maps with the distance function

$$d(A_1, A_2) = \sup_{u \in \Pi_1(\mathcal{D})} \|A_1(u) - A_2(u)\|_{L(\mathbb{X}_1, \mathbb{X}_2)}.$$

$\mathcal{T}_L$ is complete in this metric.

We also define a map $\mathcal{F} : \mathcal{T}_L \to \mathcal{T}_L$ as

$$\mathcal{F}(A)(u_T) = [Q_1 + Q_2 A(u)] \, [P_1 + P_2 A(u)]^{-1}. \tag{5.5}$$

We shall prove that $\mathcal{F}$ is a contraction and the unique fixed point $\mathcal{F}(A) = A$ is the derivative of h. This will imply that $\mathcal{M}_\varepsilon$ is at least of class C^1.

We will use the following lemmas to estimate the terms in (5.5).

LEMMA 5.1. *There is $\varepsilon_* > 0$ so that for $\varepsilon \in [0, \varepsilon_*]$*
(a) $\sup_{u \in \Pi_1(\mathcal{D})} \|P_1 - e^{-\varepsilon T \partial_x}\|_{L(\mathbb{X}_1, \mathbb{X}_1)} = O(\varepsilon)$,
(b) $\sup_{u \in \Pi_1(\mathcal{D})} \|P_2\|_{L(\mathbb{X}_1, \mathbb{X}_2)} = O(\varepsilon)$,
(c) $\sup_{u \in \Pi_1(\mathcal{D})} \|Q_1\|_{L(\mathbb{X}_2, \mathbb{X}_1)} \le L/4$,
(d) $\sup_{u \in \Pi_1(\mathcal{D})} \|Q_2\|_{L(\mathbb{X}_2, \mathbb{X}_2)} \le 1/8$.

Proof. The proof entails estimating the growth of derivatives using the equation of variations. The arguments are direct but tedious so we will omit a few details. The main point is that despite the singular perturbation, we can control the derivatives with knowledge of the limit $\varepsilon = 0$ provided we account for the unbounded terms properly (e.g., as in statement (a) of the lemma).

We start by redefining $S(t, s; u_0)$, $t, s \in \mathbf{R}$, as the solution operator to the linear nonautonomous differential equation (4.32) with

$$f_1(t) = f(e^{-\varepsilon t \partial_x} u_0). \tag{5.6}$$

Notice that Lemma 4.6 remains valid with this definition of f_1. The mild formulation is the obvious analogue of (4.33)–(4.34) provided we redefine $F(u)$ as the skew-symmetric multiplication operator whose only nonzero terms are

$$\begin{aligned} F_{13} = -F_{31} &= \mathrm{Re}(f(u(s)) - f(e^{-\varepsilon s \partial_x} u_0)), \\ F_{23} = -F_{32} &= \mathrm{Im}(f(u(s)) - f(e^{-\varepsilon s \partial_x} u_0)). \end{aligned} \tag{5.7}$$

The estimate (4.36) shows that $F(u(s))$ is uniformly small on $[0, T]$. Differentiating (4.33) with respect to the initial point u_0, we obtain linear integral equations that the derivatives must satisfy.

$$\begin{aligned} D_{u_0} u(t) = e^{-\varepsilon t \partial_x} &+ \varepsilon \int_0^t e^{-\varepsilon(t-s)\partial_x} \left(D_1 g D_{u_0} u(s) + D_2 g D_{u_0} v(s) \right) ds \\ &+ \varepsilon \int_0^t e^{-\varepsilon(t-s)\partial_x} D_3 g D_{u_0} w(s) ds \end{aligned} \tag{5.8}$$

and

$$\begin{pmatrix} D_{u_0}v(t) \\ D_{u_0}w(t) \end{pmatrix} = D_{u_0}S(t,0;u_0)\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} + \int_0^t D_{u_0}(S(t,s;u_0))\left(\begin{pmatrix} 0 \\ \lambda+1 \end{pmatrix} + F(u(s))\begin{pmatrix} v(s) \\ w(s) \end{pmatrix}\right) ds + \int_0^t S(t,s;u_0)DF(u(s))D_{u_0}u(s)\begin{pmatrix} v(s) \\ w(s) \end{pmatrix} ds + \int_0^t S(t,s;u_0)F(u(s))\begin{pmatrix} D_{u_0}v(s) \\ D_{u_0}w(s) \end{pmatrix} ds. \tag{5.9}$$

The derivative of $S(t,0;u_0)$ is computed from its definition in (4.32). Since $S(t,0;u_0)$ is defined by the solution to a system of linear nonautonomous equations, its derivative $D_{u_0}S(t,0;u_0)$ is a linear map from $\mathbb{X}_1 \to L(\mathbb{X}_2, \mathbb{X}_2)$ defined for any $u_1 \in \mathbb{X}_1$ by

$$D_{u_0}S(t,0;u_0)u_1 = \int_0^t S(t,s;u_0)(D_{u_0}G(s,0;u_0)u_1)S(s,0;u_0)ds,$$

where $G(t,0;u_0)$ is the matrix defined on the right-hand side of (4.32) with $f_1(t)$ redefined as in (5.6). It follows from Lemma 4.1 and Lemma 4.6 that $\|D_{u_0}S(t,0;u_0)\| \leq 5te^{-\beta t}$. Thus the "linear" part of $D_{u_0}(v(t),w(t))$ is bounded for all u_0 and for all $t \in [0,T]$ by some constant C_9. By the choice of L (see Remark 4.6) $C_9 \leq L/8$. The nonlinear part in (5.9) (i.e., the terms with F and DF) are $O(\varepsilon)$ by Lemma 4.6 and the estimate (4.36). The nonlinear terms in (5.8) are also $O(\varepsilon)$ since $D_i g$ is uniformly bounded (see Lemma 4.1). Thus, one may prove that a solution to the equation of variations (5.8) and (5.9) exists for sufficiently small ε_* by a contraction mapping argument as in the proof of Proposition 4.8. Then Gronwall estimates show that $\sup_{t\in[0,T]} \max(\|D_{u_0}u(t)\|, \|D_{u_0}(v(t),w(t))\|) \leq C(T,Q_i)$ for all $(u_0,v_0,w_0) \in \mathcal{D}$ so that for all $\varepsilon \in [0,\varepsilon_*]$,

$$\|D_{u_0}u(T) - e^{-\varepsilon T\partial_x}\|_{L(\mathbb{X}_1,\mathbb{X}_1)} \leq \varepsilon C(T,R_i).$$

This proves (a). Similarly, the deviation of Q_1 from its linear part is $O(\varepsilon)$ and for small ε we have (c). Estimates (b) and (d) are obtained from the equation of variations for the derivative in (v_0,w_0). These are

$$D_{(v_0,w_0)}u(t) = \varepsilon\int_0^t e^{-\varepsilon(t-s)\partial_x}\left[D_1gD_{(v_0,w_0)}u(s) + D_2gD_{(v_0,w_0)}v(s)\right] ds + \varepsilon\int_0^t e^{-\varepsilon(t-s)\partial_x} D_3gD_{(v_0,w_0)}w(s)ds,$$

$$\begin{pmatrix} D_{(v_0,w_0)}v(t) \\ D_{(v_0,w_0)}w(t) \end{pmatrix} = S(t,0;u_0) + \int_0^t S(t,s;u_0)DF(u(s))D_{(v_0,w_0)}u(s)ds + \int_0^t S(t,s;u_0)F(u(s))\begin{pmatrix} D_{(v_0,w_0)}v(s) \\ D_{(v_0,w_0)}w(s) \end{pmatrix} ds.$$

Again, the nonlinear terms are $O(\varepsilon)$ so these equations can be solved by a contraction mapping argument. Gronwall estimates show that $\|D_{(v_0,w_0)}u(t)\|$ and $\|D_{(v_0,w_0)}(v(t), w(t)) - S(t,0;u_0)\|$ are $O(\varepsilon)$ with constants that depend only on R_i and T. But $\|S(t,0;u_0)\|$ is exponentially decaying by Lemma 4.6, and since T has been chosen so large that $e^{-\beta T/2} = 1/32$, we may further reduce ε_* to obtain (b) and (d). ∎

LEMMA 5.2. *There is $\varepsilon_* > 0$ so that for each $\varepsilon \in [0, \varepsilon_*]$ and $A \in \mathcal{T}_L$ we have*

$$\sup_{u \in \Pi_1(\mathcal{D})} \|[P_1 + P_2 A(u)]^{-1}\|_{L(\mathbb{X}_1, \mathbb{X}_1)} \leq 1 + O(\varepsilon) \leq 2.$$

Proof. We fix $u \in \Pi_1(\mathcal{D})$ and write

$$\begin{aligned}[P_1 + P_2 A]^{-1} &= [e^{-\varepsilon T \partial_x} - (e^{-\varepsilon T \partial_x} - P_1 - P_2 A)]^{-1} \\ &= e^{\varepsilon T \partial_x} [\mathrm{Id} - e^{\varepsilon T \partial_x}(e^{-\varepsilon T \partial_x} - P_1 - P_2 A)]^{-1}.\end{aligned}$$

This suggests that we write the inverse as a Neumann series. If $A \in \mathcal{T}_L$, then its norm is bounded by L. Thus by Lemma 5.1,

$$\|e^{-\varepsilon T \partial_x} - P_1 + P_2 A(u)\| \leq \|e^{-\varepsilon T \partial_x} - P_1\| + L\|P_2\| \leq C\varepsilon. \tag{5.10}$$

Also note that $e^{\varepsilon T \partial_x}$ is an isometry on $\mathbb{X}_1$. Thus, the Neumann series converges for $[P_1 + P_2 A]^{-1}$ for ε_* sufficiently small, and the norm of the sum does not exceed $1 + O(\varepsilon)$, which for sufficiently small ε is less than 2. ∎

Let $A \in \mathcal{T}_L$. Then we obtain

$$\|\mathcal{F}(A)\| \leq (\|Q_1\| + L\|Q_2\|)(1 + O(\varepsilon))$$

by the previous lemma. Furthermore, by Lemma 5.1, $\|Q_1\| + \|Q_2\| L \leq L/2$. Thus, for ε_* sufficiently small $\|\mathcal{F}(A)\| \leq L$, and hence $\mathcal{F}$ is well defined. The next proposition shows that for sufficiently small ε_*, it is in fact a contraction.

PROPOSITION 5.1. *There is $\varepsilon_* > 0$ such that for $\varepsilon \in [0, \varepsilon_*]$ the mapping $\mathcal{F} : \mathcal{T}_L \to \mathcal{T}_L$ is a contraction.*

Proof. We let $A, B \in \mathcal{T}_L$, fix $u \in \Pi_1(\mathcal{D})$, and let $u_T = \Pi_1 \circ \varphi_T(u, h(u))$. Then

$$\begin{aligned}&\mathcal{F}(A)(u_T) - \mathcal{F}(B)(u_T) = Q_2(A(u) - B(u))[P_1 + P_2 A(u)]^{-1} \\ &+(Q_1 + Q_2 B(u))\left([P_1 + P_2 A(u)]^{-1} - [P_1 + P_2 B(u)]^{-1}\right).\end{aligned} \tag{5.11}$$

Applying Lemmas 5.1 and 5.2 to the first term we have

$$\|Q_2(A(u) - B(u))[P_1 + P_2 A(u)]^{-1}\| \leq \frac{2}{8}\|A(u) - B(u)\|. \tag{5.12}$$

We use the identity $(M - A)^{-1} - (M - B)^{-1} = (M - A)^{-1}(A - B)(M - B)^{-1}$ and Lemma 5.2 to estimate the second term in (5.11)

$$\begin{aligned}&(Q_1 + Q_2 B(u))\left([P_1 + P_2 A(u)]^{-1} - [P_1 + P_2 B(u)]^{-1}\right) \\ &\leq 4(\|Q_1\| + L\|Q_2\|)\|P_2\|\|A(u) - B(u)\| \leq 2L\|P_2\|\|A(u) - B(u)\| \\ &\leq C\varepsilon\|A(u) - B(u)\| \leq \frac{1}{2}\|A(u) - B(u)\|\end{aligned} \tag{5.13}$$

for sufficiently small ε_*. Thus, $\|\mathcal{F}(A)(u_T) - \mathcal{F}(B)(u_T)\| \leq 3/4\|A(u) - B(u)\|$. Since u was arbitrary, this proves the lemma. ∎

To complete the proof that $\mathcal{M}_\varepsilon$ is C^1, we must show that the unique fixed point of $\mathcal{F}$ is indeed the derivative Dh. This step is essentially the same as Proposition 7 in Fenichel's paper [11], so the proof is omitted.

5.3. C^k smoothness. Higher order smoothness will be proven using the following bootstrapping argument of Fenichel [11]. The unique fixed point $A = \mathcal{F}(A)$ can be realized as the limit of a sequence of iterates $A^n = \mathcal{F}^n A^0$ with $A^0 = 0$. For any $u \in \Pi_1(\mathcal{D})$,

$$A^{n+1}(u_T) = [Q_1 + Q_2 A^n(u)][P_1 + P_2 A^n(u)]^{-1}. \tag{5.14}$$

Since h is C^1, the maps $P_i = P_1(u, h(u))$ and Q_i are differentiable, so A^{n+1} is differentiable if A^n is. Thus, to show that the limit A is differentiable it suffices to show that the sequence $\{DA^n\}$ converges in the space $C(\Pi_1(\mathcal{D}), L^2(\mathbb{X}_1, \mathbb{X}_2))$.

We will show this with estimates similar to those of Proposition 5.1. From the proof of Proposition 5.1 (in particular, (5.12) and (5.13) with $A^{n-1} = A$ and $A^n = B$) we see that the principal term in the contraction estimate at the nth step is

$$\begin{aligned} &\|(P_1 + P_2 A^{n-1})^{-1}\| \left(\|Q_2\| + \|P_2\| \|Q_1 + Q_2 A^n\| \|(P_1 + P_2 A^n)^{-1}\|\right) \\ &\quad \le (1 + C\varepsilon)\left(\frac{1}{8} + C\varepsilon\right) := \alpha_1 \end{aligned} \tag{5.15}$$

by Lemma 5.1 and Lemma 5.2. Higher order derivatives can be obtained in the same way. Differentiating (5.14) we obtain

$$\begin{aligned} DA^{n+1}(u_T)Du_T(u) &= Q_2 DA^n(u)[P_1 + P_2 A^n(u)]^{-1} \\ &-[Q_1 + Q_2 A^n][P_1 + P_2 A^n]^{-1} P_2 DA^n [P_1 + P_2 A^n]^{-1} + \text{lower order terms}, \end{aligned} \tag{5.16}$$

where the lower order terms do not involve derivatives in A. Thus, the principal term in the contraction estimate is now

$$\begin{aligned} &\left(\|Q_2\| + \|P_2\| \|Q_1 + Q_2 A^n\| \|(P_1 + P_2 A^n)^{-1}\|\right) \|(P_1 + P_2 A^{n-1})^{-1}\|^2 \\ &\quad \le (1 + C\varepsilon)^2 \left(\frac{1}{8} + C\varepsilon\right) := \alpha_2. \end{aligned} \tag{5.17}$$

For ε_* sufficiently small, $\alpha_2 < 1$ holds for $0 \le \varepsilon \le \varepsilon_*$. Let $a_n = \sup_u \|DA^{n+1}(u_T) - DA^n(u_T)\|$. It follows from (5.16) and (5.17) that

$$a_{n+1} \le \alpha_2 a_n + r_n,$$

where r_n is a remainder term obtained from the differences in lower order terms. r_n diminishes to zero as n increases since A^n converges. Thus, for any $\eta > 0$ there exists an N such that $r_n \le \eta$ for all $n \ge N$. Hence,

$$a_{N+m} \le \alpha_2^m a_N + \frac{\eta}{1 - \alpha_2},$$

and thus $\limsup_{n \to \infty} a_n \le \eta/(1 - \alpha_2)$. Since η was arbitrary, $a_n \to 0$, and it follows that the sequence $\{DA^n\}$ converges. Thus, $\mathcal{M}_\varepsilon$ is of class C^2.

We now proceed inductively. To show that $\mathcal{M}_\varepsilon$ is C^k assuming that it is C^{k-1}, it is sufficient to show that the sequence $\{D^k A^n\}$ converges. Each term in the sequence is of the form

$$\begin{aligned} D^k A^{n+1}(u_T)Du_T(u) &= Q_2 D^k A^n(u)[P_1 + P_2 A^n(u)]^{-k} \\ &-[Q_1 + Q_2 A^n][P_1 + P_2 A^n]^{-1} P_2 D^k A^n [P_1 + P_2 A^n]^{-k} + \text{terms of order } k-1, \end{aligned}$$

and the principal term in the contraction estimate is bounded by

$$(1 + C\varepsilon)^k \left(\frac{1}{8} + C\varepsilon\right) := \alpha_k.$$

For $\varepsilon_*(k)$ sufficiently small, $\alpha_k < 1$ for all $\varepsilon \in [0, \varepsilon_*]$ and the sequence $\{D^k A^n\}$ is convergent. Thus, $\mathcal{M}_\varepsilon$ is of class C^k. The manifold is not C^∞ since it is clear that $\varepsilon_*(k)$ must decrease to zero as k increases arbitrarily. This completes the proof of the C^k smoothness and the proof of Theorem 4.4.

6. Geometric singular perturbation theory.

6.1. Notation. In this section we need to distinguish carefully between the flow for different values of ε. To emphasize this, we will use the superscript ε. For example, φ_t^ε denotes the flow with a particular choice of ε, $(u^\varepsilon(t), v^\varepsilon(t), w^\varepsilon(t))$ denotes a trajectory, and $\mathcal{A}_\varepsilon$ denotes the attractor for φ_t^ε. We use the same notation for the modified and unmodified flow, but the flow under consideration will be clear from the context.

6.2. Reduced dynamics and the slaving principle. Theorem 4.4 provides a rigorous decomposition of the flow and a justification of the "slaving principle." First consider the modified flow. Since $\mathcal{M}_\varepsilon$ is invariant, any trajectory on it must satisfy

$$u(t) = e^{-\varepsilon t \partial_x} u(0) + \varepsilon \int_0^t e^{-\varepsilon(t-s)\partial_x} g(u(s), h^\varepsilon(u(s)))ds, \tag{6.1}$$

$$h_v^\varepsilon(u(t)) = e^{-(1+i\delta)t} h_v^\varepsilon(u(0)) + \mu \int_0^t e^{-(1+i\delta)(t-s)} f(u(s)) h_w^\varepsilon(u(s))ds, \tag{6.2}$$

$$\begin{aligned} h_w^\varepsilon(u(t)) = {}& e^{-\gamma_\| t} h_w^\varepsilon(u(0)) + (\lambda + 1)(1 - e^{-\gamma_\| t}) \\ & - \mu \int_0^t e^{-(t-s)\gamma_\|} \operatorname{Re}\,(f(u(s))^* h_v^\varepsilon(u(s)))ds. \end{aligned} \tag{6.3}$$

Thus, the slow dynamics decouples from the fast dynamics. This is only half the story: we have established the existence of a reduced equation that is a *functional* differential equation, but we have not prescribed a formula to compute the reduced equation. Theorem 4.4 proves the existence of a family of invariant manifolds $\{\mathcal{M}_\varepsilon\}_{\varepsilon \in [0, \varepsilon_*]}$. In Fenichel's theory [12] these manifolds $\mathcal{M}_\varepsilon$ fit together smoothly in ε and there is a global center manifold given as a function $h(\varepsilon, u) = h^\varepsilon(u)$. Thus we may expand $h^\varepsilon(u) = h(0, u) + D_1 h(0, u)\varepsilon + R(u, \varepsilon)$, where $R = o(\varepsilon)$. In infinite dimensions the situation is considerably more delicate. The issue is, of course, the unbounded term $\varepsilon \partial_x$. For flows that are close in the C^1 topology, it usually follows from an implicit function theorem, or the proof of the existence of the invariant manifold, that the unperturbed and perturbed manifolds are close. In the presence of unbounded perturbations the convergence of h^ε to h^0 is expressed in the following theorem.

THEOREM 6.1. $\|h^\varepsilon(u) - h^0(u)\| \to 0$ *uniformly on compact sets.*

Proof. The only information we have is that $\mathcal{M}_\varepsilon$ is invariant under the flow φ_t^ε. Thus the proof will rely on the closeness of φ_t^ε to φ_t^0. We fix $u \in \Pi_1(\mathcal{D}_0)$ and let $u(0) = u$. We will estimate the difference $h^\varepsilon(u) - h^0(u)$ by using the integral equations (6.1)–(6.3). When $\varepsilon = 0$, (6.1)–(6.3) reduce to the algebraic equations $u(t) \equiv u$ and

$$\begin{pmatrix} 1 & \delta & -\mu \operatorname{Re} f(u) \\ -\delta & 1 & -\mu \operatorname{Im} f(u) \\ \mu \operatorname{Re} f(u) & \mu \operatorname{Im} f(u) & \gamma_\| \end{pmatrix} \begin{pmatrix} \operatorname{Re} h_v^0(u) \\ \operatorname{Im} h_v^0(u) \\ h_w^0(u) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma_\|(\lambda + 1) \end{pmatrix}, \tag{6.4}$$

or, more briefly,

$$B(u)h^0(u) = (0, 0, \gamma_{\|}(\lambda + 1))^T, \tag{6.5}$$

where $B(u)$ is the multiplication operator in (6.4). This is the expression that was computed in (3.1). For every $x \in S^1$, the matrix $B(u(x))$ is invertible since

$$\det(B(u(x))) = \gamma_{\|}(1 + \delta^2) + \mu^2|f(u)(x)|^2 \geq \gamma_{\|}(1 + \delta^2) \tag{6.6}$$

is uniformly bounded away from zero. Thus, $B(u)$ is invertible and its inverse, $B(u)^{-1}$, is the multiplication operator in $L(\mathbb{X}_2, \mathbb{X}_2)$ defined for every $x \in S^1$ by the matrix $B(u(x))^{-1}$. From the algebraic formula for the inverse of a matrix, and the estimate (6.6), we see that

$$\sup_{u \in \Pi_1(\mathcal{D})} \|B(u)^{-1}\| \leq C_B(R_1). \tag{6.7}$$

Define the function $\Psi : \mathbb{X} \to \mathbb{X}_2$ by

$$\Psi(u, v, w) = B(u) \begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} 0 \\ \gamma_{\|}(\lambda + 1) \end{pmatrix}. \tag{6.8}$$

Notice that $\Psi(u, v, w) = 0$ if and only if $(v, w) = h^0(u)$. Let K be a compact subset of $\Pi_1(\mathcal{D})$. We will show that $\lim_{\varepsilon \downarrow 0} \sup_{u \in K} \|\Psi(u, h^\varepsilon(u))\| = 0$. This implies the theorem since

$$h^\varepsilon(u) - h^0(u) = B(u)^{-1}B(u)h^\varepsilon(u) - B(u)^{-1}(0, \gamma_{\|}(\lambda + 1))^T = B(u)^{-1}\Psi(u, h^\varepsilon(u)), \tag{6.9}$$

and hence from (6.7) we obtain

$$\sup_{u \in K} \|h^\varepsilon(u) - h^0(u)\| \leq C_B \sup_{u \in K} \|\Psi(u, h^\varepsilon(u))\|.$$

We will consider the components of Ψ separately. The first two components of Ψ are $(1 + i\delta)h^\varepsilon_v(u) - \mu f(u)h^\varepsilon_w(u)$ (taking the real and imaginary components together). We estimate this term using (6.2). We start with the initial condition $(u, v, w)(0) = (u, h^\varepsilon(u))$ and then calculate that for any $t > 0$,

$$\begin{aligned}(1 - e^{-(1+i\delta)t})\left[-(1 + i\delta)h^\varepsilon_v(u) + \mu f(u)h^\varepsilon_w(u)\right] &= (1 + i\delta)\Big[h^\varepsilon_v(u^\varepsilon(t)) - h^\varepsilon_v(u) \\ &\quad -\mu \int_0^t e^{-(1+i\delta)(t-s)} \left(f(u^\varepsilon(s))h^\varepsilon_w(u^\varepsilon(s)) - f(u)h^\varepsilon_w(u)\right) ds\Big]\end{aligned} \tag{6.10}$$

(since u is a constant, we can take $\mu f(u)h^\varepsilon_w(u)$ under the integral sign). Notice that

$$\begin{aligned}\mathrm{Lip}(fh^\varepsilon_w) &\leq \left(\sup_u \|f(u)\|\right) \mathrm{Lip}(h^\varepsilon_w) + \left(\sup_u \|h^\varepsilon_w(u)\|\right) \mathrm{Lip}(f) \\ &\leq 2R_1L + 2R_2 \cdot 5 := C_{10}\end{aligned}$$

by Lemma 4.1 and the definition of $\mathcal{S}_L$. Thus, we obtain from (6.10) that

$$\begin{aligned}&\|(1 + i\delta)h^\varepsilon_v(u) - \mu f(u)h^\varepsilon_w(u)\| \\ &\leq \frac{|1 + i\delta|}{|1 - e^{-(1+i\delta)t}|} \left[L\|u^\varepsilon(t) - u\| + \mu \int_0^t C_{10}e^{-(t-s)}\|u^\varepsilon(s) - u\|ds\right] \\ &\leq C_{11} \left[\frac{\|u^\varepsilon(t) - u\|}{|1 - e^{-(1+i\delta)t}|} + \frac{1 - e^{-t}}{|1 - e^{-(1+i\delta)t}|} \sup_{s \in [0,t]} \|u^\varepsilon(s) - u\|\right].\end{aligned}$$

Notice that $(1-e^{-t})/|1-e^{-(1+i\delta)t}|$ is uniformly bounded for all $t>0$. And for t in any fixed domain $(0,T]$ we have $1/|1-e^{-(1+i\delta)t}| \le C(T)/t$. Thus we find that

$$\|(1+i\delta)h_v^\varepsilon(u)-\mu f(u)h_w^\varepsilon(u)\| \le C\left[\frac{\|u^\varepsilon(t)-u\|}{t}+\sup_{s\in[0,t]}\|u^\varepsilon(s)-u\|\right]. \tag{6.11}$$

A similar calculation shows that we obtain the same result for the second component of Ψ. Thus, we find

$$\|\Psi(u,h^\varepsilon(u))\| \le C\left[\frac{\|u^\varepsilon(t)-u\|}{t}+\sup_{s\in[0,t]}\|u^\varepsilon(s)-u\|\right]. \tag{6.12}$$

Finally, we use (6.1) to estimate the difference $u^\varepsilon(t)-u$. The difference consists of two parts, the deviation from the linear part of the flow $e^{-\varepsilon t\partial_x}u$ and the deviation of the linear flow from the nonlinear flow. Precisely,

$$\|u^\varepsilon(t)-u\| \le \|u^\varepsilon(t)-e^{-\varepsilon t\partial_x}u\|+\|e^{-\varepsilon t\partial_x}u-u\| \le C\varepsilon t+\|e^{-\varepsilon t\partial_x}u-u\|.$$

Inserting this estimate in (6.12) we have

$$\|\Psi(u,h^\varepsilon(u))\| \le C\left[\frac{\|e^{-\varepsilon t\partial_x}u-u\|}{t}+\sup_{s\in[0,t]}\|e^{-\varepsilon t\partial_x}u-u\|+\varepsilon(1+t)\right]. \tag{6.13}$$

For fixed t and u, the right-hand side of (6.13) goes to zero as $\varepsilon\downarrow 0$. Next suppose that we fix t but consider u ranging over a compact subset K. Since functions in K are equicontinuous, $\sup_{u\in K}\|e^{-\varepsilon t\partial_x}u-u\|\to 0$ as $\varepsilon\downarrow 0$. □

The estimate (6.13) highlights why the convergence of h^ε to h^0 is not any better than uniform convergence on compact subsets. Since t is a free parameter, the estimate is best when we take the infimum with respect to t. Since the flow is continuous in t we must have $\sup_{s\in[0,t]}\|u^\varepsilon(s)-u\|\to 0$ as $t\to 0$. But the first term in (6.12) may not have a limit. The reason is that $\lim_{t\downarrow 0}\|e^{-\varepsilon t\partial_x}u-u\|/t$ does not exist for most functions (in the sense of category). If u is C^1, then we find that

$$\|\Psi(u,h^\varepsilon(u))\| \le C\left(\|Du\|_\infty\varepsilon+\sup_{s\in[0,t]}\|Du\|_\infty\varepsilon t+\varepsilon(1+t)\right),$$

and since t is a free parameter, we take the infimum over t to find

$$\|\Psi(u,h^\varepsilon(u)) \le C(\|Du\|_\infty+1)\varepsilon. \tag{6.14}$$

Another example of more rapid convergence is provided by taking K to be a bounded subset of $C^{0,\alpha}(S^1;\mathbf{C})$, the space of Hölder continuous functions with modulus $\alpha\in(0,1]$. In this case we find that

$$\sup_{u\in K}\|e^{-\varepsilon t\partial_x}u-u\| \le H\varepsilon^\alpha t^{\alpha-1},$$

where H is the maximum Hölder seminorm of the functions in K. Then we take the infimum in t on both sides of (6.13) to find that

$$\sup_{u\in K}\|h^\varepsilon(u)-h^0(u)\| \le C\varepsilon^\alpha.$$

6.3. Formal asymptotic expansions. Equations (6.1)–(6.3) are also the starting point for a formal asymptotic expansion. Theorem 6.1 shows that we can control the remainder only if $u(s)$ has some smoothness in x. However, a *formal* asymptotic expansion may be obtained by using the invariance of $\mathcal{M}_\varepsilon$. Make the ansatz

$$h^\varepsilon(u) = h^0(u) + \varepsilon h_1(u) + \varepsilon^2 h_2(u) + \cdots. \tag{6.15}$$

Substituting this ansatz in (6.1)–(6.3) and matching the powers of ε we obtain after some calculations that

$$h_n(u) = c_n(u)\partial_x^n u + d_n(u, \partial_x u, \ldots, \partial_x^{n-1} u), \quad n \geq 1, \tag{6.16}$$

where $c_n(u)(x)$ depends only on $u(x)$. This expansion suggests that $h^\varepsilon(u)(x)$ actually depends on the germ of u at x. Thus, we expect $h^\varepsilon(u)$ to have a nonlocal dependence on u. The expansion also suggests that the reduced equation (6.17) is not hyperbolic because $h_n(u)$ includes higher order diffusive and dispersive terms. In fact, the reduced equation cannot be hyperbolic for if it were, there would be no asymptotic smoothing on the attractor. Similar questions arise in hyperbolic conservation laws with relaxation. We refer especially to the article by Chen, Levermore, and Liu, section 2 of which contains the same geometric description of formal reductions in the context of conservation laws [5].

6.4. Regular dynamics. We can now revert to a description of the unmodified Maxwell–Bloch equations in the slow (and natural) time scale. Changing the time scale to $\tau = \varepsilon t$ we have for all $u(0) \in \Pi_1(\mathcal{D}_0)$ and $\tau \geq 0$,

$$u(\tau) = e^{-\kappa\tau} e^{-\tau\partial_x} u(0) + \frac{\kappa}{\mu}\int_0^\tau e^{-\kappa(\tau-s)} e^{-(\tau-s)\partial_x} h_v^\varepsilon(u(s))ds. \tag{6.17}$$

We have used the positive invariance of $\Pi_1(\mathcal{D}_0)$ and the fact that $g(u, v, w)$ reduces to v within the domain $\mathcal{D}_0$. To make the comparison with the formal reduction precise, we shall write (6.17) as

$$\begin{aligned} u(\tau) = e^{-\kappa\tau} e^{-\tau\partial_x} u(0) &+ \frac{\kappa}{\mu}\int_0^\tau e^{-\kappa(\tau-s)} e^{-(\tau-s)\partial_x} h_v^0(u(s))ds \\ &+ \frac{\kappa}{\mu}\int_0^\tau e^{-\kappa(\tau-s)} e^{-(\tau-s)\partial_x} (h_v^\varepsilon(u(s)) - h_v^0(u(s)))ds. \end{aligned} \tag{6.18}$$

The attractor $\mathcal{A}_\varepsilon$ is an invariant set contained in $\mathcal{M}_\varepsilon$. On the attractor, the reduction is valid uniformly in time. Applying (6.14) to $u \in \Pi_1(\mathcal{A}_\varepsilon)$ we have

$$\begin{aligned} &\left\| u(\tau) - e^{-\kappa\tau} e^{-\tau\partial_x} u(0) - \frac{\kappa}{\mu}\int_0^\tau e^{-\kappa(\tau-s)} e^{-(\tau-s)\partial_x} h_v^0(u(s)) \right\| \\ &\leq \frac{\kappa}{\mu}\int_0^\tau e^{-\kappa(\tau-s)} \|h^\varepsilon(u(s)) - h^0(u(s))\| ds \leq C\varepsilon \left(\sup_{u\in\mathcal{A}_\varepsilon} \|Du\|_\infty + 1 \right) \end{aligned}$$

for all τ. Unfortunately, this isn't enough as the estimates of section 4 in [9] show that $\sup_{u\in\mathcal{A}_\varepsilon} \|Du\|_\infty = O(1/\varepsilon)$. Furthermore, based on numerical evidence we expect that this estimate is sharp. In several parameter regimes the Lorenz ODEs have periodic solutions with arbitrarily large period. These solutions in turn imply the existence of traveling wave solutions to the Maxwell–Bloch equations with gradients of $O(1/\varepsilon)$. In fact, estimating $\sup_{u\in\mathcal{A}_\varepsilon} \|\partial_x^n u\|_\infty$ we find that the series (6.16) diverges even on the attractor.

6.5. Change of stability under perturbation. The guiding philosophy of geometric singular perturbation for ODEs is that normally hyperbolic manifolds within the formally reduced flow persist for the perturbed flow, provided the critical manifold is normally hyperbolic. As an example of this, Fenichel proved a theorem of Anosov on the persistence of periodic orbits for a singularly perturbed ODE [12]. A simpler example is to consider hyperbolic fixed points. Let the reduced flow have an exponentially attracting fixed point, and let the critical manifold be exponentially attracting. Then Theorem 12.1 in [12] shows that the fixed point persists for $\varepsilon > 0$ and remains attracting.

Even this simple assertion is false for PDE; i.e., the unbounded perturbation may change the stability type of a fixed point within the persisting slow manifold. For simplicity suppose $\delta = 0$ and consider only real (u, v, w). In this case the reduced equation is a gradient dynamical system with a double well potential, and $u = 1$ is a spatially homogenous equilibrium of the reduced equation. It is attracting since it lies at the minimum of the well. A calculation reveals that the point $(u, v, w) = (1, h^0(1)) = (1, \mu, 1)$ is an equilibrium of the full equations for all $\varepsilon > 0$. Nevertheless, it need not retain the stability type of the $\varepsilon = 0$ limit. Risken and Nummedal [25] showed that the fixed point is unstable for large λ for all positive ε and the number of linearly unstable modes diverges like $1/\varepsilon$. Thus the divergence between the formal limit and the full system is dramatic for small ε.

6.6. Conclusions. We have developed a geometric method of studying the singularly perturbed Maxwell–Bloch equations. The main merit of this method is that it rigorously separates the dynamics of this problem into slow and fast evolution. The geometric principles underlying the method are simple and thus it should be of use in other problems. However, the Maxwell–Bloch equations have several simplifying features and there are often many technical difficulties inherent in a rigorous analysis of PDEs with multiple scales. Thus transporting these ideas to other PDEs will be a difficult (but rewarding) task. Moreover, we have shown that global invariant manifolds with infinite dimension and codimension arise naturally in evolution equations with two scales. One may rigorously find reduced equations for such systems, but these are functional differential equations, and naive approximations to these equations seem to fail. There are several subtle features in geometric singular perturbation theory in infinite dimensions, and the Maxwell–Bloch equations illustrate some of these in a setting with few technicalities.

There have been several recent developments in geometric singular perturbation theory for PDE. We mention the work by Li et al. [21], Haller [17], and Zeng [29] on the damped and driven nonlinear Schrödinger equation. The motivation and methods are different there: in that case the $\varepsilon = 0$ limit is integrable, and a lot of effort is expended in solving problems associated with nonhyperbolicity and weak hyperbolicity. We also mention that Hale, Raugel, Sell, and coworkers have studied PDE in thin domains (see the references in [24]).

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