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# Geometry and chaos near resonant equilibria of 3-DOF Hamiltonian systems

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**Abstract**

In this paper we study the dynamics near resonant elliptic equilibria in three-degree-of-freedom Hamiltonian systems. The resonances we consider have multiplicity two, and the corresponding local normal form for the equilibrium is integrable at cubic order. We prove the existence of families of 3-tori and whiskered 2-tori with nearby chaotic dynamics in the quartic normal form. The whiskers of the 2-tori intersect in a non-trivial way giving rise to *multi-pulse* homoclinic and heteroclinic connections. These connections survive in the full system as orbits homoclinic to invariant 3-spheres.

## 1. Introduction

Suppose that the quadratic part $H_2$ of a three-degree-of-freedom (3-DOF) smooth Hamiltonian $H = H_2 + \tilde{H}$ is of the form

$$H_2 = \tfrac{1}{2}\sum_{k=1}^{3} \omega_k(q_k^2 + p_k^2), \qquad (1.1)$$

which is characterized by the frequency vector $\omega = (\omega_1, \omega_2, \omega_3)$. This frequency is said to be resonant if we can find a nonzero integer vector $n = (n_1, n_2, n_3)$ such that

$$\langle \omega, n \rangle = 0 \qquad (1.2)$$

is satisfied. We usually speak about a *strong resonance* if there exists an integer vector $n$ verifying (1.2) with $|n| = |n_1| + |n_2| + |n_3| \leq 4$. One can also introduce the term *full resonance* (or *multiplicity-two resonance*) which means that there are two linearly independent integer vectors $n$ and $\tilde{n}$ satisfying the relation (1.2).

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### 1.1. The nature of dynamics near resonant elliptic equilibria

It is well known that the six-dimensional phase space of (1.1) is foliated by a three-parameter family of 3-tori, which best manifest themselves in action-angle variables (see, e.g., Arnold [4]). System (1.1) also has three families of distinguished motions, usually called *normal modes*, which are periodic orbits with only one nonvanishing action. As we know from the appropriate version of the KAM theory (see, e.g., Arnold et al. [5]), in a neighborhood of the origin $q = p = 0$ most of the 3-tori typically persist under the effect of the higher order terms in $\tilde{H}$ provided $\omega$ is not strongly resonant. In these cases, close to the origin, the observed perturbed dynamics is reminiscent of the unperturbed one, at least for finite times.

As it was soon revealed by computer experiments following the appearance of KAM-type results, the dynamics around a strongly resonant equilibrium may differ significantly from the picture described above (see, e.g., Ford and Waters [20] and Ford and Lunsford [21]). In particular, one can observe significant short-time deviation from the solutions of (1.1). Trajectories of $H$ starting close to the linear normal modes of the quadratic part $H_2$ may leave and return on time scales much shorter than those mentioned above. Moreover, plotting the action values of these trajectories, one experiences irregular patterns in the change of action, which is usually referred to as *energy transfer* between different modes (cf. Ford and Waters [20], Ford and Lunsford [21], Van der Aa and Sanders [57,58], etc.) Though this terminology is descriptive, it is not quite accurate since the linear normal modes of $H_2$ do not necessarily persist as *nonlinear normal modes* under the effect of $\tilde{H}$. Hence, the first natural question one might ask about the nature of full resonances should be about the fate of the linear normal modes and the possible creation of new periodic orbits.

A general answer to this question is given by the work of Weinstein [60] (see also Moser [48] and Ito [30]). If $H_2$ is definite, his results guarantee the existence of at least three distinct periodic orbits on every energy surface $H$ = const. For more specific results one has to appeal to the method of normal forms (see, e.g., Arnold et al. [5], Sanders and Verhulst [54]) and simplify the general Hamiltonian $H = H_2 + \tilde{H}_3$ for the purposes of the analysis. This is achieved through a smooth near-identity change of variables which puts $H$ to the Birkhoff normal form

$$H = H_2 + H_3 + H_4 + \cdots + H_r + \mathcal{O}(r+1), \tag{1.3}$$

where $H_j$ is a homogeneous polynomial of order $j$ in the new coordinates $(q', p')$ with $\{H_2, H_j\} = 0$, $\{\ ,\ \}$ denoting the Poisson-bracket. If we truncate (1.3) at some order less than $r$ to obtain a Hamiltonian $\bar{H}$, the corresponding system has two independent first integrals: $\bar{H}$ and $H_2$. If $\omega$ is not fully resonant, we can always find one more independent integral (see Arnold et al. [5]); hence the normal form is integrable if truncated at any finite order. This effectively means that system (1.3) exhibits a resonance only between two of the frequencies and can be analyzed by the methods developed for 2-DOF resonant Hamiltonians (see Arnold et al. [5], Churchill et al. [8,9], Sanders and Verhulst [54], and the references cited therein). To obtain inherently 3-DOF effects, one therefore needs to assume that the frequency $\omega$ is fully resonant, in which case the truncated normal form is not automatically integrable.

### 1.2. Previous work on 3-DOF resonant normal forms

To describe how complex a given fully resonant normal form is, we may speak about a *genuine kth order resonance* if (1.3) truncated at order $k+2$ (and not below) exhibits full coupling between all the three degrees of freedom (for a precise definition see Sanders and Verhulst [54]). A systematic study of periodic solutions and their stability for genuine first order resonances was carried out in Van der Aa [58] (see also Sanders and Verhulst [54]). Related results for symmetric systems appeared in, e.g., Montaldi et al. [45,46], and the references therein, which also address the question of persistence of periodic solutions for the full system (1.3). Parallel to this, one can also see an increasing interest in the integrability of fully resonant normal forms. The picture arising from the works of Martinet et al. [43], Van der Aa and Sanders [57], Van der Aa [58], Van der Aa and Verhulst et al. [59], is that genuine first order resonances (truncated at cubic order) do not seem to have a third independent integral in general. Nice exceptions are the 1:2:2 resonance and resonances with discrete symmetries. For these integrable cases the methods of symplectic and Poisson reductions (Abraham and Marsden [1]) gave insights into the foliation of the phase space (see Cushman [11] and Kummer [38]), making use of related results for two degrees of freedom (cf. Kummer [37], Cushman and Rod [12], Churchill et al. [9], Knobloch et al. [34] ). The work of Kummer [38] extends this point of view to $n$ degrees of freedom and also gives a persistence theorem for a class of $n-1$-tori arising from the truncated normal form. Recently Hoveijn [29] gave a nice classification of the possible reduced phase spaces of integrable 3-DOF resonances using the theory of singular reduction. He also proved the existence of invariant spheres in these systems.

As a rule, most of the above results seem to focus on regular behavior (periodic orbits, 2-tori) in 3-DOF resonances. The first reference pointing towards irregularity appears to be Duistermaat [15] with a proof that the 1:1:2 resonance is typically nonintegrable. Duistermaat shows an infinite branching of complex continuation of manifolds of periodic orbits with constant frequency, which is known to be an obstruction to integrability (see Arnold et al. [5]).

A more geometric discussion of another resonance, the 1:2:3 is given by Hoveijn and Verhulst [28], who consider a given fourth order normal form and present numerical results showing the existence of an orbit homoclinic to a relative equilibrium. Using Melnikov's method Hoveijn [29] completed this study by proving that the stable and unstable manifolds of the relative equilibrium intersect transversally for a special choice of the quartic normal form terms. As Hoveijn points out, the algebraic splitting of separatrices in the truncated 1:2:3 normal form is an indication that the chaotic behavior occurring near 3-DOF resonant equilibria is much stronger than near 2-DOF resonant equilibria, for which the local normal form is integrable up to any order of truncation.

A general study of the dynamics near multi-degree-of-freedom resonant equilibria appears in Delshams [13] (see also de la Llave and Wayne [40]). Delshams considers equilibria with a *simple* resonance and establishes the existence of whiskered tori for the truncated normal form in the vicinity of the equilibrium. His calculations indicate the survival of these tori in the full system with their whiskers exhibiting exponentially small splittings.

### 1.3. A special class of multiplicity two resonances

In this paper we would like to go one step further in exploring the mechanism and onset of chaos near 3-DOF resonant equilibria. We consider fully resonant systems for which the resonance relationship (1.2) is satisfied with $|n_1| = 2$, $|n_2| = 1$. (This resonance is usually referred to as *Fermi resonance* in the physics literature.) In the usual terminology, we assume that the resonance has a *generator* of the form $\bar{n} = (2,1,0)$ or $(2,-1,0)$. Rescaling the frequencies by $\omega_1$, we then obtain the new frequency vector $\tilde{\omega} = (1, \pm 2, \tilde{\omega}_3)$, where $\tilde{\omega}_3$ is a nonzero rational number. We want to ensure that the corresponding normal form is integrable when truncated at cubic order and intend to treat higher order normal form terms as perturbations on this integrable structure. To obtain an integrable cubic normal form we may make various assumptions. *Either* we require $|\tilde{\omega}_3| \geq 5$ or $|\tilde{\omega}_3| = 2$ to hold (see hypothesis (H2i) of Section 3.2) *or* we assume that the full Hamiltonian $H$ is close to being discrete symmetric at cubic order (see (H2ii) of Section 3.2). A large number of multiplicity two resonances satisfy one of these assumptions and can be cast in the same type of normal form. In particular, we can treat the genuine first order resonances 1:2:1, 1:2:3, and 1:2:4 with appropriate discrete symmetries, and the 1: 2: 2 without any symmetry assumed. All the genuine second order resonances with cubic generator $\bar{n}$ also qualify (i.e., the 1: 2: 6, 2: 4: 3, 3: 6: 1, and the 3: 6: 2), and, we repeat, all resonances of the form 1: 2: $\omega_3$, $\omega_3 \geq 5$ are covered without symmetry assumptions. Further, in all the cases listed here and above, the signs of individual frequencies may be negative.

### 1.4. Main results

Our study is based on the global knowledge of the integrable geometry of the cubic normal form. This enables us to establish the survival of two families of three-dimensional tori on most energy surfaces of the full system. We also pay special attention to a family of invariant 3-spheres of the integrable cubic truncation which have four-dimensional homoclinic manifolds. Each 3-sphere is filled with a two-parameter family of periodic solutions. Under the effect of higher-order normal form terms these periodic orbits are all destroyed, but in the typical case new invariant 2-tori are created by the quartic normal form terms on a four-dimensional invariant set $\mathcal{M}_\varepsilon$ that perturbs from the family of 3-spheres.

The tori are connected through heteroclinic and homoclinic orbits, which are not amenable to Melnikov-type methods because the 2-tori are *created by the perturbation* acting on the integrable limit. This introduces a singular perturbation problem that can be dealt with using an appropriate version of the *energy-phase method* which is developed for 2-DOF Hamiltonian and dissipative systems in Haller and Wiggins [24]. Using this method we obtain criteria for the existence of *multi-pulse* connections between the whiskered 2-tori which pass repeatedly near $\mathcal{M}_\varepsilon$ before approaching a torus in backward or forward time. Along the multi-pulse solutions the stable and unstable whiskers of the corresponding 2-tori intersect at an angle of $\mathcal{O}(\varepsilon)$ which enables us to prove the existence of Smale horseshoes and chaotic dynamics on most energy levels. This implies the existence of observable irregular behavior on energy levels close to the resonant equilibrium, *even if the equilibrium is Lyapunov-stable*. Although it is most likely true, we are not able to prove at this point that the invariant 2-tori persist under the effect of the "tail" of the normal form (cf. Section 7.4). Instead, we show how the multi-pulse solutions survive and asymptote to certain invariant 3-spheres. In fact, the multi-pulse orbits turn out to be transverse homoclinic orbits to these spheres in the full system.

The organization of this paper is as follows. In Section 2 we describe our main assumptions and the resonances we study. Section 3 contains a detailed description of the dynamics of the integrable cubic normal form for these resonances. In Section 4 we introduce new coordinates that are more appropriate for the study of the effect of higher order normal forms on the integrable truncation. In Section 5 we describe our basic tool for the study of resonant manifolds, which is a version of the 2-DOF energy-phase method developed in Haller and Wiggins [24]. In Section 6 we use this method to obtain multi-pulse solution sets in higher-order truncations of the normal form that connect 2-dimensional whiskered tori with two different time scales. In Section 6 we study what remains of the structures of the quartic truncation under the effect of the tail of the normal form. In particular, we analyze the fate of families of 3-tori using Arnold's results on properly degenerate Hamiltonians (see, e.g., Arnold et al. [5]) and also study the fate of the multi-pulse solution sets. Finally, in Section 8 we summarize our results and their applications and comment on aspects of chaos and diffusion near the resonant equilibria we studied.

## 2. Set-up and assumptions

We are concerned with Hamiltonians of the form

$$H(q,p) = H_2(q,p) + \tilde{H}(q,p), \tag{2.1}$$

where $(q,p)$ are canonical coordinates on the phase space $(\mathbb{R}^6, \omega)$ with $\omega = dq \wedge dp$. In (2.1) $H_2$ is a quadratic polynomial and $\tilde{H}$ satisfies $D\tilde{H}(0) = 0$ and $D^2\tilde{H}(0) = 0$. Throughout this paper we assume that $H$ is a $C^{r+1}$ function with $r \geq 6$. This smoothness requirement is imposed on the Hamiltonian by one of the tools we use: a version of the KAM theorem which requires a Hamiltonian of class $C^7$ for 3-DOF systems, as shown in Pöschel [52].

With the notation $x = (q,p)$ the Hamiltonian equations associated with (2.1) take the form

$$\dot{x} = J_6 DH(x), \quad J_6 = \begin{pmatrix} 0 & Id_3 \\ -Id_3 & 0 \end{pmatrix}, \tag{2.2}$$

where $Id_3 \in \mathbb{R}^{3\times 3}$ is the identity matrix. We assume that $x = 0$ is an elliptic fixed point of system (2.2) and $J_6 D^2 H_2(0)$ is semisimple, in which case a linear canonical transformation puts $H_2$ in the form

$$H_2(q,p) = \tfrac{1}{2}\sum_{k=1}^{3} \omega_k (q_k^2 + p_k^2). \tag{2.3}$$

Here $\Omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3 - \{0\}$ is the frequency vector of the linear flow generated by $H_2$. We introduce the canonical change of variables

$$z = q - ip, \quad \bar{z} = q + ip, \quad z, \bar{z} \in \mathbb{C}^3, \tag{2.4}$$

with its inverse defined as $T_1: (z, \bar{z}) \mapsto (q,p)$, and consider the transformed system on the phase space $(\mathbb{C}^3, \Omega)$ with $\Omega = \frac{1}{2}\mathrm{Im}(d\bar{z} \wedge dz)$. Letting $z \to \varepsilon z$, $\bar{z} \to \varepsilon \bar{z}$ with $\varepsilon > 0$ small, and dividing (2.1) by $\varepsilon^2$, we arrive at the Hamiltonian

$$H(z, \bar{z}; \varepsilon) = H_2(z, \bar{z}) + \varepsilon \tilde{H}(z, \bar{z}; \varepsilon), \tag{2.5}$$

with

$$H_2(z,\bar{z}) = \tfrac{1}{2}\sum_{k=1}^{3} \omega_k |z_k|^2, \quad \tilde{H}(z,\bar{z};\varepsilon) = \sum_{3 \leq |l|+|m| \leq r+1} \varepsilon^{|l|+|m|-3} h_{lm} z^l \bar{z}^m + \mathcal{O}(\varepsilon^{r-1}), \tag{2.6}$$

where $h_{lm} = \bar{h}_{ml} \in \mathbb{C}$ for $l, m \in \mathbb{N}^3$, and $|l| = l_1 + l_2 + l_3$. In (2.6) we used the usual notation $z^p := z_1^{p_1} z_2^{p_2} z_3^{p_3}$.

### 2.1. *The type of the resonance*

To describe the resonances we study, we introduce the resonant module

$$M = \{n \in \mathbb{Z}^3 \mid \langle \omega, n \rangle = 0\}. \tag{2.7}$$

We then assume that, after a possible reindexing of the variables $z_k, \bar{z}_k$,

(H1) Either $(2,1,0) \in M$ or $(2,-1,0) \in M$ holds *and* $\dim M = 2$.

(H2) If $s \in M$ and $|s_1|/|s_2| \neq 2$, then *one of the following* is satisfied:

(i) $|s| > 4$,

(ii) For any $l, m \in \mathbb{N}^3$ with $|l| + |m| = 3$, $l - m = s$, we have $h_{lm} = 0$.

Here (H1) means that the first two frequencies satisfy the strong resonance relationship $|\omega_1|:|\omega_2| = 1:2$ *and* there is one more independent resonance relationship among the three frequencies. If (H2i) holds then $(2, \pm 1, 0)$ is the unique third order generator of the resonant module $M$. This is satisfied by all resonances of the form $1:\pm 2:\omega_3$ with $|\omega_3| \geq 5$, and by a number of other resonances, like the 1:2:6, 2:4:3, 3:6:1, 3:6:2, 3:6:10, etc. If, alternatively, (H2ii) holds then $(2, \pm 1, 0)$ is not the only third order generator of $M$. We then require certain coefficients of $\tilde{H}$ to vanish, which usually translates into the assumption that the Hamiltonian (2.5) is close to having a discrete symmetry at cubic order. (One can also assume $h_{lm} = \nu\varepsilon$ which requires the symmetry breaking terms to be less in order than the symmetric cubic terms.) We note that all genuine first order resonances in 3-DOF Hamiltonians (i.e., $1:\pm 2:\omega_3$, $|\omega_3| = 1, 2, 3, 4$) with appropriate weakly broken discrete symmetries satisfy hypotheses (H1) and (H2ii) (see Sanders and Verhulst [54] for the definition of genuine resonances and discussion on the effect of discrete symmetries).

### 2.2. *The resonant normal form*

Based on hypothesis (H1) we can rescale the frequencies by letting

$$\omega_1 = 1, \quad \omega_2 = 2, \quad \omega_3 \to \frac{\omega_3}{\omega_1}, \tag{2.8}$$

where we have set the sign of $\omega_2$ positive for convenience. Although the basic results are the same, some of the geometry is different in the case $\omega_2 = -2$. We believe that, based on the material presented below, it is straightforward to make the necessary modifications for that case. We finally note that (H1) implies that the rescaled frequency $\omega_3$ is a rational number.

In order to simplify the Hamiltonian (2.5) further, we apply a near-identity canonical change of variables which puts our system in Birkhoff normal form up to some order $\rho$ with $4 \leq \rho \leq r+1$ (see Arnold [4] or Sanders and Verhulst [54], etc.). Using (H1),(H2), and the rescaling (2.8), we can write the normalized part of the Hamiltonian in the form

$$\tilde{H} = H_2 + \varepsilon H_3 + \varepsilon^2 \tilde{H}_4 + \varepsilon^3 H_5 + \cdots + \varepsilon^{\rho-2} H_\rho, \tag{2.9}$$

where

$$H_2 = \tfrac{1}{2}|z_1|^2 + |z_2|^2 + \tfrac{1}{2}\omega_3|z_3|^2, \quad H_3 = \tfrac{1}{2}a\mathrm{Re}\,(z_1^2\bar{z}_2), \quad a \in \mathbb{R}, \tag{2.10}$$

and

$$\tilde{H}_4 = H_4 + \hat{H}_4,$$

with

$$\{H_2, H_j\} = 0, \quad 3 \leq j \leq r, \quad \{H_2, \tilde{H}_4\} = 0, \tag{2.11}$$

where $\{\ ,\ \}$ is the canonical Poisson bracket. In (2.10) $H_j$ contains resonant terms of the form $z^l\bar{z}^m$ with $|l| + |m| = j$, $l - m \in M$. We note that, in general, the normalization procedure yields a term of the form $H_3 = \mathrm{Re}\,(Az_1^2\bar{z}_2)$, $A \in \mathbb{C}$ and one needs to apply an additional symplectic change of coordinates $z_1 \to \exp(-i\arg(A)/2)z_1$, $\bar{z}_1 \to \exp(-i\arg(A)/2)\bar{z}_1$, to bring $H_3$ to the form in (2.10) with $a = 2|A|$ (see Kummer [38]). From this point on we will assume that the nondegeneracy condition $a \neq 0$ holds.

The expression $\hat{H}_4$ in (2.11) contains some possible quadratic detuning of $H_2$ from the exact resonance and cubic terms $h_{lm}z^l\bar{z}^m$ of the normal form with $h_{lm} = \mathcal{O}(\varepsilon)$, which may arise in the case of hypothesis (H2).

We remark that for $\omega = (1 : \pm 2 : \pm 2)$ system (2.5) can be put in the form (2.10) even if (H2) is not satisfied (see Kummer [38]).

With the normal form transformed into the form (2.9), integrability at cubic order is clear. This can be seen due to the fact that $z_3$ "separates" from the $z_1 - z_2$ dynamics leaving a 2-DOF Hamiltonian system, and all 2-DOF *truncated* Hamiltonian normal forms are integrable by construction (since $\{H, H_2\} = 0$).

Realistic physical systems are never in perfect resonance so it is important to allow for the presence of some detuning from the resonance. In this paper the detuning is assumed to be of the form $\varepsilon H_d = \varepsilon H_{d0} + \mathcal{O}(\varepsilon^2)$ with

$$H_{d0} = \tfrac{1}{2} d |z_1|^2.$$

It can be easily verified that, when considered up to order $\mathcal{O}(\varepsilon)$, the normal form is still integrable with the addition of this term. In particular,

$$H_c = H_2 + \varepsilon H_{d0} + \varepsilon H_3, \quad J_1 = \tfrac{1}{2}|z_1|^2 + |z_2|^2, \quad J_2 = \tfrac{1}{2}|z_3|^2 \tag{2.12}$$

are integrals for the Hamiltonian system defined by $H_c$. Note that we allow the detuning to be fairly large in the $(z_1, \bar{z}_1)$ degree of freedom for which it has the same order of magnitude as the nonlinear terms in the flow generated by $H_c$. This is required, e.g., for the chemical application described in Haller and Wiggins [25]. From now on we assume that the $\mathcal{O}(\varepsilon^2)$ and higher-order terms in the detuning have been incorporated in $\tilde{H}_4$.

## 3. Integrability and geometry of the cubic normal form: symplectic reduction

In this section we briefly discuss the geometry of the cubic, detuned normal form

$$H_c = \tfrac{1}{2}(1 + \varepsilon d)|z_1|^2 + |z_2|^2 + \tfrac{1}{2}\omega_3 |z_3|^2 + \varepsilon \tfrac{1}{2} a \operatorname{Re}(z_1^2 \bar{z}_2). \tag{3.1}$$

We study this Hamiltonian using symplectic reduction, for which the standard terminology can be found in Abraham and Marsden [1]. The results we present in Subsection 3.1 are essentially adaptations of those of Kummer [37,38], Churchill et al. [9], and Cushman [11], to the class of resonances under consideration.

### *3.1. Set-up for symplectic reduction*

Let us introduce the notation $P = \mathbb{C}^3$ and consider the symplectic action of the Lie group $G = \mathbb{T}^2 = S^1 \times S^1$ on $(P, \Omega)$ given by

$$\Phi : G \times P \to P, \quad (g_1, g_2, z_1, z_2, z_3) \mapsto (e^{-ig_1} z_1, e^{-2ig_1} z_2, e^{-ig_2} z_3), \tag{3.2}$$

and $\phi_g : P \to P$ denotes the induced mapping with a particular group element $g$ held fixed. We define the mapping

$$J : P \to \mathfrak{g}^* \simeq \mathbb{R}^2, \quad (z_1, z_2, z_3) \mapsto (\tfrac{1}{2}|z_1|^2 + |z_2|^2, \tfrac{1}{2}|z_3|^2), \tag{3.3}$$

where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of $G$. For any fixed $\xi \in \mathfrak{g}$ we also introduce the map $\hat{J}(\xi) : P \to \mathbb{R}^2$, $\hat{J}(\xi)(z) = \langle J(z), \xi \rangle$, where $\langle\ ,\ \rangle$ denotes the natural pairing between elements of $\mathfrak{g}^*$ and $\mathfrak{g}$. Using the definition of $J$ we have

$$\hat{J}(\xi)(z) = (\tfrac{1}{2}|z_1|^2 + |z_2|^2)\xi_1 + \tfrac{1}{2}|z_3|^2 \xi_2. \tag{3.4}$$

Let us also define for any $\xi \in \mathfrak{g}$ the vectorfield $\xi_P: P \to TP$ by

$$\xi_P(z) = \left.\frac{d}{dt}\right|_{t=0} \Phi(\exp t\xi, z), \quad \xi \in \mathfrak{g}, \quad z \in P,$$

which is just the infinitesimal generator of the action corresponding to by $\xi \in \mathfrak{g}$. A straightforward calculation shows that

$$\xi_P(z) = (-i\xi_1 z_1, -2i\xi_1 z_2, -i\xi_2 z_3),$$

which together with (3.4) yields

$$i_{\xi_P(z)}\Omega(z) = d\hat{J}(\xi)(z), \quad z \in P,$$

with $i_{\xi_P}\Omega$ denoting the interior product of the vectorfield $\xi$ with the symplectic form $\Omega$. Consequently, $J$ is a momentum mapping for the action $\Phi$. Since $G$ is Abelian, $J$ is $Ad^*$-equivariant, i.e., the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\Phi_g} & P \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle J} \\ \mathfrak{g}^* & \xrightarrow[Ad^*_{g^{-1}}]{} & \mathfrak{g}^* \end{array} \tag{3.5}$$

with the map $Ad^*_{g^{-1}}: \mathfrak{g}^* \to \mathfrak{g}^*$, $\langle Ad^*_{g^{-1}}\mu, \xi\rangle = \langle \mu, Ad_g\xi\rangle$, for all $g \in G$, $\mu \in \mathfrak{g}^*$, and $\xi \in \mathfrak{g}$.

With the momentum mapping in hand, the program of symplectic reduction can now be carried out. Any nonzero $\mu \in \mathfrak{g}^*$ is a regular value of the momentum mapping, from which it follows that $J^{-1}(\mu)$ is a smooth manifold. The reduced phase space is defined as $P_\mu = J^{-1}(\mu)/G_\mu$, which is well-defined since $J$ is $Ad^*$-equivariant. From the general theory, if the action of the isotropy subgroup $G_\mu = \{g \in G | Ad^*_{g^{-1}}\mu = \mu\} = G$ on $J^{-1}(\mu)$ is free and proper, then $P_\mu$ is a smooth manifold. Since $G$ is compact, the action is proper, but it is not free ($G$ has nonidentity elements leaving $(0, z_2, z_3) \in J^{-1}(\mu)$ fixed). Hence the Marsden–Weinstein reduction theorem does not guarantee that $P_\mu$ is a smooth manifold, and it is indeed not. As we will see, $P_\mu$ has a singularity that has some very interesting dynamical consequences.

### *3.1.1. Realization of the reduced phase space*

Next we describe an explicit realization of the reduced phase space $P_\mu$ in a form due to Churchill et al. [9] (see also Knobloch et al. [34]). Consider the "Euler-variables"

$$W_1 = \mathrm{Re}(z_1^2\bar{z}_2), \quad W_2 = \mathrm{Im}(z_1^2\bar{z}_2), \quad W_3 = \tfrac{1}{2}|z_1|^2, \tag{3.6}$$

which, by direct substitution, can be shown to satisfy the relation

$$W_1^2 + W_2^2 = 4W_3^2(J_1(z) - W_3). \tag{3.7}$$

Let $Z_\mu \subset \mathbb{R}^3$ denote the zero set of the function $f_\mu: \mathbb{R}^3 \to \mathbb{R}$ defined as

$$f_\mu(W_1, W_2, W_3) = W_1^2 + W_2^2 - 4W_3^2(\mu_1 - W_3). \tag{3.8}$$

This two-dimensional surface in $\mathbb{R}^3$ is a realization of the reduced phase space $P_\mu$ (see Churchill et al. [9] or Cushman [11] for details). It is not hard to see from (3.8) that $P_\mu$ is homeomorphic to $S^2$ and that it has a singular point, or "pinch" at the origin of the $W$-space. Thus, $Z_\mu$ is a pinched sphere, with the $W_3$ axis as an axis of symmetry (see Fig. 1). Note that the pinch occurs where the action of $G$ on $J^{-1}(\mu)$ is not free.

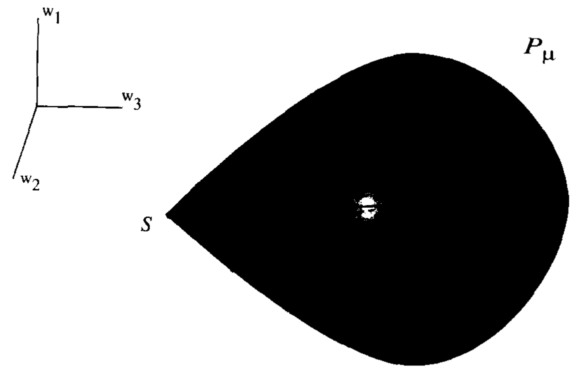


Fig. 1. The reduced phase space $P_\mu$.

*3.1.2. Dynamics on the reduced phase space*

Churchill et al. [9] show that the reduced flow of (3.1) on $P_\mu$ is given by the Euler-like equations

$$\dot{W} = \nabla H_c \times \nabla f_\mu, \tag{3.9}$$

where $H_c$ is restricted to $P_\mu$, i.e.,

$$H_c = \mu_1 + \omega_3 \mu_2 + \varepsilon d W_3 + \varepsilon \tfrac{1}{2} a W_1.$$

Explicitly, the reduced vector field is given by

$$\begin{aligned} \dot{W}_1 &= -\varepsilon 2 d W_2, \\ \dot{W}_2 &= \varepsilon 2 \Big( d W_1 - a \left( W_3^2 - 2 W_3 \left( \mu_1 - W_3 \right) \right) \Big), \\ \dot{W}_3 &= \varepsilon a W_2. \end{aligned} \tag{3.10}$$

One can readily verify that the function $dW_3 + \frac{1}{2} a W_1$ is an integral for (3.10). Hence the trajectories of (3.10) are given by the intersection of the planes $dW_3 + \frac{1}{2} a W_1 = \text{const.}$ with the pinched sphere $Z_\mu$. Therefore the trajectories are either closed curves or isolated points. We illustrate these orbits in Fig. 1. Note that for $\mu_1 - W_3 - d^2/a^2 > 0$, (3.10) has a homoclinic orbit connecting the pinch to itself. This orbit satisfies

$$W_1 = -(2d/a) W_3, \quad W_2 = 2 W_3 \sqrt{\mu_1 - W_3 - d^2/a^2}, \quad 0 < W_3 < \mu_1 - d^2/a^2. \tag{3.11}$$

*3.1.3. Reconstruction of orbits in the full six-dimensional phase space*

Using the fact that for any $\mu \in \mathrm{g}^*$ a subset of $J^{-1}(\mu)$ is a 2-torus bundle over $P_\mu - \{0\}$, we can reconstruct invariant structures in the phase space of (3.1) based on our knowledge of the orbits of the reduced system (3.10). Closed curves not passing through the pinch correspond to 3-tori in the full phase space, and the two elliptic fixed points at the top and bottom of $P_\mu$ correspond to 2-tori in the full phase space. The pinch requires a separate analysis in order to determine its manifestation in the full phase space, and we turn to this next.

*3.2. Singularity from the symplectic reduction*

Consider the set of points in a given energy surface $\{H_c = h\}$ that are mapped to the singularity via the quotient projection $\pi_\mu : P \to P_\mu$. This is an invariant set, denoted by $M^h \subset P$, with $h > 0$, which is given by

$$M^h = \bigcup_{\mu_1+\mu_2=h} \pi_\mu^{-1}(0).$$

Using (3.6) and (2.4), one can immediately see that the image of $M^h$ under the diffeomorphism $T_1$ (defined after (2.4)) satisfies

$$T_1(M^h) = \{ (q,p) \in \mathbb{R}^6 \mid q_1 = p_1 = 0,\ q_2^2 + p_2^2 + \tfrac{1}{2}\omega_3(q_3^2 + p_3^2) = h \}. \tag{3.12}$$

This shows that for $\omega_3 > 0$, $M^h$ is diffeomorphic to $S^3$, while for $\omega_3 < 0$ it is a three-dimensional hyperbolic surface of revolution. In either case, $M^h$ is connected to itself by a four dimensional homoclinic manifold $W^h$. This follows from the nature of the reduced dynamics discussed in Section 3.1.2.

### 3.2.1. Orbit space reduction of $M^h$

One can check that for any $h > 0$, $M^h$ is entirely filled with periodic orbits. Two of these closed orbits are distinguished: they are the (nonlinear) *normal modes* of $H_c$ given by

$$N_2^h = \{ z \in \mathbb{C}^3 \mid z_1 = z_3 = 0,\ |z_2| = \sqrt{h} \},$$
$$N_3^h = \{ z \in \mathbb{C}^3 \mid z_1 = z_2 = 0,\ |z_3| = \sqrt{2h/\omega_3} \}.$$

These two normal modes survive from the quadratic Hamiltonian $H_2$ under the effect of the cubic terms in (3.1). This can be seen by noting that $M^h$ is an invariant manifold, and the dynamics on $M^h$ is, from (3.12), that of two linear oscillators which, by hypothesis (H1), are in $2{:}\,\omega_3$ resonance. With the exception of $N_3^h$, all the periodic orbits in $M^h$ have two-dimensional stable and unstable manifolds, which foliate $W^h$ into a two-parameter family of cylindrical surfaces.

Since $M^h$ is a manifold of periodic orbits, the flow of the Hamiltonian (3.1) restricted to $M^h$ can be considered as the action of the group $S^1$ on $M^h$. We can then define $F^h = M^h/S^1$, the quotient space corresponding to this action, with the usual quotient projection $\pi_F\colon M^h \to F^h$. In other words, $F^h$ is the orbit space of the periodic solutions contained in $M^h$. From (3.12) we see that the set $T_1(M^h) \simeq M^h$ can be considered as the three-dimensional energy surface for a Hamiltonian system of two linear oscillators which, by (H1), are in $2{:}\,\omega_3$ resonance. Accordingly, $F^h$ can be viewed as the reduced phase space for these oscillators with respect to the resonant $2{:}\,\omega_3$ action of $S^1$. As is shown by Churchill et al. [9], this reduced phase space is homeomorphic to $S^2$. Furthermore, if $\kappa_1$ and $\kappa_2$ are relatively prime positive integers with

$$\frac{2}{|\omega_3|} = \frac{\kappa_1}{\kappa_2}, \tag{3.13}$$

then $F^h$ is a pinched sphere with a $\kappa_1$-order singularity at its north pole, and with a $\kappa_2$-order singularity at its south pole (e.g., $\kappa_1 = 1,2,3$ would mean no singularity, conical singularity, and cusp singularity, respectively). We summarize the observations of this section in the following proposition.

*Proposition 3.1.* Suppose that $\omega_3 > 0$ holds. Then on any energy surface $H_c = h$ (with $h > 0$) of the integrable Hamiltonian system defined by $H_c$ there exists an invariant set $M^h$ defined by $z_1 = \bar{z}_1 = 0$, which is diffeomorphic to $S^3$. The set $M^h$ is entirely filled with periodic orbits. Furthermore,

(i) Any invariant subset $M_0^h \subset M^h$ that does not contain the third normal mode $N_3^h$, is normally hyperbolic and is connected to itself by a four-dimensional homoclinic manifold $W_0^h$.

(ii) The orbit space $F^h = M^h/S^1$ of periodic solutions in $M^h$ is homeomorphic to a 2-sphere with a $\kappa_1$-order singularity at its north pole, and with a $\kappa_2$ order singularity at its south pole (see (3.13)).

The notion of normal hyperbolicity is defined and discussed in detail in Appendix B. We note that invariant spheres similar to $M^h$ have been found recently in a large class of Hamiltonian resonances by Hoveijn [29]. The methods we use in the following sections can be used to study perturbations of those spheres as well if they admit a homoclinic structure similar to $W^h$.

## 4. The "blown-up" cubic normal form and the formulation of the perturbation problem

The goal of this section is to introduce coordinates which are suited to the study of what happens to the manifolds $M^h$ and $W^h$ in the normal form (2.9) under the perturbative effects of the terms $\varepsilon^2\tilde{H}_4 + \cdots + \varepsilon^{r-2}H_r + \mathcal{O}(\varepsilon^{r-1})$.

### *4.1. The blow-up transformation*

Let us first introduce action-angle variables for the quadratic part $H_2$ of (2.9) by letting

$$z_k = \sqrt{2I_k}e^{i\phi_k}, \quad \bar{z}_k = \sqrt{2I_k}e^{-i\phi_k}, \quad k = 1,2,3, \tag{4.1}$$

with the inverse change of variables $T_2\colon (I,\phi) \mapsto (z,\bar{z})$. We apply a further canonical change of variables

$$\begin{aligned} &\psi_1 = \phi_1, && K_1 = I_1 + 2I_2 + \omega_3 I_3, \\ &\psi_2 = \phi_3 - \omega_3\phi_1, && K_2 = I_3, \\ &x_1 = \sqrt{2I_2}\sin(\phi_2 - 2\phi_1), && x_2 = \sqrt{2I_2}\cos(\phi_2 - 2\phi_1), \end{aligned} \tag{4.2}$$

with the inverse coordinate transformation being $T_3\colon (x,K,\psi) \mapsto (I,\phi)$. In this final set of variables our Hamiltonian (3.1) takes the form

$$H_c(x,K,\psi) = K_1 + \varepsilon(d + ax_2)(K_1 - \omega_3 K_2 - |x|^2). \tag{4.3}$$

The corresponding Hamiltonian vectorfield is smooth and defined on the set

$$\mathcal{P} = \{(x, K_1, \psi_1, K_2, \psi_2) |\ x \in \mathbb{R}^2, K \in \mathbb{R}^2, \psi \in \mathbb{T}^2\}, \tag{4.4}$$

with the symplectic form

$$\omega = dx_1 \wedge dx_2 + d\psi \wedge dK, \tag{4.5}$$

but it is related to system (3.1) only in the domain

$$\bar{\mathcal{P}} = \{(x,K,\psi) \in \mathcal{P} |\ K_1 - \omega_3 K_2 \geq 0,\ K_2 \geq 0\}. \tag{4.6}$$

The vector field corresponding to the cubic truncated Hamiltonian $H_c$ takes the form

$$\begin{aligned} \dot{x}_1 &= \varepsilon[a(K_1 - \omega_3 K_2 - x_1^2 - x_2^2) - 2(d + ax_2)x_2], \\ \dot{x}_2 &= \varepsilon 2x_1(d + ax_2), \\ \dot{K}_2 &= 0, \\ \dot{\psi}_2 &= -\varepsilon\omega_3(d + ax_2), \\ \dot{K}_1 &= 0, \\ \dot{\psi}_1 &= 1 + \varepsilon(d + ax_2). \end{aligned} \tag{4.7}$$

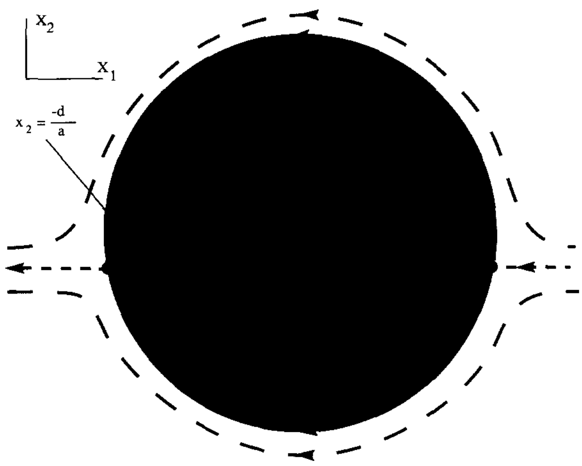


Fig. 2. Phase portrait of the $x$-component of (4.7). The shaded region indicates the region where the orbits of the blown-up normal form will be related to orbits in the original normal form and the boundary of the shaded region is given by $x_1^2 + x_2^2 = K_1 - \omega_3 K_2$, $K_1 \geq 0$.

Note that the $x$ component of (4.7) decouples from the rest of the equations since $K$ does not change in time. It is a simple phase plane analysis, coupled with the use of the Hamiltonian (4.3), to verify that for $K_1 - \omega_3 K_2 \geq 0$ and $K_2 \geq 0$ the phase portrait for the $x$-equations appears as in Fig. 2.

Our main goal with the sequence of transformations (4.1),(4.2) is to "blow up" the singularity of the reduced phase space. This is achieved, as we will see next, by extending the transformations (4.1),(4.2) to the domain $I_1 = 0$, in which case (4.1) is not a diffeomorphism any longer and the angle variable $\phi_1$ is not well defined. It should also be emphasized that since $M^h$ is characterized by $I_1 = 0$, we have to study the effect of this singular transformation if we want to relate our later results on the perturbation of $M^h$ back to the original normal form (2.9).

### *4.2. The phase space of the "blown-up" normal form*

Broadly speaking, there are two types of geometric structures, or invariant manifolds, for the normal form (4.7). *Normally elliptic* invariant manifolds containing families of elliptic 2 or 3-tori and *normally hyperbolic* invariant manifolds that contain families of resonant "whiskered" tori. The special form of (4.7) makes it particularly easy to determine these structures.

*Proposition 4.1.* The system (4.7) has two three-parameter families of elliptic 3-tori and two two-parameter families of elliptic 2-tori.

*Proof.* This result follows easily from the structure of the $x$-component of (4.7). The 3-tori are the Cartesian product of the periodic orbits of the $x$-component of (4.7) with $\psi_2$ and $\psi_1$. The three parameters are $H_c$, $K_2$, and $K_1$. The elliptic 2-tori are the Cartesian product of the elliptic fixed points of the $x$-component of (4.7) with $\psi_2$ and $\psi_1$. The two parameters are $K_2$ and $K_1$. □

Next we consider homoclinic and heteroclinic structures that arise in the blown-up normal form.

*Proposition 4.2.* Suppose that $\omega_3 > 0$ is satisfied. Then on a fixed five-dimensional energy surface $H_c(x, K, \psi) = h$ with $h - d^2/a^2 > 0$, the following hold:

(i) $T_3^{-1} \circ T_2^{-1}(M^h) = \mathcal{D}^h \cup \mathcal{M}^h$ where the set $\mathcal{D}^h$ satisfies

$$\mathcal{D}^h = \left\{ (x, K, \psi) \in \mathcal{P} \,\middle|\, |x|^2 + \omega_3 K_2 = h,\ x_2 \neq -d/a,\ K_1 = h \right\};$$

hence $\mathcal{D}^h$ is diffeomorphic to the disjoint union of two four-dimensional open disks. The set $\mathcal{M}^h$ consists of two connected components given by

$$\mathcal{M}_{0,1}^h = \left\{ (x, K, \psi) \in \mathcal{P} \,\middle|\, x_2 = -d/a,\ x_1 = -\sqrt{h - d^2/a^2 - \omega_3 K_2},\ K_1 = h \right\},$$

and

$$\mathcal{M}_{0,2}^h = \left\{ (x, K, \psi) \in \mathcal{P} \,\middle|\, x_2 = -cd/a,\ x_1 = +\sqrt{h - d^2/a^2 - \omega_3 K_2},\ K_1 = h \right\}.$$

Hence, each component of $\mathcal{M}^h$ is diffeomorphic to $S^2 \times S^1$. Furthermore, $\mathcal{M}^h$ is filled with two-dimensional invariant tori. All these whiskered tori carry a resonant flow as they are all filled with periodic orbits. The manifold $\mathcal{D}^h$ consists of orbits of (4.3) that are forward and and backward asymptotic to periodic orbits on the whiskered tori in $\mathcal{M}^h$.

(ii) The manifold $\mathcal{W}^h = T_3^{-1} \circ T_2^{-1}(W^h)$ satisfies

$$\mathcal{W}^h = \left\{ (x, K, \psi) \in \mathcal{P} \,|\, x_2 = -d/a,\ |x_1| < \sqrt{h - \omega_3 K_2 - d^2/a^2},\ K_1 = h \right\};$$

thus $\mathcal{W}^h$ is diffeomorphic to $B^3 \times S^1$ where $B^3$ is the open unit ball in $\mathbb{R}^3$. Furthermore, $\mathcal{W}^h$ is filled with a three-parameter family of orbits positively and negatively asymptotic to periodic orbits in $\mathcal{M}^h$.

(iii) Let $\pi_{\mathcal{F}}: \mathcal{M}^h \to \mathcal{F}^h = \mathcal{M}^h/S^1$ be the quotient projection from $\mathcal{M}^h$ to its orbit space $\mathcal{F}^h$, and let $Q^h$ be defined through the diagram

$$\begin{array}{ccc} \mathcal{M}^h & \xrightarrow{T_2 \circ T_3} & M^h \\ \pi_{\mathcal{F}} \downarrow & & \downarrow \pi_F \\ \mathcal{F}^h & \xrightarrow{Q^h} & F^h \end{array}$$

In other words, let $Q^h$ be the map between the quotient spaces $\mathcal{F}^h$ and $F^h$ induced by $T_2 \circ T_3$. Then $\mathcal{F}^h$ is diffeomorphic to $S^2$. Moreover, if $M_0^h$ is a compact subset of $M^h$ which does not contain the periodic orbits $N_2^h$ and $N_3^h$, then $\mathcal{F}^h$ has a bounded subset $\mathcal{F}_0^h$ consisting of two connected components and not containing $\mathcal{N}_2^h = (Q^h)^{-1} \circ \pi_F(N_2^h)$ and $\mathcal{N}_3^h = (Q^h)^{-1} \circ \pi_F(N_3^h)$. Furthermore, the map $Q^h$ restricts to a $\kappa_1$-fold smooth covering map onto $\pi_F(M_0^h)$ on any of the two connected components of $\mathcal{F}_0^h = \pi_{\mathcal{F}}(\mathcal{M}_0^h)$.

*Remark.* The superscript $h$ in the notation for the various invariant manifolds indicates that we fix $K_1 = h$.

*Proof.* The proof of statements (i),(ii) is a direct computation based on the definition of $T_2$ and $T_3$, which we omit. To prove (iii) we first note that the change of variables (4.1) puts the vectorfield corresponding to (3.1) into the form

$$\begin{aligned}
\dot{I}_1 &= \varepsilon 2 a I_1 \sqrt{2 I_2} \sin(2\phi_1 - \phi_2), \\
\dot{I}_2 &= -\varepsilon a I_1 \sqrt{2 I_2} \sin(2\phi_1 - \phi_2), \\
\dot{I}_3 &= 0, \\
\dot{\phi}_1 &= 1 + \varepsilon a \sqrt{2 I_2} \cos(2\phi_1 - \phi_2) + \varepsilon d,
\end{aligned}$$

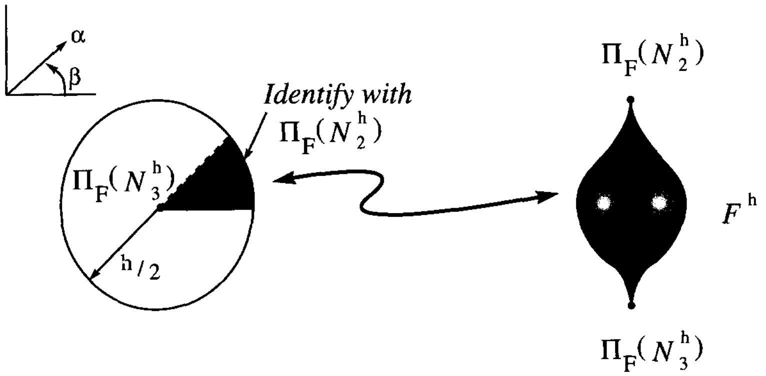


Fig. 3. Parametrization of the orbit space $F^h$.

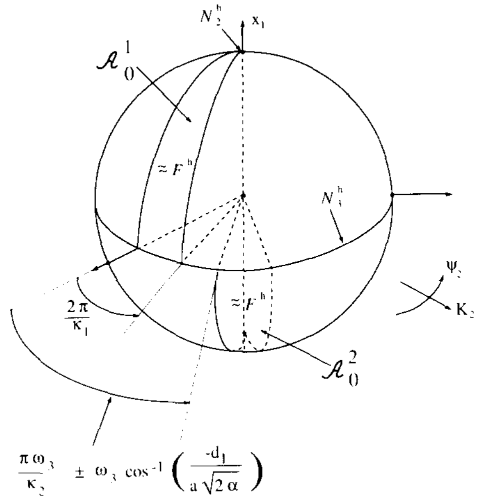

Fig. 4. Parametrization of the orbit space $\mathcal{F}^h$.

$$\dot{\phi}_2 = 2 + \varepsilon a \frac{I_1}{\sqrt{2I_2}} \cos(2\phi_1 - \phi_2),$$

$$\dot{\phi}_3 = \omega_3. \tag{4.8}$$

First we obtain a parametrization of $F^h$. Straightforward calculations show that the periodic orbits in $T_2^{-1}(M^h)$ (described by $I_1 = 0$) can be labelled by the two parameters

$$\alpha = I_2 = \tfrac{1}{2}(q_2^2 + p_2^2) \in [0, \tfrac{1}{2}h], \quad \beta = \omega_3\phi_2 - 2\phi_3 \mod \omega_3\frac{2\pi}{\kappa_2}, \tag{4.9}$$

which, combined with the definition of $T_2$, gives a parametrization of $F^h$. The parametrization is singular on $\pi_F(N_3^h)$ and $\pi_F(N_2^h)$ in the sense that all points of $F^h$ with $\alpha = 0$ should be identified with $\pi_F(N_3^h)$ and all points with $\alpha = h/2$ should be identified with $\pi_F(N_2^h)$. We illustrate this in Fig. 3.

Next we seek a representation of $\mathcal{F}^h$. Using (4.2) and (4.9) we can see that on the set $\mathcal{M}^h$ the coordinates $(x, K, \psi)$ and the parameters $(\alpha, \beta)$ satisfy the following relationships:

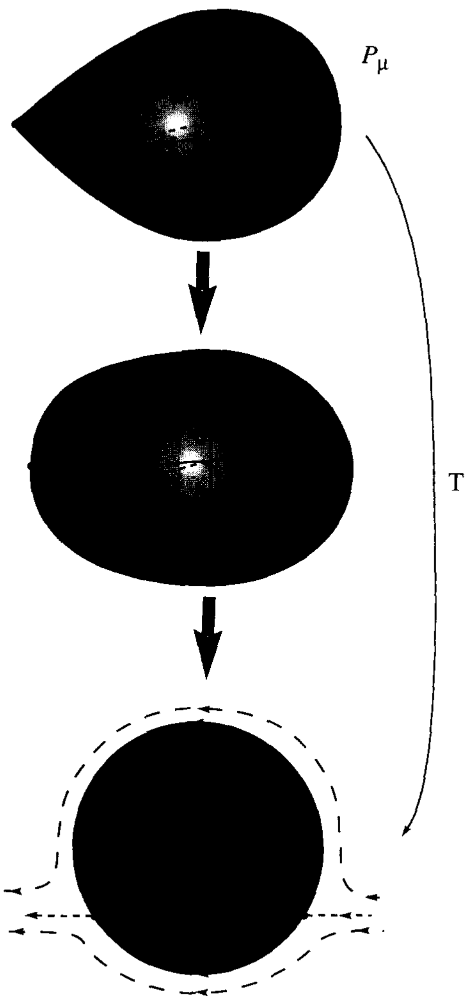


Fig. 5. The effect of the blow-up transformation on the reduced phase space $\mathcal{P}_\mu$.

$$\begin{aligned} \dot{\psi}_1 &= 1, & K_1 &= h, \\ \psi_2 &= -\beta/2 \pm \tfrac{1}{2}\omega_3 \cos^{-1}\left(-d/a\sqrt{2\alpha}\right), & K_2 &= (h - 2\alpha)/\omega_3, \\ x_1 &= \pm\sqrt{2\alpha - d^2/a^2}, & x_2 &= -d/a. \end{aligned} \tag{4.10}$$

Factoring out the uniform rotation in the $\psi_1$ coordinate in the above equations is equivalent to passing to the quotient space $\mathcal{F}^h$. Performing this, we find from (4.10) that $\mathcal{F}^h$ can be identified with the set defined by the equation

$$x_1^2 + \omega_3 K_2 = h - d^2/a^2 \tag{4.11}$$

for $h - d^2/a^2 > 0$. But (4.11) represents a 2-sphere in the $(x_1, K_2, \psi_2)$ space (viewing $K_2$-$\psi_2$ as polar coordinates, as shown in Fig. 4).

Choosing a compact set $M_0^h$ as in statement (iii) of the proposition, (4.9) and (4.10) show that $\mathcal{F}_0^h = \pi_{\mathcal{F}}(\mathcal{M}_0^h)$ has two components: one on the northern hemisphere $(x_1 > 0)$ and one on the southern hemisphere $(x_1 < 0)$ of $\mathcal{F}^h$. We denote these two components by $\mathcal{A}_0^1$ and $\mathcal{A}_0^2$, respectively, and note that both can be globally parametrized by the variables $(K_2, \psi_2)$. From (4.10) we in fact obtain a coordinate representation of $Q^h|\mathcal{A}_0^1 \cup \mathcal{A}_0^2$ in the form

$$Q^h: \mathcal{F}_0^h = \mathcal{A}_0^1 \cup \mathcal{A}_0^2 \longrightarrow \pi_F(M_0^h) \subset F^h,$$
$$e(K_2, \psi_2) \mapsto \left(\tfrac{1}{2}(h - K_2\omega_3), -2\psi_2 - \omega_3 \cos^{-1}\left(\frac{-d}{a\sqrt{2\alpha}}\right) \operatorname{sign} x_1\right) = (\alpha, \beta). \tag{4.12}$$

We see from (4.12) that two relative equilibria $p_1 \neq p_2$ in $\mathcal{F}^h - \left(\pi_F(\mathcal{N}_2^h) \cup \pi_F(\mathcal{N}_3^h)\right)$ represent the same periodic solution in $M^h$ if and only if $Q^h(p_1) = Q^h(p_2)$, i.e., one of the following holds:

(1) $p_1$ and $p_2$ lie on the same hemisphere of $\mathcal{F}^h$ (i.e., their $x_1$ coordinates have the same sign), their $K_2$ coordinates are the same, and their $\psi_2$ coordinates differ by some integer multiple of

$$\tilde{\psi}_2 = \omega_3 \frac{\pi}{\kappa_2} = \frac{2\pi}{\kappa_1}. \tag{4.13}$$

This implies that $Q^h$ restricts to a $\kappa_1$-fold smooth covering map onto $\pi_F(M_0^h)$ on any of the two connected components of $\mathcal{F}_0^h = \pi_{\mathcal{F}}(\mathcal{M}_0^h)$.

(2) $p_1$ and $p_2$ lie on different hemispheres of $\mathcal{F}^h$, their $K_2$ coordinates are the same, and the difference in their $\psi_2$ coordinates satisfies

$$\pi n \frac{\omega_3}{\kappa_2} \pm \omega_3 \cos^{-1}\left(\frac{-d}{a\sqrt{2\alpha}}\right), \tag{4.14}$$

for some integer $n$. □

Based on this proposition we show the effect of the blow-up transformation on the original reduced phase space in Fig. 5.

### *4.3. Perturbations of the integrable, cubic normal form*

Throughout Sections 5–8 we will analyze the effect of perturbations to (4.7). We postpone the discussion of the persistence for the 3-tori described in Proposition 4.1 to Section 7.1 since it is a direct application of a result of Arnold [2] combined with some of the calculations in Kummer [36]. We do not address the question of persistence of the elliptic 2-tori of Proposition 4.1 for two main reasons. First, there are no KAM-type results in the literature that would apply to them. (Note that the results of Moser [47] and Pöschel [53] on lower-dimensional elliptic tori do not apply to our situation because in the limit $\varepsilon \mapsto 0$ one frequency on the tori vanishes). Second, the family of 2-tori forms an isolated, codimension-two set in the six-dimensional phase space before perturbation, hence it does not have a detectable influence on typical motions.

The other main issue we will study is the effect of higher order perturbations on the hyperbolic structure described in Proposition 4.2. This turns out to be a more complicated problem because the hyperbolic structure is degenerate in several respects. First, it completely disappears in the limit $\varepsilon = 0$. Second, for $\varepsilon > 0$ its "strength" of hyperbolicity is only of the order $\mathcal{O}(\varepsilon)$. Third, it is completely filled with resonant 2-tori, all of which will be seen destroyed by the perturbation. To overcome these problems, we proceed in two steps. First we consider the perturbation arising from a finite number of higher-order normalized terms. The second step is to consider terms of all orders, i.e., consideration of the effect of the "tail" of the normal form.

Accordingly, we first rewrite the full Hamiltonian (2.9), normalized through $\mathcal{O}(\varepsilon^{\rho-2})$, in terms of the $(x, K, \psi)$ coordinates:

$$\bar{H}(x, K, \psi_2; \varepsilon) = H_c(x, K) + \varepsilon^2 \bar{H}_4(x, K, \psi_2) + \cdots + \varepsilon^{\rho-2} H_\rho(x, K, \psi_2). \tag{4.15}$$

Note that the quantities listed in (4.15) do not depend on $\psi_1$, which follows from the fact that the symplectic transformations defined in (4.1),(4.2) preserve the bracket relations in (2.11).

The underlying idea of our study will be the following: the Hamiltonian (4.15) has an unbroken $S^1$ symmetry corresponding to rotations in the $\psi_1$ coordinate which allows a reduction to a 2-DOF subsystem. This practically means considering (after rescaling the time by $t \to t/\varepsilon$) the Hamiltonian system

$$\begin{aligned} \dot{x} &= J_2 D_x H_0(x, K_2) + \varepsilon J_2 D_x H_1(x, K_2, \psi_2; \varepsilon), \\ \dot{K}_2 &= -\varepsilon D_{\psi_2} H_1(x, K_2, \psi_2; \varepsilon), \\ \dot{\psi}_2 &= D_{K_2} H_0(x, K_2) + \varepsilon D_{K_2} H_1(x, K_2, \psi_2; \varepsilon), \end{aligned} \tag{4.16}$$

with

$$\begin{aligned} H_0(x, K) &= (d + a x_2)(K_1 - \omega_3 K_2 - |x|^2), \\ H_1(x, K, \psi_2; \varepsilon) &= \tilde{H}_4(x, K, \psi_2) + \sum_{j=5}^{\rho} \varepsilon^{j-3} H_j(x, K, \psi_2). \end{aligned} \tag{4.17}$$

System (4.16) derives from the Hamiltonian $H = H_0 + \varepsilon H_1$ through the symplectic form

$$\omega^h = dx_1 \wedge dx_2 + d\psi_2 \wedge dK_2, \tag{4.18}$$

on the phase space

$$\mathcal{P}^h = \{ (x, K_2, \psi_2) \mid x \in \mathbb{R}^2,\ K_2 \in \mathbb{R}^+,\ \psi_2 \in S^1 \}, \tag{4.19}$$

with $K_1 = h$ fixed.

For the invariant manifolds $\mathcal{M}^h$ and $\mathcal{W}^h$ this reduction means a passage to the quotient spaces $\mathcal{F}^h$ and $\mathcal{W}^h/S^1$, respectively. Since $\mathcal{M}^h$ is entirely filled with periodic orbits that coincide with the orbits of the symmetry group, the 2-DOF reduced system has a 2-manifold of (relative) equilibria, which we identify with $\mathcal{F}^h$. This invariant 2-sphere is connected to itself by a three-dimensional homoclinic manifold which we identify with $\mathcal{W}^h/S^1$. Any subset of $\mathcal{F}^h$ not containing the poles and the equator therefore appears as a normally hyperbolic 2-manifold of equilibria which is connected to some other subset of $\mathcal{F}^h$ through the appropriate subset of the homoclinic 3-manifold $\mathcal{W}^h/S^1$.

We would like to find out what happens to the set $\mathcal{F}^h$ and its homoclinic structure $\mathcal{W}^h/S^1$ under the effect of the terms of $\mathcal{O}(\varepsilon)$ and higher in (4.16). Since $\mathcal{F}^h$ is entirely filled with equilibria, we are faced with a singular perturbation problem which is not amenable to usual Melnikov-type global perturbation methods (see, e.g., Wiggins [62] for a survey of such methods). Instead, we use a version of the *energy-phase method* developed for 2-DOF systems in Haller and Wiggins [24]. This method can be used to show the existence of orbits homoclinic or heteroclinic to invariant sets on *slow manifolds* that perturb from the original manifolds of fixed points or *resonant manifolds*. The homoclinic or heteroclinic orbits obtained this way are nontrivial: they may make repeated passages near, and departures from, slow manifolds before they start approaching their slow limit sets.

Another complication in our analysis is due to the fact that the set $\mathcal{F}^h$ is not normally hyperbolic. In order to apply the energy-phase method, we have to consider two disjoint normally hyperbolic subsets of $\mathcal{F}^h$ which appear as two 2-manifolds of equilibria with heteroclinic connections in the system defined by (4.15) on $\mathcal{P}^h$. (These subsets will be $\mathcal{A}_0^1$ and $\mathcal{A}_0^2$ defined in the proof of Proposition 3.1.) We will analyze the effect of higher-order perturbations on these subsets of $\mathcal{F}^h$, but first discuss an appropriate heteroclinic version of the energy-phase method in the next section. This involves no new ideas compared to Haller and Wiggins [24] (where the homoclinic case was considered), but the formulation of some of the results will be somewhat different.

## 5. Perturbation of resonant manifolds: the energy-phase method

In this section we describe a specialized form of the *energy-phase method* developed in Haller and Wiggins [24] that can be used to study the effect of perturbations on normally hyperbolic invariant manifolds of equilibria with a heteroclinic structure.

### 5.1. Setting and initial assumptions

We consider 2-DOF Hamiltonian systems of the form

$$\begin{aligned} \dot{x} &= J_2 D_x H_0(x,I) + \varepsilon J_2 D_x H_1(x,I,\phi;\varepsilon), \\ \dot{I} &= -\varepsilon D_\phi H_1(x,I,\phi;\varepsilon), \\ \dot{\phi} &= D_I H_0(x,I) + \varepsilon D_I H_1(x,I,\phi;\varepsilon), \end{aligned} \tag{5.1}$$

on the phase space $\tilde{\mathcal{P}} \subset \mathbb{R}^2 \times \mathbb{R} \times S^1$ equipped with the symplectic form $\tilde{\omega} = dx_1 \wedge dx_2 + d\phi \wedge dI$. In (5.1) we use the notation

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.2}$$

We assume that the Hamiltonian $H = H_0 + \varepsilon H_1$ is $C^{r+1}$ smooth in its arguments with $r \geq 3$, and

(A1) There exist $I_1, I_2 \in \mathbb{R}$, $I_1 < I_2$, such that for $\varepsilon = 0$ any for any $I \in [I_1, I_2]$, the $x$-component of system (5.1) has two hyperbolic fixed points, $\bar{x}^1(I)$ and $\bar{x}^2(I)$, connected by a cycle of heteroclinic trajectories, $x^{h,+1}(t,I)$, $x^{h,-1}(t,I)$ and $x^{h,0}(t,I)$, with

$$\begin{aligned} &\lim_{t\to+\infty} x^{h,+1}(t,I) = \lim_{t\to+\infty} x^{h,-1}(t,I) = \bar{x}^1(I), \\ &\lim_{t\to-\infty} x^{h,+1}(t,I) = \lim_{t\to-\infty} x^{h,-1}(t,I) = \bar{x}^2(I), \\ &\lim_{t\to-\infty} x^{h,0}(t,I) = \bar{x}^1(I), \quad \lim_{t\to+\infty} x^{h,0}(t,I) = \bar{x}^2(I). \end{aligned}$$

We assume that the heteroclinic orbits are arranged in the $x_1 - x_2$ plane in a manner that they can be smoothly deformed into the arrangement shown in Fig. 6.

(A2) For every $I \in [I_1, I_2]$

$$D_I H_0\big(\bar{x}^j(I), I\big) = 0, \quad j = 1,2. \tag{5.3}$$

It follows from assumption (A1) that system (5.1) possesses two two-dimensional normally hyperbolic invariant manifolds (with boundary) defined as

$$\mathcal{A}_0^j = \{(x,I,\phi) \in \tilde{\mathcal{P}} \mid x = \bar{x}^j(I), I \in [I_1,I_2], \phi \in S^1\}, \quad j = 1,2\ . \tag{5.4}$$

These sets are the images of the annulus $A = [I_1, I_2] \times S^1$ under the embeddings

$$g_0^j : A \to \mathcal{A}_0^j \subset \tilde{\mathcal{P}}, \quad (I,\phi) \mapsto (\bar{x}^j(I), I, \phi). \tag{5.5}$$

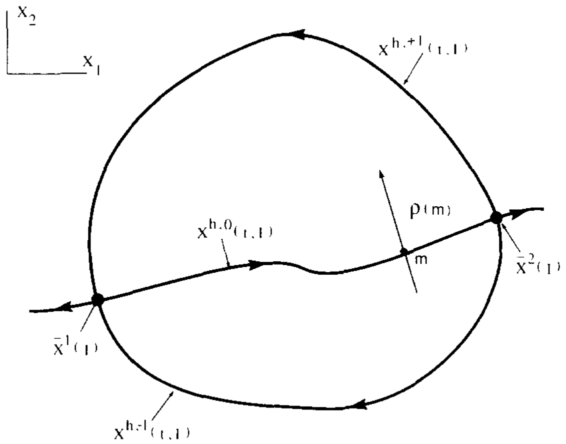

Fig. 6. Geometry associated with the heteroclinic cycle.

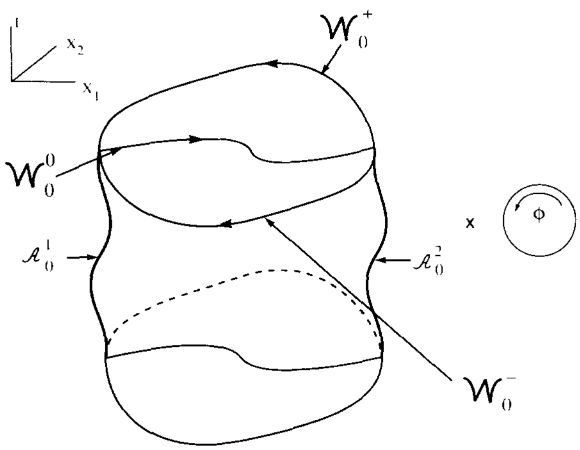

Fig. 7. The manifolds $\mathcal{W}_0^0$, $\mathcal{W}_0^+$, and $\mathcal{W}_0^-$ .

Note that $\mathcal{A}_0^1$ has a three-dimensional unstable manifold $W^u(\mathcal{A}_0^1)$ which coincides with the three-dimensional stable manifold $W^s(\mathcal{A}_0^2)$ along two branches, to form the heteroclinic manifolds $\mathcal{W}_0^+$ and $\mathcal{W}_0^-$. Assumption (A1) also implies that there exists another heteroclinic manifold in the phase space, defined as $\mathcal{W}_0^0 = W^s(\mathcal{A}_0^2) \cap W^u(\mathcal{A}_0^1)$ (see Fig. 7).

As a consequence of assumption (A2), the manifolds $\mathcal{A}_0^j$, $j = 1,2$, are entirely filled with equilibria. Solutions of (5.1) in $\mathcal{W}_0^0$ and $\mathcal{W}_0^\pm$ are heteroclinic connections between these equilibria. An important quantity associated with these heteroclinic connections is the net change of the coordinate $\phi$ along them. This change will be referred to as the *phase shift*, and denoted by $\Delta\phi^{+1}$, $\Delta\phi^0$, and $\Delta\phi^{-1}$ for orbits in $\mathcal{W}_0^+$, $\mathcal{W}_0^0$ and $\mathcal{W}_0^-$, respectively. From (5.1) we easily find that

$$\Delta\phi^k(I) = \int_{-\infty}^{+\infty} D_I H_0(x^{h,k}(t,I),I)\, dt, \quad k = +1, 0, -1. \tag{5.6}$$

## 5.2. Slow manifolds

Since, for $j = 1,2$, $\mathcal{A}_0^j$ is a compact, normally hyperbolic invariant manifold, for small $\varepsilon > 0$, system (5.1) has a two-dimensional invariant manifold $\mathcal{A}_\varepsilon^j$, which is $\mathcal{O}(\varepsilon)$ $C^r$-close to $\mathcal{A}_0^j$, and is still a $C^r$ embedding of the annulus $A$ through a map

$$g_\varepsilon^j : A \to \mathcal{A}_\varepsilon^j \subset \tilde{\mathcal{P}}, \quad (I,\phi) \mapsto (x_\varepsilon^j(I,\phi), I, \phi) = (\bar{x}^j(I) + \varepsilon \tilde{x}^j(I,\phi;\varepsilon), I, \phi). \tag{5.7}$$

Let $i_\varepsilon^j : \mathcal{A}_\varepsilon^j \hookrightarrow \tilde{\mathcal{P}}$ be the inclusion map of $\mathcal{A}_\varepsilon^j$ with $\varepsilon \geq 0$. Then it can be shown that for small $\varepsilon \geq 0$, $(\mathcal{A}_\varepsilon^j, (i_\varepsilon^j)^*\tilde{\omega})$ is a symplectic 2-manifold with

$$(i_\varepsilon^j)^*\omega = (1 + \mathcal{O}(\varepsilon))d\phi \wedge dI.$$

(The notation $(i_\varepsilon^j)^*\tilde{\omega}$ refers to the pull-back of the form $\tilde{\omega}$ under the map $i_\varepsilon^j$.)
On $(\mathcal{A}_\varepsilon^j, (i_\varepsilon^j)^*\tilde{\omega})$ the vector field (5.1) derives from the *restricted Hamiltonian*

$$\mathcal{H}_\varepsilon^j = H|\mathcal{A}_\varepsilon^j = (i_\varepsilon^j)^* H = h_0 + \varepsilon \mathcal{H}^j + \mathcal{O}(\varepsilon^2), \tag{5.8}$$

with

$$h_0 = H_0|\mathcal{A}_0^j = \text{const.}, \quad \mathcal{H}^j(I,\phi) = H_1(\bar{x}^j(I), I, \phi; 0). \tag{5.9}$$

Eq. (5.8) justifies the usual terminology of *slow manifold* for a manifold perturbing from a set of equilibria since it shows that the characteristic time scale of motions on $\mathcal{A}_\varepsilon^j$ is of order $\mathcal{O}(\varepsilon)$. We call $\mathcal{H}^j(I,\phi)$ the *reduced Hamiltonian* corresponding to the manifold $\mathcal{A}_0^j$, and consider it to be defined on the annulus $A$. We say that an orbit $\gamma \subset A$ of some Hamiltonian system defined on $A$ is an *internal orbit* if it is either a periodic orbit or an orbit homoclinic to a hyperbolic fixed point, *and* it is bounded away from $\partial A$. Similarly, an orbit $\gamma_\varepsilon \in \mathcal{A}_\varepsilon^j$ of the restricted Hamiltonian $\mathcal{H}_\varepsilon^j$ is called an *internal orbit* if $(g_\varepsilon^j)^{-1}(\gamma_\varepsilon)$ is an internal orbit of the Hamiltonian $(g_\varepsilon^j)^*\mathcal{H}_\varepsilon^j$ on $(A, (g_\varepsilon^j)^*\tilde{\omega})$. By definition, internal orbits are structurally stable with respect to small Hamiltonian perturbations, hence for small $\varepsilon$ the internal orbits of the reduced Hamiltonian $\mathcal{H}^j$ give rise to $\mathcal{O}(\varepsilon)$ $C^r$-close internal orbits of $(g_\varepsilon^j)^*\mathcal{H}_\varepsilon^j$ (see (5.8)). We also note that for small nonzero $\varepsilon$ we have persisting, locally invariant 3-manifolds $W^s_{\rm loc}(\mathcal{A}_\varepsilon^j)$ and $W^u_{\rm loc}(\mathcal{A}_\varepsilon^j)$, that are $\mathcal{O}(\varepsilon)$ $C^r$-close to $W^s_{\rm loc}(\mathcal{A}_0^j)$ and $W^u_{\rm loc}(\mathcal{A}_0^j)$, respectively (see Fenichel [17] or Appendix C.1). As usual, $W^s_{\rm loc}(\mathcal{A}_\varepsilon^j)$ and $W^u_{\rm loc}(\mathcal{A}_\varepsilon^j)$ can be extended to globally defined invariant sets $W^s(\mathcal{A}_\varepsilon^j)$ and $W^u(\mathcal{A}_\varepsilon^j)$.

In the following we give conditions for the existence of *N-pulse heteroclinic orbits* connecting the two manifolds $\mathcal{A}_\varepsilon^1$ and $\mathcal{A}_\varepsilon^2$ to one another.

*Definition 5.1.* An *N-pulse heteroclinic orbit* is an orbit that is negatively asymptotic to an orbit in $\mathcal{A}_\varepsilon^1$, and makes $N-1$ passages through a neighborhood of $\mathcal{A}_\varepsilon^2$ before reentering this neighborhood for the $N$th, and final, time. Then it asymptotically approaches an orbit in $\mathcal{A}_\varepsilon^2$.

We discuss the existence of $N$-pulse heteroclinic orbits under the simplifying assumption

(A3) Consider system (5.1) but only with the leading order term $H_1(x, I, \phi; 0)$ in the perturbation Hamiltonian $H_1$. We assume that under this perturbation, for any $\varepsilon > 0$ small, $W^s(\mathcal{A}_\varepsilon^1) = W^u(\mathcal{A}_\varepsilon^2)$, i.e., $\mathcal{W}_0^+$ and $\mathcal{W}_0^-$ possibly deform into the heteroclinic manifolds $\mathcal{W}_\varepsilon^+$ and $\mathcal{W}_\varepsilon^-$, but do not break.

Note that this assumption forbids the existence of homoclinic connections to orbits in either $\mathcal{A}_\varepsilon^1$ or $\mathcal{A}_\varepsilon^2$ under the effect of the leading order terms in the perturbation. To illustrate Definition 5.1 and assumption (A3), we show schematically the $(x_1, x_2)$-projections of a simple, a 2-pulse, and a 3-pulse heteroclinic connection between the

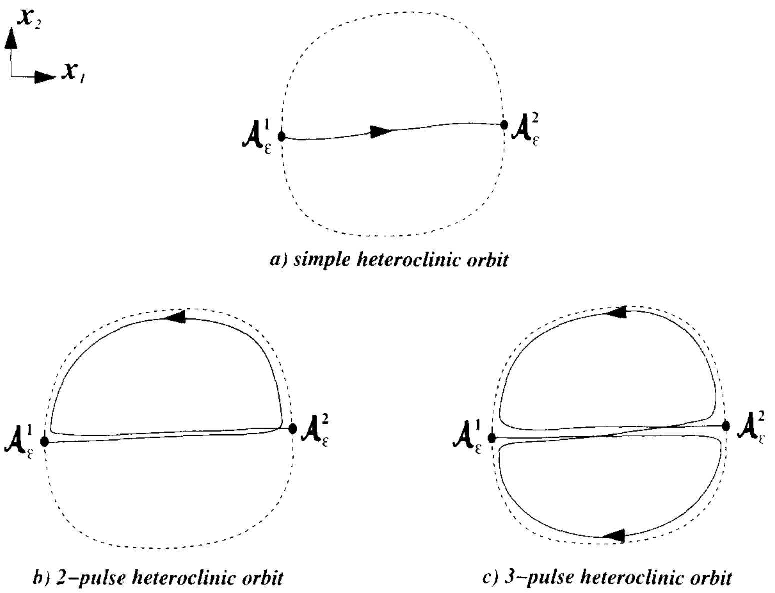


Fig. 8. Simple and multi-pulse heteroclinic orbits between the slow manifolds.

two slow manifolds in Fig. 8. We finally note that assumption (A3) is not necessary for the general theory, but greatly simplifies the statement of the results. Moreover, it will be shown to hold for the perturbation problem for the 2-DOF subsystem outlined in the previous section.

### 5.3. *Invariant foliations of $W^u(\mathcal{A}^1_\varepsilon)$ and $W^s(\mathcal{A}^2_\varepsilon)$*

One of the key tools used in the energy-phase method is foliations of the stable (respectively unstable) manifold of $\mathcal{A}^j_\varepsilon$ by one-dimensional $C^r$ curves, which were first constructed by Fenichel [18] (see also Appendix C.1). These curves are referred to as *fibers* and each fiber intersects $\mathcal{A}^j_\varepsilon$ in a unique point called the *basepoint* of the fiber. Thus, the foliation is a 2-parameter family of one-dimensional $C^r$ curves, that are also $C^r$ with respect to the basepoint. The important feature of these fibers is that points on a fiber correspond to initial conditions that asymptotically approach as $t \to \infty$ (respectively $t \to -\infty$) the trajectory on $\mathcal{A}^j_\varepsilon$ that passes through the basepoint of the fiber. We use the notation $f^s_\varepsilon(q)$ for a stable fiber contained in $W^s(\mathcal{A}^2_\varepsilon)$ which has basepoint $q \in \mathcal{A}^2_\varepsilon$. Similarly, $f^u_\varepsilon(p)$ denotes an unstable fiber contained in $W^u(\mathcal{A}^1_\varepsilon)$. The properties of fibers enable us to identify lower-dimensional invariant manifolds within stable and unstable manifolds. For instance, if $\gamma_\varepsilon \subset \mathcal{A}^1_\varepsilon$ is a slow orbit then it has its own unstable manifold $W^u(\gamma_\varepsilon)$ which is simply the union of all unstable fibers which have their basepoints lying on $\gamma_\varepsilon$. Since the time scales of motions in $\gamma_\varepsilon$ are much longer than those of motions in $W^u(\gamma_\varepsilon)$, the basepoint $p \in \gamma_\varepsilon$ of an unstable fiber $f^u_\varepsilon(p)$ also serves as "take-off" point for any solution in $W^u(\gamma_\varepsilon)$ that intersects $f^u_\varepsilon(p)$: Such a solution slowly asymptotes to $\gamma_\varepsilon$ in backward time but leaves abruptly a neighborhood of $\gamma_\varepsilon$ in forward time near the point $p$. Similarly, the basepoints of stable fibers can be thought of as "landing" points for the solutions intersecting those fibers. We give a precise formulation of all these properties in Appendix C.

### 5.4. Energy-difference functions, jump sequences, and pulse numbers

We now describe our main tools that can be used to follow solutions in the unstable manifold $W^u(\gamma_\varepsilon)$ of some slow internal orbit $\gamma_\varepsilon \subset \mathcal{A}^1_\varepsilon$. These tools will enable us to establish intersections between $W^u(\gamma_\varepsilon)$ and $W^s_{\text{loc}}(\mathcal{A}^2_\varepsilon)$ that lead to multi-pulse heteroclinic orbits described in Definition 5.1.

Suppose that $\gamma^1_\varepsilon \subset \mathcal{A}^1_\varepsilon$ is an internal orbit. Then, as described in the previous subsection, $W^u(\gamma^1_\varepsilon)$ contains all the unstable fibers $f^u_\varepsilon(p)$ whose basepoints $p \equiv g^1_\varepsilon(I_p, \phi_p)$ lie on $\gamma^1_\varepsilon$. This implies that any solution $u(t)$ that intersects $f^u_\varepsilon(p)$ "takes off" from the slow manifold $\mathcal{A}^1_\varepsilon$ near the point $p$. By standard Gronwall estimates for $\varepsilon > 0$ small, $u(t)$ will enter a neighborhood $U_0$ of the other slow manifold $\mathcal{A}^2_\varepsilon$. Upon entry it may or may not intersect the local stable manifold $W^s_{\text{loc}}(\mathcal{A}^2_\varepsilon)$. To check whether this intersection occurs, we can monitor the energy difference between $u(t)$ and the locally closest trajectory $s_1(t)$ in $W^s_{\text{loc}}(\mathcal{A}^2_\varepsilon)$. It can be shown (see Haller and Wiggins [24]) that $s_1(t)$ intersects a stable fiber $f^s_\varepsilon(q_1)$ such that the basepoint of that fiber satisfies

$$q_1 = g^2_\varepsilon(I_p + \mathcal{O}(\sqrt{\varepsilon}), \phi_p + \Delta\phi^0(I_p) + \mathcal{O}(\sqrt{\varepsilon})).$$

Then it is not hard to see from (5.8) that the leading order difference in the energies of $u(t)$ and $s_1(t)$ is $\varepsilon\Delta^1\mathcal{H}(I_p, \phi_p)$ with

$$\Delta^1\mathcal{H}(I, \phi) = \mathcal{H}^2(I, \phi + \Delta\phi^0(I)) - \mathcal{H}^1(I, \phi),$$

since the fibers have necessarily the same energies as their basepoints. It follows that if the function $\Delta^1\mathcal{H}(I, \phi)$ has a transverse zero $(I_p, \phi_p)$ falling on $\gamma^1_\varepsilon$ (i.e., $p = g^1_\varepsilon(I_p, \phi_p) \in \gamma^1_\varepsilon$) then the unstable manifold $W^u(\gamma^1_\varepsilon)$ contains a solution that takes off from $\gamma^1_\varepsilon$ near the point $p$ and lands on the slow manifold $\mathcal{A}^2_\varepsilon$ near the point $q_1$ defined above. This establishes a simple transverse heteroclinic connection between $\gamma^1_\varepsilon$ and a slow orbit $\gamma^2_\varepsilon \subset \mathcal{A}^2_\varepsilon$. In particular, if $\gamma^1_\varepsilon$ is periodic and all slow orbits on $\mathcal{A}^2_\varepsilon$ passing near $q_1$ are periodic then we obtain a "fast" transverse heteroclinic connection between two slow periodic orbits.

In general, however, the function $\Delta^1\mathcal{H}$ need not have any zero falling on the orbit $\gamma^1_\varepsilon$. This means that for $\varepsilon > 0$ small enough, $W^u(\gamma^1_\varepsilon)$ leaves the neighborhood $U_0$ of $\mathcal{A}^2_\varepsilon$ without intersecting $W^s_{\text{loc}}(\mathcal{A}^2_\varepsilon)$. Looking at Fig. 7 we see that $W^u(\gamma^1_\varepsilon)$ may exit $U_0$ in the direction of one of the two persisting heteroclinic manifolds $\mathcal{W}^+_\varepsilon$ or $\mathcal{W}^-_\varepsilon$ (see assumption (A3)). The exit direction turns out to depend on the sign of $\Delta^1\mathcal{H}|\gamma^1_0$ where $\gamma^1_0$ is an orbit of the reduced Hamiltonian $\mathcal{H}^1$ passing through the point $(I_p, \phi_p)$. This dependence can be described in the following way. Let $\rho(m)$ denote the unit vector normal to the heteroclinic manifold $\mathcal{W}^0_0$ at $m \in \mathcal{W}^0_0$, which points in the direction of $\mathcal{W}^+_0$ (see Fig. 6). We define the quantity

$$\sigma = \text{sign}\langle D_x H_0(m), \rho(m)\rangle. \tag{5.10}$$

Note that by assumption (A1), $\sigma$ is independent of the choice of $m$. For $\sigma = +1$ the orbits inside the region bounded by $\mathcal{W}^0_0 \cup \mathcal{W}^+_0$ have higher energies than the orbits inside the region bounded by $\mathcal{W}^0_0 \cup \mathcal{W}^-_0$. For $\sigma = -1$ the opposite holds. We also introduce the parameter

$$\chi_1(\gamma^1_0) = -\sigma\text{sign}\,(\Delta^1\mathcal{H}|\gamma^1_0).$$

After a little thinking one realizes that for $\varepsilon > 0$ sufficiently small, $\chi_1(\gamma^1_0) = +1$ implies that $W^u(\gamma^1_\varepsilon)$ exits $U_0$ in the direction of $\mathcal{W}^+_0$ and $\chi_1(\gamma^1_0) = -1$ implies that $W^u(\gamma^1_\varepsilon)$ exits $U_0$ in the direction of $\mathcal{W}^-_0$. This again follows from the fact that $\varepsilon\Delta^1\mathcal{H}|\gamma^1_\varepsilon$ is a function that approximates the leading order difference in energy between solutions in $W^u(\gamma^1_\varepsilon)$ and the solutions closest to them in $W^s_{\text{loc}}(\mathcal{A}^2_\varepsilon)$.

Depending on its exit direction, $W^u(\gamma^1_\varepsilon)$ is guided back to the slow manifold $\mathcal{A}^1_\varepsilon$ by either the unbroken heteroclinic manifold $\mathcal{W}^+_\varepsilon$ or by its counterpart $\mathcal{W}^-_\varepsilon$. Then it passes near $\mathcal{A}^1_\varepsilon$ and subsequently enters the neighborhood $U_0$ of $\mathcal{A}_\varepsilon^2$. This time the leading order energy-differences between solutions in $W^u(\gamma_\varepsilon^1)$ and $W_{\text{loc}}^s(\mathcal{A}_\varepsilon^2)$ turn out to be given by $\varepsilon\Delta^2\mathcal{H}(I,\phi)$ where $(I,\phi)$ are the coordinates of take-off points on $\gamma_0^1$ and

$$\Delta^2\mathcal{H}(I,\phi) = \mathcal{H}^2(I,\phi + 2\Delta\phi^0(I) + \Delta\phi^{\chi_1(\gamma_0^1)}(I)) - \mathcal{H}^1(I,\phi).$$

Again, if $\Delta^2\mathcal{H}$ has a transverse zero on $(I_p,\phi_p)$ on $\gamma_0^1$, then it follows that $\gamma_\varepsilon^1$ is connected through a transverse *2-pulse* heteroclinic orbit (see Definition 5.1) to a slow orbit $\gamma_\varepsilon^2 \subset \mathcal{A}_\varepsilon^2$ that passes near the point $q_2 = g_\varepsilon^2(I_p,\phi_p + 2\Delta\phi^0(I_p) + \Delta\phi^{\chi_1(\gamma_0^1)}(I_p))$. If $\Delta^2\mathcal{H}$ has no zeros on $\gamma_0^1$, then we define the constant

$$\chi_2(\gamma_0^1) = -\sigma\text{sign}\,(\Delta^2\mathcal{H}|\gamma_0^1),$$

and the sign of this constant again gives us the correct exit direction for $W^u(\gamma_\varepsilon^1)$ from $U_0$.

We can now repeat this construction recursively. Assuming that $\gamma_0^1$ does not contain any zeros of the functions $\Delta^1\mathcal{H}$, $\Delta^2\mathcal{H}, \ldots, \Delta^{n-1}\mathcal{H}$, for the $n$th entry of $W^u(\gamma_\varepsilon^1)$ into $U_0$ we define the *nth order energy-difference function*

$$\Delta^n\mathcal{H}(I,\phi) = \mathcal{H}^2\left(I,\phi + n\Delta\phi^0(I) + \sum_{l=1}^{n-1}\Delta\phi^{\chi_l(\gamma_0^1)}(I)\right) - \mathcal{H}^1(I,\phi). \tag{5.11}$$

If this function has a transverse zero $(I_p,\phi_p)$ on $\gamma_0^1$, then we conclude the existence of a transverse, *n-pulse* heteroclinic connection between $\gamma_\varepsilon^1 \subset \mathcal{A}_\varepsilon^1$ (that contains the take-off point $p = g_\varepsilon^1(I_p,\phi_p)$ after perturbation) and another slow orbit $\gamma_\varepsilon^2 \subset \mathcal{A}_\varepsilon^2$ which passes $\mathcal{O}(\sqrt{\varepsilon})$-close to the approximate landing point

$$q_n = g_\varepsilon^2\left(I_p,\phi_p + n\Delta\phi^0(I_p) + \sum_{l=1}^{n-1}\Delta\phi^{\chi_l(\gamma_0^1)}(I_p)\right).$$

Then we call the sequence $\chi(\gamma_0^1) = \{\chi_l(\gamma_0^1)\}_{l=1}^{n-1}$ the *jump sequence* associated with $\gamma_\varepsilon^1$ because it describes how the $n$-pulse orbit jumps between neighborhoods of the two heteroclinic manifolds $\mathcal{W}_\varepsilon^+$ and $\mathcal{W}_\varepsilon^-$: If $\chi_l(\gamma_0^1) = +1$ the orbit makes its $l$th pulse near $\mathcal{W}_\varepsilon^+$, while for $\chi_l(\gamma_0^1) = -1$ the orbit makes its $l$th pulse near $\mathcal{W}_\varepsilon^-$. For example, for the 2-pulse heteroclinic orbit shown in Fig. 8b the jump sequence is simply $\chi_1 = +1$, while for the 3-pulse heteroclinic orbit in Fig. 8c the jump sequence is $\chi_1 = -1$, $\chi_2 = +1$.

Notice that for any internal orbit $\gamma_0^1$ of the reduced Hamiltonian $\mathcal{H}^1$ the sequence of energy-difference functions and the jump sequence are well defined. Both sequences may be infinite, which means that for $\varepsilon > 0$ small enough, there are no "finite-pulse" heteroclinic orbits backward asymptotic to $\gamma_0^1$. If the sequences are finite, i.e., they terminate at an index $N$, then we conclude the existence of an $n$-pulse orbit backward asymptotic to $\gamma_\varepsilon^1$. We then call the number $N \equiv N(\gamma_0^1)$ the *pulse number* of $\gamma_0^1$. We summarize this construction in the following theorem.

*Theorem 5.1.* Let us assume that (A1)–(A3) hold. Suppose that for an internal orbit $\gamma_0^1 \subset A$ of the reduced Hamiltonian $\mathcal{H}^1$,

(A4) $N \equiv N(\gamma_0^1) < \infty$,

(A5) Let $Z_-^N \subset A$ be the transverse zero set of $\Delta^N\mathcal{H}$. Suppose that $Z_-^N$ intersects $\gamma_0^1$ transversally in a point $b_1 = (I_p,\phi_p)$ and $\gamma_0^2$ is an internal orbit of the reduced Hamiltonian $\mathcal{H}^2$ that contains the point $(I_p,\phi_{q_N})$ with

$$\phi_{q_N} = \phi_p + N\Delta\phi^0(I_p) + \sum_{l=1}^{N-1}\Delta\phi^{\chi_l(\gamma_0^1)}(I_p).$$

Then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,

(i) There exists an $N$-pulse heteroclinic orbit $y_\varepsilon^N$ which is backward asymptotic to an internal orbit $\gamma_\varepsilon^1 \in \mathcal{A}_\varepsilon^1$ and forward asymptotic to an internal orbit $\gamma_\varepsilon^2 \in \mathcal{A}_\varepsilon^2$. Moreover, $(g_\varepsilon^1)^{-1}(\gamma_\varepsilon^1)$ and $\gamma_0^1$ are locally $\mathcal{O}(\varepsilon)$ $C^r$-close near the point $(I_p, \phi_p)$, and $(g_\varepsilon^2)^{-1}(\gamma_\varepsilon^2)$ and $\gamma_0^2$ are locally $\mathcal{O}(\varepsilon)$ $C^r$-close near the point $(I_p, \phi_{q_N})$.

(ii) $y_\varepsilon^N$ lies in the intersection of $W^u(\gamma_\varepsilon^1)$ and $W^s(\gamma_\varepsilon^2)$, which is transversal within the energy surface $\{H = h\}$ with $h = H|\gamma_\varepsilon^1 = H|\gamma_\varepsilon^2$. The manifolds $W^u(\gamma_\varepsilon^1)$ and $W^s(\gamma_\varepsilon^2)$ intersect along $y_\varepsilon^N$ at an angle of $\mathcal{O}(1)$ as $\varepsilon \to 0$.

(iii) If $\chi_l(\gamma_0^1) = \pm 1$, then $y_\varepsilon^N$ makes its $l$th pulse near the heteroclinic manifold $\mathcal{W}_\varepsilon^\pm$.

(iv) The manifolds $W^u(\mathcal{A}_\varepsilon^1)$ and $W^s(\mathcal{A}_\varepsilon^2)$ intersect along the solution $y_\varepsilon^N$ with $\mathcal{O}(\varepsilon)$ transversality.

*Proof.* The theorem follows from our previous discussion and the results in Haller and Wiggins [24]. □

*Remark 5.1.* Notice the peculiarity of statement (ii) stating that the stable and unstable manifolds of the slow orbits $\gamma_\varepsilon^1$ and $\gamma_\varepsilon^2$ intersect with $\mathcal{O}(1)$ transversality as a result of an order $\mathcal{O}(\varepsilon)$ perturbation. On the surface, this contradicts elementary facts from perturbation theory. However, one should not forget that we are not in a regular but in a singular perturbation context: the slow periodic orbits together with their stable and unstable manifolds are *created by the perturbation*. So although the stable and unstable manifolds of $\gamma_\varepsilon^1$ and $\gamma_\varepsilon^2$ keep intersecting at an angle of $\mathcal{O}(1)$ as $\varepsilon \to 0$, they suddenly disappear at the singular limit of $\varepsilon = 0$.

*Remark 5.2.* In most applications the $n$th order energy-difference functions and the jump sequences are independent of the choice of internal orbit $\gamma_0^1$. (They are always independent in the homoclinic version of the energy-phase method, see Haller and Wiggins [24].) In that case one does not have to go through the recursive construction we described and the energy-difference functions and the jump sequence can be written down immediately. In that case the application of the energy-phase method simplifies to finding the zero set $Z_-^n$ of $\Delta^n\mathcal{H}$ for all $n$ and identifying the internal orbits of $\mathcal{H}^1$ that have transverse intersection with, say, $Z_-^N$ but no intersections with $Z_-^1, \ldots, Z_-^{N-1}$. In this manner one can obtain a global characterization of the internal orbits of $\mathcal{H}^1$ in terms of their pulse numbers and hence a classification of all existing multi-pulse orbits in a given problem. We study such a case in Haller and Wiggins [25] where the theory developed in this paper is applied to low energy oscillations of the water molecule.

*Remark 5.3.* Using Melnikov's method combined with a version of the exchange lemma of Jones et al. [32] described in Tin [56] and Jones et al. [32], one can find further multi-pulse orbits in the exponentially small vicinities of single-pulse orbits homoclinic to slow manifolds. This is shown in Kaper and Kovačič [33] for the case when the slow manifold $\mathcal{A}_\varepsilon^j$ arises in the blow-up of a resonance band on a two-dimensional normally hyperbolic invariant manifold.

## 6. Dynamics in the 3-DOF truncated normal form

In this section we first study the dynamics of the 2-DOF subsystem (4.16) using the energy-phase method, then discuss the immediate implications of these results for the 3-DOF truncated normal form which is generated by the Hamiltonian $\bar{H}$ in (4.15).

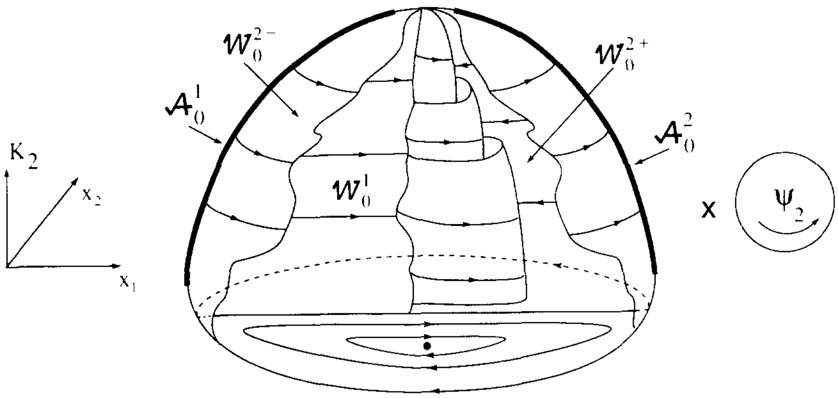


Fig. 9. The invariant manifolds of system (4.16) for $\varepsilon = 0$.

### *6.1. Invariant manifolds and multi-pulse orbits in the 2-DOF subsystem*

Substituting $(I, \phi)$ for $(K_2, \psi_2)$ in (4.16), we see that the 2-DOF subsystem (4.16) is exactly of the form (5.1). We fix some small but arbitrary $\lambda > 0$ and define the annulus

$$A = [\lambda, (1/\omega_3)(h - d^2/a^2) - \lambda] \times S^1. \tag{6.1}$$

Then

$$\mathcal{A}_0^1 = \{ (x, K_2, \psi_2) \in \mathcal{P}^h \mid x_1 = -\sqrt{h - \omega_3 K_2 - d^2/a^2},\ x_2 = -d/a,\ (K_2, \psi_2) \in A \},$$
$$\mathcal{A}_0^2 = \{ (x, K_2, \psi_2) \in \mathcal{P}^h \mid x_1 = +\sqrt{h - \omega_3 K_2 - d^2/a^2},\ x_2 = -d/a,\ (K_2, \psi_2) \in A \}, \tag{6.2}$$

are two normally hyperbolic 2-manifolds connected through the heteroclinic manifolds

$$\mathcal{W}_0^0 = \{ (x, K_2, \psi_2) \in \mathcal{P}^h \mid |x_1| < \sqrt{h - \omega_3 K_2},\ x_2 = -d/a,\ (K_2, \psi_2) \in A \},$$
$$\mathcal{W}_0^+ = \{ (x, K_2, \psi_2) \in \mathcal{P}^h \mid x_1^2 + x_2^2 = h - \omega_3 K_2,\ x_2 > -d/a,\ (K_2, \psi_2) \in A \},$$
$$\mathcal{W}_0^- = \{ (x, K_2, \psi_2) \in \mathcal{P}^h \mid x_1^2 + x_2^2 = h - \omega_3 K_2,\ x_2 < -d/a,\ (K_2, \psi_2) \in A \}. \tag{6.3}$$

We illustrate the geometry of these invariant manifolds in Fig. 9 (cf. Fig. 7). From (5.3)–(5.5) and (6.2) we see that for fixed $K_1$ the embedding of the manifold $\mathcal{A}_0^j$ can be written as

$$g_0^j(K_2, \psi_2) = \Big( (-1)^j \sqrt{K_1 - \omega_3 K_2 - d^2/a^2},\ -d/a,\ K_2,\ \psi_2 \Big), \quad j = 1, 2. \tag{6.4}$$

It follows from our earlier discussion that assumption (A1) is satisfied for system (4.16) (see Fig. 2). Furthermore, as we see from Fig. 2, the heteroclinic orbits for our system have the geometry of Fig. 6. Also, from (4.16) and (6.2), we see that $D_{K_2} H_0(x, K) = 0$ on $\mathcal{A}_0^1$ and $\mathcal{A}_0^2$ for $K_2 \in [\lambda, (1/\omega_3)(K_1 - d^2/a^2) - \lambda]$. Hence assumption (A2) is also satisfied. As a result, all the sets and quantities of the energy-phase method discussed at the beginning of Section 5 can be defined for system (4.16) with the substitution $(I, \phi) \to (K_2, \psi_2)$, and $(\tilde{\mathcal{P}}, \tilde{\omega}) \to (\mathcal{P}^h, \omega^h)$.

The phase shifts defined in (5.6) can be easily computed (see Appendix A) and are found to be

$$\Delta\psi_2^0(K_2) = 0,$$
$$\Delta\psi_2^{+1}(K_2) = -\omega_3\pi + \omega_3\cos^{-1}\left(\frac{-d}{a\sqrt{h-\omega_3K_2}}\right),$$
$$\Delta\psi_2^{-1}(K_2) = \omega_3\cos^{-1}\left(\frac{-d}{a\sqrt{h-\omega_3K_2}}\right), \tag{6.5}$$

where the superscript $\pm1$ refers to the phase shift on $\mathcal{W}_0^{\pm}$, and the subscript 0 refers to the phase shift on $\mathcal{W}_0^0$.

The *restricted Hamiltonians* $\mathcal{H}_\varepsilon^j$ defined in (5.8) take the form

$$\mathcal{H}_\varepsilon^j(K_2,\psi_2) = h + \varepsilon^2\mathcal{H}^j(K_2,\psi_2;K_1) + \mathcal{O}(\varepsilon^3), \tag{6.6}$$

with the *reduced Hamiltonians* being specifically

$$\mathcal{H}^j(K_2,\psi_2;K_1) = \tilde{H}_4\Big((-1)^j\sqrt{K_1-\omega_3K_2-d^2/a^2},\ -d/a,\ K_1,\ K_2,\ \psi_2\Big),\quad j=1,2. \tag{6.7}$$

Throughout this section $K_1 = h$ is regarded as a fixed parameter in the function $\mathcal{H}^j$. Note that by our discussion at the end of the proof of part (iii) of Proposition 4.2 (see (4.13),(4.14)), for any $k \in \mathbb{Z}$ we have

$$\mathcal{H}^j(K_2,\psi_2 + k\tilde{\psi}_2) = \mathcal{H}^j(K_2,\psi_2),\quad j=1,2, \tag{6.8}$$
$$\mathcal{H}^1(K_2,\psi_2) = \mathcal{H}^2\big(K_2,\psi_2 + \pi k\omega_3/\kappa_2 \pm \omega_3\cos^{-1}(-d/a\sqrt{h-\omega_3K_2})\big). \tag{6.9}$$

Next, we give a sufficient condition under which assumption (A3) in Section 5 is satisfied for the 2-DOF subsystem (4.16).

*Lemma 6.1.* Suppose that

(A3′) the resonant module $M$ defined in (2.7) contains no element of the form $(1,n_2,n_3)$ with the integers $n_1$ and $n_2$ satisfying $|n_2|+|n_3|=3$.

Then assumption (A3) is satisfied for the 2-DOF subsystem (4.16).

*Proof.* Since assumption (A3) involves statements on invariant manifolds for the leading order perturbation in system (4.16), throughout the proof all perturbed invariant manifolds will be defined with respect to quartic perturbation $\varepsilon^2\tilde{H}_4$ of the cubic normal form Hamiltonian.

It is easy to verify that assumption (A3) implies the quartic complex normal form $H_2(z,\bar{z}) + \varepsilon H_3(z,\bar{z}) + \varepsilon^2\tilde{H}_4(z,\bar{z})$ to have an invariant manifold satisfying $z_1 = \bar{z}_1 = 0$. Using the coordinate transformations introduced in (4.1),(4.2), we see that this invariant manifold is given by $K_1 - \omega_3K_3 - |x|^2 = 0$ in terms of the $(x,K,\psi)$ coordinates. This implies that for any fixed $K_1 = h$ the Hamiltonian system generated by $H_c(x,K) + \varepsilon^2\tilde{H}_4(x,K,\psi_2)$ has a four-dimensional invariant manifold $\mathcal{W}$ which is diffeomorphic to $S^3\times S^1$ and satisfies the equation

$$x_1^2 + x_2^2 + \omega_3K_2 = h. \tag{6.10}$$

We first want to show that under the effect of the leading order perturbation Hamiltonian $\varepsilon^2\tilde{H}_4$, the perturbed invariant manifolds $\mathcal{A}_\varepsilon^j$ must be contained in $\mathcal{W}$.

Suppose the contrary, i.e., suppose that, say, $\mathcal{A}_\varepsilon^2$ contains a point $p$ such that $p \notin \mathcal{W}$. We recall that $\mathcal{A}_\varepsilon^2$ is $\mathcal{O}(\varepsilon)$ $C^r$-close to $\mathcal{A}_0^2$, and the stable fibers in $W^s_{\text{loc}}(\mathcal{A}_\varepsilon^2)$ are $\mathcal{O}(\varepsilon)$ $C^r$-close to stable fibers in $W^s_{\text{loc}}(\mathcal{A}_0^2)$. Now the stable fibers in $W^s_{\text{loc}}(\mathcal{A}_0^2)$ intersect the manifold $\mathcal{W}$ transversely at their basepoints with $\mathcal{O}(1)$ transversality. As a result, the stable fiber $f_\varepsilon^s(p)$ must intersect the manifold $\mathcal{W}$ transversely. By assumption, the intersection point is not the basepoint $p$ of this fiber. But this implies that $W^s_{\text{loc}}(\mathcal{A}_\varepsilon^2)$ intersects $\mathcal{W}$ transversally near $f_\varepsilon^s(p)\cap\mathcal{W}$.

But this contradicts the invariance of $\mathcal{W}$ hence $\mathcal{A}_\varepsilon^2 \subset \mathcal{W}$ must hold. A similar argument shows that $\mathcal{A}_\varepsilon^1 \subset \mathcal{W}$ must also hold.

Now observe that $\mathcal{W}$ also contains the manifold $W^u(\mathcal{A}_0^2)$ and this implies that $W^u(\mathcal{A}_\varepsilon^2)$ is $\mathcal{O}(\varepsilon)$ $C^r$-close to $\mathcal{W}$. Therefore, $\mathcal{W}$ and $W^u_{\mathrm{loc}}(\mathcal{A}_\varepsilon^2)$ both have the properties that they are locally invariant in forward time, their closures contain $\mathcal{A}_\varepsilon^2$, and they are are $\mathcal{O}(\varepsilon)$ $C^r$-close to $W^u_{\mathrm{loc}}(\mathcal{A}_0^2)$. But the existence theory of unstable manifolds of normally hyperbolic invariant manifolds guarantees that the only manifold that has all these properties in a neighborhood of $\mathcal{A}_\varepsilon^2$ is the local unstable manifold $W^u_{\mathrm{loc}}(\mathcal{A}_\varepsilon^2)$. Consequently, $\mathcal{W}$ and $W^u(\mathcal{A}_\varepsilon^2)$ coincide near the slow manifold $\mathcal{A}_\varepsilon^2$. Then, by invariance, they must coincide globally on $\mathcal{W} - (\mathcal{A}_\varepsilon^1 \cup \mathcal{A}_\varepsilon^2)$. A similar argument shows that $W^s(\mathcal{A}_\varepsilon^1) \equiv \mathcal{W} - (\mathcal{A}_\varepsilon^1 \cup \mathcal{A}_\varepsilon^2)$, which implies that $W^u(\mathcal{A}_\varepsilon^2) \equiv W^s(\mathcal{A}_\varepsilon^1)$. □

We note that (A3′) is *always* satisfied if $\kappa_1$ is odd (cf. (3.13)). If (A3′) is not satisfied in a given application, one can replace it by some alternative assumption (e.g., by an appropriate form of reversibility assumption on the quartic normal form terms) that also implies (A3).

To apply the energy-phase method we need to compute the energy-difference functions as defined in (5.11). Using (6.7) we directly obtain that

$$\Delta^n \mathcal{H}(K_2,\psi_2;K_1) = \tilde{H}_4\left(\sqrt{K_1-\omega_3 K_2 - d^2/a^2},\ -d/a,\ K_1,\ K_2,\ \psi_2 + \sum_{l=0}^{n-1}\Delta\psi^{\chi_l(\gamma_0^1)}(K_2)\right)$$
$$-\tilde{H}_4\left(-\sqrt{K_1-\omega_3 K_2 - d^2/a^2},\ -d/a,\ K_1,\ K_2,\ \psi_2\right), \tag{6.11}$$

where $\chi_0(\gamma_0^1) \equiv 0$ and the jump sequence $\{\chi_l(\gamma_0^1)\}_l$ can be computed for a given $\tilde{H}_4$ and $\gamma_0^1$, as we described in Section 5. After evaluating (6.11) for a concrete application, we can use Theorem 5.1 directly to show the existence of transverse multi-pulse heteroclinic orbits in system (4.16). The choice of $\gamma_0^1$ then depends on what parts of the slow manifolds $\mathcal{A}_\varepsilon^j$ carry the types of motions we are interested in.

*Remark 6.1.* We note that since the jump sequence $\{\chi_l(\gamma_0^1)\}$ is defined via "open" conditions, it will be the same for a family $\Gamma^N$ of internal orbits that contains $\gamma_0^1$. As a result, the same energy-difference function defined in (6.11) can be used for this whole family of internal orbits to find transverse, $N$-pulse heteroclinic orbits backward asymptotic to members of the family. Each member in the family $\Gamma^N$ intersects $Z_-^N$, the zero set of the corresponding $N$th order energy-difference function, transversally. Note that the order of this transversality is $\mathcal{O}(1)$, i.e., members of the family $\Gamma^N$ intersect the zero set $Z_-^N$ transversally at an angle of $\mathcal{O}(1)$ as $\varepsilon \to 0$.

*Remark 6.2.* It is shown in Haller and Wiggins [24] that the zero sets $Z_-^N$ are $\mathcal{O}(\sqrt{\varepsilon})$ $C^1$-close to a curve of basepoints of stable fibers, that intersect the $N$-pulse orbits backward asymptotic to the family $\Gamma^N$ described in Remark 6.1. In other words, based on our discussion in Section 5.3, $Z_-^N$ approximates the set of approximate take-off points for the $N$-pulse orbits as they leave the family $\Gamma^N$ in forward time.

### *6.2. Implications for the 3-DOF truncated normal form*

We now discuss what the results of the previous theorem for the 2-DOF subsystem mean for the blown-up, 3-DOF *truncated* normal form

$$\begin{aligned}
\dot{x}_1 &= \varepsilon[a(K_1 - \omega_3 K_2 - x_1^2 - x_2^2) - 2(d + ax_2)x_2] + \varepsilon^2 D_{x_2} H_1(x, K, \psi_2; \varepsilon), \\
\dot{x}_2 &= \varepsilon 2x_1(d + ax_2) - \varepsilon^2 D_{x_1} H_1(x, K, \psi_2; \varepsilon), \\
\dot{K}_2 &= -\varepsilon^2 D_{\psi_2} H_1(x, K, \psi_2; \varepsilon), \\
\dot{\psi}_2 &= -\varepsilon\omega_3(d + ax_2) + \varepsilon^2 D_{K_2} H_1(x, K, \psi_2; \varepsilon), \\
\dot{K}_1 &= 0, \\
\dot{\psi}_1 &= 1 + \varepsilon(d + ax_2) + \varepsilon^2 D_{K_1} H_1(x, K, \psi_2; \varepsilon)
\end{aligned} \tag{6.12}$$

with $H_1$ defined in (4.17). Subsequently, we show that similar results hold for the original truncated complex normal form (2.9).

Recall that system (6.12) derives from the Hamiltonian (4.15) on the phase space $(\mathcal{P}, \omega)$. Since $K_1$ is an integral for this system, any invariant set $\mathcal{S}$ in the phase space $(\mathcal{P}^h, \omega^h)$ is manifested as an invariant set $\mathcal{S}^*$ in the $K_1 = h$ hypersurface of the phase space $(\mathcal{P}, \omega)$, such that $\mathcal{S}^*$ is diffeomorphic to $S^1 \times \mathcal{S}$. Now letting $K_1 = h$ vary in some open interval $U \subset \mathbb{R}^+$, we obtain a one-parameter family of invariant sets of the form $U \times \mathcal{S}^*$. In particular, for $\varepsilon > 0$, we obtain the four-dimensional, normally hyperbolic invariant manifolds

$$\bar{\mathcal{M}}_\varepsilon^j \equiv U \times S^1 \times \mathcal{A}_\varepsilon^j, \quad j = 1, 2. \tag{6.13}$$

which are given by the two $C^r$ embeddings

$$\bar{G}_\varepsilon^j : U \times S^1 \times A \to \mathcal{P}, \quad (K_1, \psi_1, K_2, \psi_2) \mapsto (K_1, \psi_1, g_\varepsilon^j(K_2, \psi_2)), \quad j = 1, 2. \tag{6.14}$$

Although they are not hyperbolic, $\bar{\mathcal{M}}_0^j$ still exist as smooth limits of manifolds, with the corresponding embeddings $\bar{G}_0^j : U \times S^1 \times A \to \mathcal{P}$, $j = 1, 2$. The manifolds $\bar{\mathcal{M}}_\varepsilon^j$ have five-dimensional stable and unstable manifolds of the form

$$W^{s,u}(\bar{\mathcal{M}}_\varepsilon^j) = U \times S^1 \times W^{s,u}(\mathcal{A}_\varepsilon^j), \quad j = 1, 2. \tag{6.15}$$

The following proposition gives us information about the dynamics on $\bar{\mathcal{M}}_\varepsilon^j$.

*Proposition 6.2.* There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the following are satisfied:

(i) Let $\bar{I}_\varepsilon^j : \bar{\mathcal{M}}_\varepsilon^j \hookrightarrow \mathcal{P}$ denote the inclusion map of $\bar{\mathcal{M}}_\varepsilon^j$, $j = 1, 2$. Then $(\bar{\mathcal{M}}_\varepsilon^j, (\bar{I}_\varepsilon^j)^*\omega)$ are invariant, symplectic 4-manifolds on which the (integrable) dynamics is generated by the restricted Hamiltonian $\bar{H}|\bar{\mathcal{M}}_\varepsilon^j \equiv (\bar{I}_\varepsilon^j)^*\bar{H}$.

(ii) For any fixed $K_1 = h \in U$ and for $j = 1, 2$, an internal orbit $\gamma_0^j$ of the reduced Hamiltonian (6.7) for the 2-DOF subsystem (4.16) yields a two-dimensional invariant manifold $\bar{\mathcal{T}}_\varepsilon^j \subset (\bar{\mathcal{M}}_\varepsilon^j \cap \{K_1 = h\})$ for system (6.12). If $\gamma_0^j$ is periodic then $\bar{\mathcal{T}}_\varepsilon^j$ is an invariant two-dimensional torus which is $\mathcal{O}(\varepsilon)$ $C^r$-close to the set $\bar{G}_0^j(\{h\} \times S^1 \times \gamma_0^j)$.

(iii) $\bar{\mathcal{T}}_\varepsilon^j$ has three-dimensional stable and unstable manifolds, denoted $W^s(\bar{\mathcal{T}}_\varepsilon^j)$ and $W^u(\bar{\mathcal{T}}_\varepsilon^j)$, respectively.

*Proof.* These results follow from our calculations in Section 6.1 and from Theorem 5.1 by noting that after rescaling time by $\varepsilon$, the $(x, K_2, \psi_2)$ equations in (6.12) coincide with the 2-DOF subsystem (4.16). □

Next, we consider how $N$-pulse heteroclinic connections in the 2-DOF subsystem are manifested in system (6.12) that is generated by the truncated Hamiltonian $\bar{H}$. First, note that under assumption (A3′) in the previous section, it follows that $W^s\left(\bar{\mathcal{M}}_\varepsilon^1\right) = W^u\left(\bar{\mathcal{M}}_\varepsilon^2\right)$ which have "upper" and "lower" components denoted $\mathcal{D}_\varepsilon^+$ and $\mathcal{D}_\varepsilon^-$, respectively. To facilitate the statement of the results, we introduce the set

$$E(h_1, h_2) = \{(x, K, \psi) \in \mathcal{P} \mid K_1 = h_1, \bar{H}(x, K, \psi_2) = h_2\}. \quad (6.16)$$

Notice that for fixed $h_1 > 0$, $|h_2| < h_1$, the set $E(h_1, h_2)$ is a four-dimensional sphere that lies in the intersection of the two five-dimensional spheres $K_1 = K_{10} =$ const. and $\bar{H} =$ const. with $\bar{H}$ defined in (4.15). Without the quartic and higher order terms in the normal form, this 4-sphere contains the three-dimensional set $\mathcal{M}^h$ (with $h = K_{10} + \varepsilon h_2$) and the bounded components of its stable and unstable manifolds. It is easy to verify from the expression for the Hamiltonian $H_c$ that the intersection of the two 5-spheres is non-transverse along $\mathcal{M}^h$ but is transverse away from $\mathcal{M}^h$.

*Theorem 6.3.* Suppose that assumption (A3′) is satisfied and that $\gamma_0^j \in \mathcal{A}_0^j$, $j = 1, 2$, are internal orbits of the 2-DOF subsystem satisfying the hypotheses of Theorem 5.1. Then there exists $\varepsilon_0 > 0$ and an open set $U \subset \mathbb{R}^+$ with $h \in U$ such that for $0 < \varepsilon < \varepsilon_0$ and $K_1 = h \in U$,

(i) The corresponding sets $\bar{\mathcal{T}}_\varepsilon^j \in \bar{\mathcal{M}}_\varepsilon^j \cap \{K_1 = h\}$, $j = 1, 2$, described in Proposition 6.2 are connected by a two-dimensional, $N$-pulse cylindrical surface $\bar{Y}_\varepsilon^N$ which is diffeomorphic to $y_\varepsilon^N \times S^1$ (see the statement of Theorem 5.1).

(ii) The solutions contained in $\bar{Y}_\varepsilon^N$ make their $l$th excursion around $\mathcal{D}_\varepsilon^+$ if $\chi_l(\gamma_0^1) = +1$, or around $\mathcal{D}_\varepsilon^-$ if $\chi_l(\gamma_0^1) = -1$.

(iii) The manifold $W^u(\bar{\mathcal{T}}_\varepsilon^1)$ intersects $W^s(\bar{\mathcal{T}}_\varepsilon^2)$ transversally within the 4-sphere $E_\varepsilon(h, \bar{H}|\bar{\mathcal{T}}_\varepsilon^1)$ with $\mathcal{O}(1)$ transversality, but the intersection is not transversal within the corresponding full energy surface $\bar{H} =$ const.

(iv) The manifolds $W^u(\bar{\mathcal{M}}_\varepsilon^1)$ and $W^s(\bar{\mathcal{M}}_\varepsilon^2)$ intersect transversally along the $N$-pulse solution set $\bar{Y}_\varepsilon^N$ with transversality of order $\mathcal{O}(\varepsilon)$.

*Proof.* Statements (i)–(iii) are immediate applications of Theorem 5.1 and Proposition 6.2. The transversal intersection of $W^u(\bar{\mathcal{M}}_\varepsilon^1)$ and $W^s(\bar{\mathcal{M}}_\varepsilon^2)$ along $\bar{Y}_\varepsilon^N$ in statement (iv) follows from the fact that the manifolds $W^u(\bar{\mathcal{T}}_\varepsilon^2)$ and $W^s(\bar{\mathcal{T}}_\varepsilon^2)$ intersect transversally within the surface $E_\varepsilon(h, \bar{H}|\bar{\mathcal{T}}_\varepsilon^1)$, hence at any point $p \in \bar{Y}_\varepsilon^N$ the tangent space $T_p W^u(\bar{\mathcal{M}}_\varepsilon^1)$ contains a one-dimensional subspace that is not contained in the three-dimensional space $T_p W^s(\bar{\mathcal{M}}_\varepsilon^2)$ (see also Remark 5.1). The order $\mathcal{O}(\varepsilon)$ transversality of the intersection follows from (*iv*) of Theorem 5.1. and the persistence theory of normally hyperbolic invariant manifolds (see, e.g., Appendix C).□

In most cases the internal orbits $\gamma_0^1$ and $\gamma_0^2$ appearing in this theorem are members of families of periodic orbits in the annulus $A$. It may also happen that they map to the same set of points in the orbit space $F^h$ (defined in Section 3.2.1) under the map $Q^h$ defined in (4.12) (see, e.g., Haller and Wiggins [25] or the potential problem considered in Van der Aa and Verhulst [59]). Thus transverse heteroclinic connections between their perturbed counterparts actually yield transverse *homoclinic* connections in system (2.9). For this case we have more specific results.

*Proposition 6.4.* Suppose that $\gamma_0^1$ and $\gamma_0^2$ are periodic orbits with the assumptions of Theorem 6.3 holding. Then for $\varepsilon > 0$ small enough,

(i) $\bar{\mathcal{T}}_\varepsilon^1$ and $\bar{\mathcal{T}}_\varepsilon^2$ are members of two-parameter families of whiskered 2-tori. The whiskers $W^u(\bar{\mathcal{T}}_\varepsilon^1)$ and $W^s(\bar{\mathcal{T}}_\varepsilon^2)$ intersect transversally within $E(h, H|\bar{\mathcal{T}}_\varepsilon^1)$, and their intersection contains a two-dimensional, $N$-pulse, cylindrical set diffeomorphic to $y_\varepsilon^N \times S^1$. Along this set the manifolds $W^u(\bar{\mathcal{T}}_\varepsilon^1)$ and $W^s(\bar{\mathcal{T}}_\varepsilon^2)$ intersect at an angle of order $\mathcal{O}(1)$ as $\varepsilon \to 0$.

Suppose further that

(A6) For some integer $k$ either $\mathcal{R}_k^+(\gamma_0^1) = \gamma_0^2$ or $\mathcal{R}_k^-(\gamma_0^1) = \gamma_0^2$ where the rotation maps $\mathcal{R}_k^\pm : A \to A$ are defined by

$$\mathcal{R}_k^{\pm}(K_2,\psi_2) = \left(K_2, \psi_2 + \pi k\frac{\omega_3}{\kappa_2} \pm \omega_3 \cos^{-1}\left(\frac{-d}{a\sqrt{2\alpha}}\right)\right) \tag{6.17}$$

(see (4.14)).

Then for $\varepsilon > 0$ small,

(ii) On energy surfaces close to the surface $\bar{H} = h \in U$ an appropriately defined four-dimensional Poincaré map of system (6.12) has invariant Cantor sets of two-dimensional annuli. On these Cantor sets the Poincaré map is topologically conjugate to a full shift on a finite number of symbols.

(iii) The truncated normal form (6.12) does not possess any nontrivial analytic integral other than $\bar{H}$ and $K_1$.

*Proof.* Statement (i) is a consequence of Theorem 6.3. To prove statement (ii) we observe that under assumption (A6) the proof of (iii) of Proposition 4.2 shows that the orbits $\gamma_0^1$ and $\gamma_0^2$ both lie in the image of a single closed curve $\mathcal{C}_\varepsilon$ under the covering map $Q^h$. In Theorem 6.5 below we prove that $\mathcal{C}_\varepsilon$ does imply the existence of an invariant torus $\bar{\mathcal{T}}_\varepsilon$ in the original problem, hence we obtain the existence of a one-parameter family of *homoclinic* orbits to $\bar{\mathcal{T}}_\varepsilon$. Now the $S^1$ symmetry of the truncated normal form is of course present in the Hamiltonian system generated by (2.9) as well, hence one can also reduce the original problem to a 2-DOF (complex) subsystem. That system then contains a periodic orbit corresponding to $\mathcal{C}_\varepsilon$ that has a transverse *homoclinic* orbit. By the Smale–Birkhoff homoclinic theorem (Smale [55]) this implies the existence of an invariant Cantor set of points in the 2-DOF complex subsystem on which an appropriately defined Poincaré map is topologically conjugate to a full shift on a finite number of symbols. For the full 3-DOF complex system this implies the existence of a Cantor set of *annuli* with similar properties. Since these Cantor sets are isolated from the normally hyperbolic invariant manifold $T_2 \circ T_3(\bar{\mathcal{M}}_\varepsilon^j)$ by construction, they map diffeomorphically into the phase space of system (6.12) under the transformation $T_3^{-1} \circ T_2^{-1}$ (see also the proof of Theorem 6.5). This completes the proof of statement (ii). Finally, statement (ii) and a theorem of Moser [49] together prove (iii). □

Notice that Proposition 6.2, Theorem 6.3, and Proposition 6.4 are statements about the *blown-up* normal form (6.12). Since this system is related to the original truncated normal form (2.9) via the singular transformation $T_2 \circ T_3$, the validity of these statements for the original normal form (2.9) is an issue we need to address separately.

*Theorem 6.5.* There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the following are satisfied for the truncated complex normal form (2.9):

(i) Let $\bar{M}_\varepsilon = T^2 \circ T^3(\bar{\mathcal{M}}_\varepsilon^1 \cup \bar{\mathcal{M}}_\varepsilon^2)$ and let $\bar{I}_\varepsilon \colon \bar{M}_\varepsilon \hookrightarrow \mathbb{C}^3$ be the inclusion map of $\bar{M}_\varepsilon$. Then $(\bar{M}_\varepsilon, \bar{I}_\varepsilon^* \Omega)$ is a four-dimensional, normally hyperbolic, symplectic invariant manifold for the Hamiltonian (2.9).

(ii) For any fixed $K_1 = h \in U$ and for $j = 1, 2$, an internal orbit $\gamma_0^j$ of the reduced Hamiltonian (6.7) for the 2-DOF subsystem (4.16) yields a two-dimensional invariant manifold $\bar{T}_\varepsilon^j = T_2 \circ T_3(\bar{\mathcal{T}}_\varepsilon^j) \subset (\bar{M}_\varepsilon \cup \{z_1 = \bar{z}_1 = 0\})$ (see (ii) of Proposition 6.2). If $\gamma_0^j$ is periodic then $\bar{T}_\varepsilon^j$ is a two-dimensional invariant torus which is $\mathcal{O}(\varepsilon)$ $C^r$-close to the set $T_2 \circ T_3 \circ \bar{G}_0^j(\{h\} \times S^1 \times \gamma_0^j)$. Since $(T_2 \circ T_3)^{-1}$ is multi-valued, there exist different tori of the form $\bar{\mathcal{T}}_\varepsilon^1$ and $\bar{\mathcal{T}}_\varepsilon^2$ that map to the same torus $\bar{T}_\varepsilon^j$ under the map $T_2 \circ T_3$.

(iii) $\bar{T}_\varepsilon^j$ has three-dimensional stable and unstable manifolds, denoted $W^s(\bar{T}_\varepsilon^j)$ and $W^u(\bar{T}_\varepsilon^j)$, respectively.

(iv) Suppose that the assumptions of Theorem 6.4 hold. Then the two-dimensional tori $\bar{T}_\varepsilon^j$ are connected by a two-dimensional, $N$-pulse heteroclinic manifold which is diffeomorphic to $y_\varepsilon^N \times S^1$ (see the statement of Proposition 6.4). Solutions in this manifold are backward asymptotic to $\bar{T}_\varepsilon^1$, leave and return to a neighborhood of the plane $z_1 = \bar{z}_1 = 0$ $N$-times, and finally approach $\bar{T}_\varepsilon^2$ asymptotically. Moreover, $W^u(\bar{T}_\varepsilon^1)$ intersects $W^s(\bar{T}_\varepsilon^2)$ transversally at an angle of order $\mathcal{O}(1)$ (as $\varepsilon \to 0$) within the set

$$E_z(h, \bar{H}|\bar{T}_\varepsilon^1) = \{(z, \bar{z}) \in \mathbb{C}^6 \mid H_2(z, \bar{z}) = h,\ \bar{H}(z, \bar{z}) = \bar{H}|\bar{T}_\varepsilon^1\}.$$

Suppose further that assumption (A6) of Proposition 6.4 are satisfied. Then

(i) $\bar{T}_\varepsilon \equiv \bar{T}_\varepsilon^1 = \bar{T}_\varepsilon^2$ is a member of a two-parameter family of invariant, whiskered tori. The whiskers $W^s(\bar{T}_\varepsilon)$ and $W^u(\bar{T}_\varepsilon)$ intersect transversally with $\mathcal{O}(1)$ transversality within the set $E_z(h, \bar{H}|\bar{T}_\varepsilon)$ in a two-dimensional, $N$-pulse, homoclinic manifold, which is diffeomorphic to $y_\varepsilon^N \times S^1$.

(ii) On energy surfaces close to the surface $\bar{H} = h \in U$ an appropriately defined four-dimensional Poincaré map for the system generated by (2.9) has invariant Cantor sets of two-dimensional annuli. On these Cantor sets the Poincaré map is homeomorphic to a full shift on a finite number of symbols.

(iii) The truncated normal form (2.9) has no nontrivial analytic integral other than $\bar{H}$ and $H_2$.

*Proof.* Statement (i) follows directly from (i) of Proposition 6.2 so we turn to the proof of (ii)–(vii).

From (4.1) and (4.2) one can verify that the coordinate representation for the inverse transformation $T_2 \circ T_3$ is given by

$$\begin{aligned} z_1 &= \sqrt{2(K_1 - \omega_3 K_2 - |x|^2)}\cos\psi_1 + i\sqrt{2(K_1 - \omega_3 K_2 - |x|^2)}\sin\psi_1, \\ z_2 &= x_2\cos 2\psi_1 - x_1\sin 2\psi_1 - i(x_1\cos 2\psi_1 + x_2\sin 2\psi_1), \\ z_3 &= \sqrt{2K_2}\cos(\psi_2 + \omega_3\psi_1) + i\sqrt{2K_2}\sin(\psi_2 + \omega_3\psi_1), \end{aligned} \tag{6.18}$$

and of course, we have the complex conjugates of these expressions for $\bar{z}_1$, $\bar{z}_2$, and $\bar{z}_3$. This inverse transformation is a composition of symplectic diffeomorphisms away from $\mathcal{W}$ (defined in (6.10)). Since the stable manifolds of the manifolds $\bar{T}_\varepsilon^2$ in system (6.12) do not intersect $\mathcal{W}$ (because $\bar{T}_\varepsilon^2 \subset \mathcal{W}$), they are mapped back diffeomorphically into $\mathbb{C}^6$. Similarly, the unstable manifolds of the manifolds $\bar{T}_\varepsilon^1$ map to diffeomorphic objects in $\mathbb{C}^6$, which proves the second part of statement (i). Also, the Smale horseshoes described in (vi) are separated from $\mathcal{W}$ by construction, so statements (vi)–(vii) follow directly from Proposition 6.4. However, the inverse transformation (6.18) is degenerate on $\mathcal{W}$, so the existence of two-dimensional whiskered tori and the related statements in (ii)–(iv) and in the first part of (v) for the normal form (2.9) do not follow immediately.

We construct the tori in question in a limit procedure that uses the invariant foliations of stable and unstable manifolds of $\bar{M}_\varepsilon$ (see also Sections 5.3 and 7 for discussion). We observe that the closure of, e.g., $T_2 \circ T_3\left(W_{\rm loc}^u(\bar{T}_\varepsilon^1)\right)$ intersects $\bar{M}_\varepsilon$ in an object diffeomorphic to $\bar{T}_\varepsilon^1$. Using the results of Fenichel [18] on the foliation of unstable manifolds (see Appendix C.1), we conclude, that this object is a set of basepoints of unstable fibers that is $C^r$-diffeomorphic to a two-dimensional torus. By the invariance properties of fibers this torus is necessarily invariant and it carries the same types of motions as $\bar{T}_\varepsilon^1$. Therefore $\bar{M}_\varepsilon$ contains an invariant torus corresponding to $\bar{T}_\varepsilon^1$ which has a three-dimensional *unstable* manifold. Using the local stable manifold $W_{\rm loc}^s(\bar{T}_\varepsilon^2)$ we obtain in the same manner that $\bar{M}_\varepsilon$ also contains an invariant torus corresponding to $\bar{T}_\varepsilon^2$ which has a three-dimensional *stable* manifold. Since each torus on $\bar{M}_\varepsilon$ has representations in both $\bar{\mathcal{M}}_\varepsilon^1$ and $\bar{\mathcal{M}}_\varepsilon^2$ (see (iii) of Proposition 4.2), we obtain that each torus in the normally hyperbolic invariant manifold $\bar{M}_\varepsilon$ has both stable and unstable manifolds. Then statements (ii)–(iv) follow from Proposition 6.2 and Theorem 6.4. □

*Remark 6.3.* We note that the same statements are true for the corresponding truncated normal form of our initial system (2.2) given in the $(p,q)$ coordinates, since the transformation $T_1$ introduced in (2.4) is a diffeomorphism. The only difference is, that the stable and unstable manifolds of the whiskered tori intersect at an angle of order $\mathcal{O}(\varepsilon)$ as a result of the scaling of variables (described after (2.4)) which we used to obtain the complex Hamiltonian system defined by (2.5).

## 7. Dynamics in the full 3-DOF system

We now return to our original 3-DOF system which appears in the $(x, K, \psi)$ coordinates in the form

$$\begin{aligned}
\dot{x} &= J_2 D_x H_c(x,K) + \varepsilon^2 J_2 D_x H_1(x,K,\psi_2;\varepsilon) + \varepsilon^{\rho-1} J_2 D_x H_T(x,K,\psi;\varepsilon),\\
\dot{K}_2 &= -\varepsilon^2 D_{\psi_2} H_1(x,K,\psi_2;\varepsilon) - \varepsilon^{\rho-1} D_{\psi_2} H_T(x,K,\psi;\varepsilon),\\
\dot{\psi}_2 &= D_{K_2} H_c(x,K) + \varepsilon^2 D_{K_2} H_1(x,K,\psi_2;\varepsilon) + \varepsilon^{\rho-1} D_{K_2} H_T(x,K,\psi;\varepsilon),\\
\dot{K}_1 &= -\varepsilon^{\rho-1} D_{\psi_1} H_T(x,K,\psi;\varepsilon),\\
\dot{\psi}_1 &= D_{K_1} H_c(x,K) + \varepsilon^2 D_{K_1} H_1(x,K,\psi_2;\varepsilon) + \varepsilon^{\rho-1} D_{K_1} H_T(x,K,\psi;\varepsilon).
\end{aligned} \tag{7.1}$$

This system derives from the Hamiltonian

$$H(x,K,\psi;\varepsilon) = H_c(x,K;\varepsilon) + \varepsilon^2 H_1(x,K,\psi_2;\varepsilon) + \varepsilon^{\rho-1} H_T(x,K,\psi;\varepsilon), \tag{7.2}$$

where $H_c = K_1 + \varepsilon H_0(x,K)$ is the cubic normal form Hamiltonian (see (4.3) and (4.17)), $H_1$ contains higher-order normalized terms (see (4.17)) and $H_T$ contains the "tail" of the normal form which we have ignored so far.

First we examine the existence of regular motions in this system which lie on the continuations of the invariant 3-tori described in Proposition 3.1. These motions can be divided into two distinct families and will be seen to occupy a set of large measure in the phase space. As a result, they have substantial influence on typical trajectories. However, they do not account for the irregular behavior which is usually observed near strongly resonant equilibria. As our results for the truncated normal form indicate, strongly irregular motions can exist in a neighborhood of the hyperbolic structure described in Proposition 4.2, which forms the boundaries of the two domains that contain 3-tori. We analyze the persistence of elements in this hyperbolic structure after discussing the continuation of 3-tori. First we examine the persistence of $\bar{\mathcal{M}}_\varepsilon^1$ and $\bar{\mathcal{M}}_\varepsilon^2$ along with their stable and unstable manifolds, and their foliations (discussed after Theorem 5.1), under the $\mathcal{O}(\varepsilon^{\rho-1})$ terms in (7.1). Then we study the persistence of invariant 3-spheres on $\bar{\mathcal{M}}_\varepsilon^1$ and $\bar{\mathcal{M}}_\varepsilon^2$, and subsequently examine what remains of the $N$-pulse heteroclinic or homoclinic connections between 2-tori. Finally, we relate our results back to the original complex Hamiltonian system (2.5) defined on the phase space $(\mathbb{C}^3, \Omega)$.

### *7.1. Persistence of 3-tori: regular motions*

We rewrite the full Hamiltonian (7.2) in the form

$$H(x,K,\psi;\varepsilon) = K_1 + \varepsilon H_0(x,K) + \mathcal{O}(\varepsilon^2), \tag{7.3}$$

with $H_0$ defined in (4.17). As we described in Proposition 4.1, without the $\mathcal{O}(\varepsilon^2)$ terms this system has two 3-parameter family of 3-tori, that correspond to the families of periodic orbits shown in Fig. 2. For fixed values of $K_1$ and $K_2$, we can express the variable $x_2$ on these periodic orbits as piecewise smooth function of $x_1$ in the form

$$x_2 = X_2(x_1; K_1, K_2, h_0),$$

where $h_0$ denotes the value of $H_0$ on a given periodic orbit. (For simplicity, we do not distinguish in our notation between the two families of periodic orbits). Now we can introduce action-angle variables for the periodic orbits using the usual construction for 1-DOF Hamiltonians (see, e.g Arnold [4]). The action-angle variables $(J_3, \Theta_3)$ are related to $x_2$ and $x_1$ via a transformation

$$x_1 = a_1(J_3, \Theta_3; K_1, K_2), \quad x_2 = a_2(J_3, \Theta_3; K_1, K_2),$$

which transforms the two-form $dx_1 \wedge dx_2$ into $d\Theta_3 \wedge dJ_3$ for fixed values of $K_1$ and $K_2$, and puts the Hamiltonian $H_0(x, K)$ to the form

$$\bar{H}_0(K_1, K_2, J_3) = H_0(a(J_3, \Theta_3; K_1, K_2), K_1, K_2). \tag{7.4}$$

To make this transformation canonical for the full system (i.e., for the case when $K_1$ and $K_2$ are allowed to vary), we also transform the variables $(K_1, K_2, \psi_1, \psi_2)$ to $(J_1, J_2, \Theta_1, \Theta_2)$ using the generating function

$$S(x_1, \psi_2, \psi_2, J_1, J_2, J_3) = \int_0^{x_1} X_2(y; J_1, J_2, \bar{H}_0(J_1, J_2, J_3)dy + \psi_1 J_1 + \psi_2 J_2,$$

which generates the transformation

$$\begin{aligned} K_1 &= J_1, & K_2 &= J_2, \\ \psi_1 &= \Theta_1 - \int_0^{x_1} \left(D_{J_1} X_2 + D_{h_0} X_2 D_{J_1} \bar{H}_0\right) dy, & \psi_2 &= \Theta_2 - \int_0^{x_1} \left(D_{J_2} X_2 + D_{h_0} X_2 D_{J_2} \bar{H}_0\right) dy. \end{aligned} \tag{7.5}$$

In this new set of variables we obtain the transformed full Hamiltonian

$$H(J) = H_{00}(J_1) + \varepsilon H_{01}(J) + \mathcal{O}(\varepsilon^2), \tag{7.6}$$

with

$$H_{00}(J_1) = J_1, \quad H_{01}(J) = \bar{H}_0(J) = H_0(a(J_3, \Theta_3; J_1, J_2), J_1, J_2). \tag{7.7}$$

This Hamiltonian is formally in the proper coordinates for the application of the KAM theory to the invariant 3-tori present without the $\mathcal{O}(\varepsilon^2)$ terms. However, $H$ becomes degenerate in the limit of $\varepsilon = 0$ because $H_{00}$ does not depend on the action variables $J_2$ and $J_3$, and as a result, all unperturbed tori are resonant. This is exactly the situation that is called *proper degeneracy* in Arnold et al. [5]. It is shown in Arnold [2] that if the term $\varepsilon H_{01}$ "removes" the degeneracy of the $\varepsilon = 0$ limit, i.e., it creates a family of 3-tori which satisfy certain regularity assumptions, then most of these newly created 3-tori will persist under the effect of $\mathcal{O}(\varepsilon^2)$ terms. Sufficient regularity conditions are isoenergetic nondegeneracy for $H_{00}$ and nondegeneracy for $H_{01}$ with respect to the $(J_2, J_3)$ variables. In our case this leads to the conditions

$$D_{J_1} H_{00} \neq 0, \quad \det(D^2_{(J_2,J_3)} H_{01}) \neq 0, \tag{7.8}$$

where $D^2_{(J_2,J_3)} H_{01}$ denotes the Hessian of $H_{01}$ with respect to the variables $J_2$ and $J_3$. The first condition is obviously satisfied, but verifying the second one leads to formidable calculations. Fortunately, some related calculations are performed in Kummer [36] for the existence of quasiperiodic motions in the truncated 1:2:3 normal form for parameter values near a symmetry. The Hamiltonian that arises there can be shown to be equivalent to our Hamiltonian $H_{01}$ for zero detuning ($d = 0$). Using this analogy we can make use of the calculations of Kummer and prove the following result.

*Lemma 7.1.* There exists $d_0 > 0$ such that for detunings with $|d| < d_0$ both nondegeneracy conditions in (7.8) are satisfied on all but a finite number of isolated, measure zero subsets of a finite number of energy surfaces $H_{01} = \text{const}$.

*Proof.* See Appendix B. □

From this lemma, using the results in Arnold et al. [5], we obtain the following theorem on the existence of quasiperiodic motions in the full 3-DOF system (7.1).

*Theorem 7.2.* There exist $\varepsilon_0, d_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and $|d| < d_0$ most invariant 3-tori of the cubic truncated normal form (4.3) survive on all but a finite number of energy surfaces in the full system (7.1) and hence in the original complex system (2.5). On each energy surface the surviving tori form two families, and carry quasiperiodic motions with one frequency of order $\mathcal{O}(1)$ and with two frequencies of order $\mathcal{O}(\varepsilon)$. The measure of the tori that do not survive tends to zero as $\varepsilon \to 0$.

We remark that Arnold's theorem is originally proved for analytic Hamiltonians, but as for other versions of the KAM theorem, smoothing techniques introduced by Moser can be employed to obtain the same result for $n$-DOF Hamiltonian which that are of class $C^{2n+1}$ (see Pöschel [52] or Arnold et al. [5]).

### 7.2. Persistence of $\bar{\mathcal{M}}_\varepsilon^j$, $W^{u,s}(\bar{\mathcal{M}}_\varepsilon^j)$, the invariant foliations, and the multi-pulse solution sets

Neither the persistence of $\bar{\mathcal{M}}_\varepsilon^j$ nor the persistence of $W^{u,s}(\bar{\mathcal{M}}_\varepsilon^j)$ with their invariant foliations are trivial issues because $\bar{\mathcal{M}}_\varepsilon^j$ is only *weakly* normally hyperbolic in system (7.1): its stable and unstable manifolds, as well as $\bar{\mathcal{M}}_\varepsilon^j$ itself disappear for $\varepsilon = 0$. So the general persistence theory for normally hyperbolic invariant manifolds (see, e.g., Wiggins [63]) does not apply immediately and some extra work is needed to show that the weakly hyperbolic $\bar{\mathcal{M}}_\varepsilon^j$ and its stable and unstable manifolds are sufficiently robust to survive the effect of the tail of the normal form. Once this is established, the survival of multi-pulse solutions follows easily since they lie in the transverse intersection of surviving manifolds. Their exact asymptotics, however, is a more difficult question which will be addressed in Sections 7.3 and 7.4.

To formulate the persistence result we need, let us consider $K_1$ values lying in some closed interval $V_1 \subset U$. As earlier, we choose some small but arbitrary $\lambda > 0$ and consider

$$K_2 \in V_2 \equiv \left[\lambda, \frac{1}{\omega_3}(K_1 - d^2/a^2) - \lambda\right].$$

We then have the following result.

*Theorem 7.3.* There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and for $j = 1, 2$,

(i) System (7.1) has an invariant, four-dimensional $C^r$ manifold $\mathcal{M}_\varepsilon^j$ given by an embedding $G_\varepsilon^j : V_1 \times S^1 \times V^2 \times S^1 \to \mathcal{P}$, which is $\mathcal{O}(\varepsilon^2)$ $C^r$-close to the map $\bar{G}_\varepsilon^j$ on its domain.

(ii) If $I_\varepsilon^j : \mathcal{M}_\varepsilon^j \hookrightarrow \mathcal{P}$ is the inclusion map of $\mathcal{M}_\varepsilon^j$ then $(\mathcal{M}_\varepsilon^j, I_\varepsilon^{j*}\omega)$ is a symplectic manifold on which the dynamics is generated by the Hamiltonian $\mathcal{H}_R^j = H|\mathcal{M}_\varepsilon^j$ with $H$ defined in (7.2).

(iii) $\mathcal{M}_\varepsilon^j$ has five dimensional stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon^j)$ and $W^u(\mathcal{M}_\varepsilon^j)$, which are $\mathcal{O}(\varepsilon^2)$ $C^r$-close to $W^s(\bar{\mathcal{M}}_\varepsilon^j)$ and $W^u(\bar{\mathcal{M}}_\varepsilon^j)$, respectively, in a neighborhood of the manifold $\mathcal{M}_\varepsilon^j$.

(iv) If the conditions of Theorem 6.3 are satisfied then $W^u(\mathcal{M}_\varepsilon^1)$ and $W^s(\mathcal{M}_\varepsilon^2)$ intersect transversally along a four-dimensional invariant set $Y_\varepsilon^N$. This set contains $N$-pulse solutions with jump sequence $\chi(\gamma_0^1)$ (see Theorem 6.3).

*Proof.* For the proof of (i) and (iii) see Appendix C. Then statement (ii) follows from the fact that $\mathcal{M}_\varepsilon^j$ is $\mathcal{O}(\varepsilon^2)$ $C^r$-close to $\bar{\mathcal{M}}_\varepsilon^j$ hence $I_\varepsilon^{j*}\omega$ remains nondegenerate, and it is trivially closed. Statement (iv) follows from (iv) of Theorem 6.3 and from the $\mathcal{O}(\varepsilon)$ $C^r$-closeness of $W_{\text{loc}}^{u,s}(\mathcal{M}_\varepsilon^j)$ to $W_{\text{loc}}^{u,s}(\bar{\mathcal{M}}_\varepsilon^j)$. □

Both the stable and unstable manifolds of $\bar{\mathcal{M}}_\varepsilon^j$ are foliated by invariant families of submanifolds, or *fibers* with properties similar to those discussed in Section 5.3. The following theorem describes the properties of the stable foliation in the context of system (7.1), but similar results hold for the unstable foliation.

*Theorem 7.4.* There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there exists a four-parameter family of curves, called the *stable fiber* that can be represented as

$$\begin{aligned} x &= f_x(x_1; K_b, \psi_b, \varepsilon), \\ K &= f_K(x_1; K_b, \psi_b, \varepsilon), \\ \psi &= f_\psi(x_1; K_b, \psi_b, \varepsilon). \end{aligned} \tag{7.9}$$

The four parameters for this family are given by the coordinates $(K_b, \psi_b)$ of points $b \in \mathcal{M}_\varepsilon^2$, and $f_x$, $f_K$, and $f_\psi$ are $C^r$ functions of these coordinates and of $x_1$ and $\varepsilon$. The point $b = G_\varepsilon^2(K_b, \psi_b)$ is referred to as the *basepoint* of the stable fiber represented by the functions $\Big(f_x(\cdot; K_b, \psi_b, \varepsilon), f_K(\cdot; K_b, \psi_b, \varepsilon), f_\psi(\cdot; K_b, \psi_b, \varepsilon)\Big)$. Furthermore,

(i) Fibers are mapped into fibers by the time $t$ flow map of system (7.1).

(ii) $W_{\text{loc}}^s(\mathcal{M}_\varepsilon^2) = \bigcup_{b \in \mathcal{M}_\varepsilon^2} \Big(f_x(x_1; K_b, \psi_b, \varepsilon), f_K(x_1; K_b, \psi_b, \varepsilon), f_\psi(x_1; K_b, \psi_b, \varepsilon)\Big)$.

(iii) Let $(K_b(t), \psi_b(t))$ be a trajectory in $\mathcal{M}_\varepsilon^2$ satisfying $(K_b(0), \psi_b(0)) = (K_b, \psi_b)$ and let $(x(t), K(t), \psi(t))$ be a trajectory in $W_{\text{loc}}^s(\mathcal{M}_\varepsilon^2)$ satisfying

$$\begin{aligned} x(0) &= f_x(x_1(0); K_b, \psi_b, \varepsilon), \\ K(0) &= f_K(x_1(0); K_b, \psi_b, \varepsilon), \\ \psi(0) &= \psi(x_1(0); K_b, \psi_b, \varepsilon), \end{aligned} \tag{7.10}$$

i.e., the trajectory starts on the fiber with basepoint $b$. Then

$$|(x(t), K(t), \psi(t)) - G_\varepsilon^j(K_b(t), \psi_b(t))| < Ce^{-\lambda t}$$

for all $t > 0$ and for some $C$, $\lambda > 0$ as long as $(K_b(t), \psi_b(t)) \in \mathcal{M}_\varepsilon^2$. In other words, trajectories starting on a stable fiber asymptotically approach the trajectory in $\mathcal{M}_\varepsilon^2$ that starts on the basepoint of the same fiber.

(iv) The $N$-pulse solution-set $Y_\varepsilon^N$ described in Theorem 7.3 intersects a 3-parameter family of unstable fibers in $W_{\text{loc}}^u(\mathcal{M}_\varepsilon^1)$ whose basepoints form a three-dimensional hypersurface $\mathcal{Z}_-^N$ in the manifold $\mathcal{M}_\varepsilon^1$. $\mathcal{Z}_-^N$ is $\mathcal{O}(\sqrt{\varepsilon})$ $C^1$-close to the set $G_\varepsilon^1(U \times S^1 \times Z_-^N)$, where $Z_-^N$ is the zero set of the energy-difference function $\Delta^N \mathcal{H}$ defined in (6.11).

(v) Based on assumption (A5) of Theorem 5.1, let us define the rotation map $\mathcal{R}_N: A \to A$ of the annulus $A$ by

$$\mathcal{R}_N(K_2, \psi_2) = \left(K_2, \psi_2 + N\Delta\psi_2^0(K_2) + \sum_{l=1}^{N-1} \Delta\psi^{x_l(\gamma_0^1)}(K_2)\right).$$

Then the $N$-pulse solution-set $Y^N$ also intersects a 3-parameter family of stable fibers in $W_{\text{loc}}^s(\mathcal{M}_\varepsilon^2)$ whose basepoints form a three-dimensional hypersurface $\mathcal{Z}_+^N$ in the manifold $\mathcal{M}_\varepsilon^2$. $\mathcal{Z}_+^N$ is $\mathcal{O}(\sqrt{\varepsilon})$ $C^1$-close to the set $G_\varepsilon^2(U \times S^1 \times \mathcal{R}_N(Z_-^N))$.

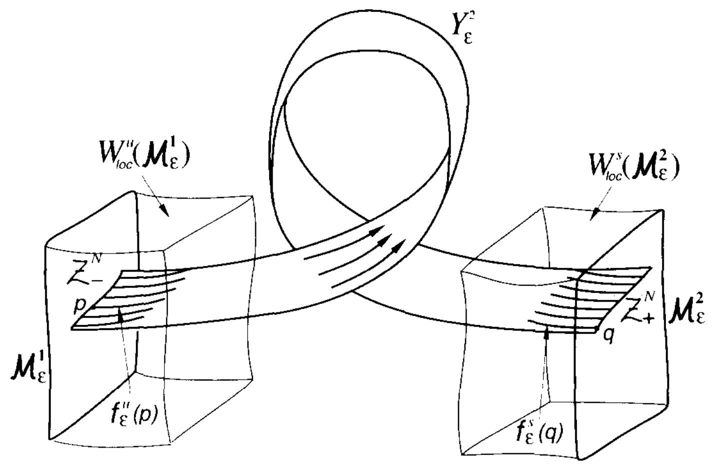


Fig. 10. Double-pulse solution set $Y_\varepsilon^2$ with the corresponding take-off and landing surfaces $\mathcal{Z}^N_-$ and $\mathcal{Z}^N_+$.

*Proof.* For the proof of (i)–(iii) see Appendix C. Statements (iv) and (v) follow from Theorem 5.1, Remark 6.2, and the fact that the stable and unstable fibers are $C^r$ functions of $\varepsilon$. □

A schematic visualization of statements (iv) and (v) can be seen in Fig. 10 for the case of a double-pulse solution set $Y_\varepsilon^2$.

### 7.3. Persistence of invariant 3-spheres on $\mathcal{M}^j_\varepsilon$

In Proposition 6.4 we described how periodic orbits of the reduced Hamiltonian $\mathcal{H}^j$ (defined in (6.7)) give rise to two-dimensional invariant tori on the four-dimensional invariant manifold $\bar{\mathcal{M}}^j_\varepsilon$ in the quartic truncated normal form. If these tori lie in a neighborhood of an equilibrium on $\bar{\mathcal{M}}^j_\varepsilon$ at which the restricted Hamiltonian $\mathcal{H}^j_\varepsilon$ is positive definite, then the energy surfaces containing the tori are diffeomorphic to the three-dimensional sphere $S^3$. While the survival of the whiskered tori on the manifold $\mathcal{M}^j_\varepsilon$ is a subtle question (see Section 7.4), the survival of the 3-spheres containing them is relatively easy to show. In this subsection we address this issue and also relate the persisting multi-pulse solution sets to the surviving spheres.

*Theorem 7.5.* Suppose that assumption (A3′) of Section 6.1 is satisfied and for $j = 1, 2$, $\gamma_0^j \subset A$ is a periodic orbit of the reduced Hamiltonian (6.7). Suppose further that $\gamma_0^j \subset A$ is a member of a family of periodic solutions encircling an equilibrium point at which the reduced Hamiltonian $\mathcal{H}^j(K_2, \psi_2; K_1)$ (see (6.7)) is positive definite. Then for $\varepsilon > 0$ sufficiently small,

(i) There exists a one-parameter family of invariant 3-spheres on the manifolds $\mathcal{M}^1_\varepsilon$ and $\mathcal{M}^2_\varepsilon$ which are are connected by $N$-pulse heteroclinic orbits contained in the set $Y_\varepsilon^N$, where $N \equiv N(\gamma_0^1)$ is the pulse number of $\gamma_0^1$ (cf. Theorems 7.3,7.4). These $N$-pulse solutions make their $l$th excursions near the set $\mathcal{W}^\pm_\varepsilon$ if $\chi_l(\gamma_0^1) = \pm 1$, where $\chi(\gamma_0^1)$ is the jump sequence associated with $\gamma_0^1$ (cf. Section 5.4).

(ii) Any two 3-spheres $\mathcal{S}^1_\varepsilon$ and $\mathcal{S}^2_\varepsilon$, that are connected by $N$-pulse orbits, have four-dimensional stable and unstable manifold. The intersection of these invariant manifolds is transverse within their energy surface. Finally, the order of splitting of $W^u(\mathcal{S}^1_\varepsilon)$ and $W^s(\mathcal{S}^2_\varepsilon)$ along an $N$-pulse heteroclinic orbit is $\mathcal{O}(1)$ as $\varepsilon \to 0$.

(iii) If assumption (A6) of Proposition 6.4 holds then an appropriate Poincaré map for system (7.1) possesses invariant Cantor sets of two-dimensional annuli. On these Cantor sets the Poincaré map is topologically conjugate to a full shift on a finite number of symbols. As a consequence, system (7.1) has no more than two nontrivial, independent, analytic integrals.

(iv) The persisting 3-spheres, the heteroclinic orbits connecting them, and the Cantor sets described in (i)–(iii) also exist in the original Hamiltonian (1.3). There the connections are all homoclinic if (A6) is satisfied. However, the order of splitting of stable and unstable manifolds of the 3-spheres is $\mathcal{O}(\varepsilon)$.

*Proof.* Let us consider the restricted normal form Hamiltonian $\bar{H}|\bar{\mathcal{M}}_\varepsilon = K_1 + \varepsilon^2\mathcal{H}^j(K_2,\psi_2,K_1) + \mathcal{O}(\varepsilon^3)$ (here we use the definition of the reduced Hamiltonian from (6.7), but consider $K_1$ as a variable). By the assumptions of the theorem, $\bar{H}|\bar{\mathcal{M}}_\varepsilon$ is positive definite on a domain containing an equilibrium close to $K_1 = 0$, $K_2 = K_{20}$, $\psi_2 = \psi_{20}$. (Note that all such equilibria are mapped back into the trivial zero equilibrium of the original Hamiltonian $H(p,q)$ since $K_1 = 0$ implies $p = q = 0$.) As a result, there exists a family of level surfaces of $\bar{H}|\bar{\mathcal{M}}_\varepsilon$ that are three-dimensional spheres embedded in the four-dimensional manifold $\bar{\mathcal{M}}^j_\varepsilon$. These spheres are the intersections of the five-dimensional energy spheres $\bar{H} = h = \text{const.}$ with the invariant manifold $\bar{\mathcal{M}}^j_\varepsilon$.

First we want to argue that these invariant spheres survive on the perturbed manifold $\mathcal{M}^j_\varepsilon$. It is clear that they survive if we include higher order normal form terms in the normal form Hamiltonian $\bar{H}$ (see (4.15)), as those still generate an integrable dynamics on the corresponding perturbation of $\bar{\mathcal{M}}^j_\varepsilon$. To establish their persistence for the full Hamiltonian $H$, it suffices to show that the 3-spheres lie in the $\mathcal{O}(1)$ transverse intersection of their energy surfaces $\{\bar{H} = \text{const.}\}$ with the manifold $\bar{\mathcal{M}}^j_\varepsilon$. Then the nonsingular energy surfaces and the manifold $\bar{\mathcal{M}}^j_\varepsilon$ perturb smoothly under the effect of the $\mathcal{O}(\varepsilon^{\rho-1})$ "tail" of the normal form, hence their (compact) intersection set perturbs smoothly to a nearby, diffeomorphic intersection set. To prove transversality, we will identify a subspace of the tangent space of $\bar{\mathcal{M}}^j_\varepsilon$ at the points of intersection which is not contained in the tangent space of the energy surface $\{\bar{H} = \text{const.}\}$ at those points.

The invariant manifold $\bar{\mathcal{M}}^j_\varepsilon$ for the truncated normal form Hamiltonian $\bar{H}$ can be written in the form

$$x_1 = \sqrt{h - d^2/a^2 - \omega_3 K_2} + \varepsilon\bar{x}_1(K,\psi) + \mathcal{O}(\varepsilon^2), \quad x_2 = -d/a + \varepsilon\bar{x}_2(K,\psi) + \mathcal{O}(\varepsilon^2). \tag{7.11}$$

In terms of the coordinates $(x,K,\psi)$, one tangent vector at any point of this manifold is given by

$$v_{\psi_2} = (0,0,0,0,0,1).$$

At the same time the gradient of the Hamiltonian $\bar{H}$ at any point of $\bar{\mathcal{M}}^j_\varepsilon$ is given by

$$D\bar{H}|\bar{\mathcal{M}}^j_\varepsilon = \varepsilon^2 DH_4|\bar{\mathcal{M}}^j_0 + \mathcal{O}(\varepsilon^3),$$

hence we obtain that at any point $p \in \bar{\mathcal{M}}^j_\varepsilon$,

$$\frac{\langle D\bar{H}, v_{\psi_2}\rangle}{|D\bar{H}||v_{\psi_2}|} = D_{\psi_2}H_4|\bar{\mathcal{M}}^j_0 + \mathcal{O}(\varepsilon). \tag{7.12}$$

Since $\gamma^j_0$ is assumed to be a member of a family of periodic orbits encircling a fixed point, it follows that on any solution on any of the spheres $D_{\psi_2}H_4 = 0$ can only hold at isolated points. Hence on any solution contained in a 3-sphere we can pick a point $p$ such that the expression in (7.12) is nonzero at $p$. Consequently, $v_{\psi_2}$ is not in the tangent space of the energy surface at $p$, i.e., $\{\bar{H} = \text{const.}\}$ intersects $\bar{\mathcal{M}}^j_\varepsilon$ transversally at $p$. Moreover, the transversality is of order $\mathcal{O}(1)$ as $\varepsilon \to 0$, as we see from (7.12). Now the flow map of the system is a diffeomorphism for any fixed finite time, hence the $\mathcal{O}(1)$ transversality between the two invariant surfaces

$\{\bar{H} = \text{const.}\}$ and $\bar{\mathcal{M}}_\varepsilon^j$ is preserved for finite times. In particular, $\mathcal{O}(1)$ transversality holds at any point of the invariant 3-spheres on $\bar{\mathcal{M}}_\varepsilon^j$. This completes the proof of the persistence of spheres on $\mathcal{M}_\varepsilon^j$.

To finish the proof of statement (i), we note that, as we pointed out in Remark 6.1, $\gamma_0^1$ is a member of a family of periodic solutions of the reduced Hamiltonian $\mathcal{H}^j(K_2, \psi_2; K_{10})$ that intersect the zero set $Z_-^N$ with $\mathcal{O}(1)$ transversality. This implies directly that a one-parameter family of the three spheres $\bar{\mathcal{S}}_\varepsilon^1$ in the truncated normal form intersects the set $\bar{G}_\varepsilon^1(U \times S^1 \times Z_-^N)$ with $\mathcal{O}(1)$ transversality within the manifold $\bar{\mathcal{M}}_\varepsilon^1$. Since the surviving spheres on $\mathcal{M}_\varepsilon^1$ deform smoothly by an order $\mathcal{O}(\varepsilon)$ amount and the "take-off" set $\mathcal{Z}_-^N$ is $\mathcal{O}(\sqrt{\varepsilon})$ $C^1$-close to $G_\varepsilon^j(U \times S^1 \times Z_-^N)$ (cf. Theorem 7.4), we obtain that $\mathcal{Z}_-^N$ is intersected by a one-parameter family of 3-spheres with transversality of $\mathcal{O}(1)$. The same argument applied to the "landing surface" $\mathcal{Z}_+^N$ of $N$-pulse solutions proves statement (i).

The local stable and unstable manifolds of any surviving 3-sphere can be constructed by taking the union of stable and unstable fibers whose basepoints are contained in the 3-sphere. Then the global stable and unstable manifolds of the sphere are obtained in the usual way by applying the flow to the local stable and unstable manifolds. The $\mathcal{O}(1)$ transversality of the intersection of these surfaces along the surviving $N$-pulse orbits again follows from the $\mathcal{O}(1)$ transversality already present in the truncated normal form, which can be seen immediately based on the geometry of the spheres and the basepoint set $\bar{\mathcal{Z}}_-^N$. This completes the proof of (ii).

Statement (iii) follows from Proposition 6.4 and from the structural stability of horseshoes under the $\mathcal{O}(\varepsilon^{\rho-1})$ perturbation given by the tail of the normal form. Finally, the proof of statement (iv) is the same as the proof of Theorem 6.5 (cf. Remark 6.3). □

### 7.4. *Persistence of whiskered tori*

In Proposition 6.4 we described how periodic orbits of the reduced Hamiltonian $\mathcal{H}^j$ (defined in (6.7)) give rise to two-dimensional invariant tori on the four-dimensional invariant manifold $\bar{\mathcal{M}}_\varepsilon^j$ in the quartic truncated normal form. Now we would like to see whether these tori continue to exist on the persisting manifold $\mathcal{M}_\varepsilon^j$ in the full system (7.1). The significance of possible surviving tori is great since they can be used to identify the exact asymptotic behavior of the multi-pulse solutions which are homoclinic to the 3-spheres described in the previous theorem. At this point we can only conclude that the multi-pulse solutions stay close to the 2-tori of the truncated normal form on time scales of order $\mathcal{O}(1/\varepsilon^{\rho-1})$.

Unfortunately, there seems to be no version of the KAM theory that can be used to show that the majority of the 2-tori survive the effect of the tail of the normal form. In particular, the results of Moser, Graff, or Zehnder (see [47,22], and [64]) require constant Lyapunov exponents along the unperturbed whiskered tori. Another reason why these results do not apply directly is that fact that for $\varepsilon = 0$ the manifold $\mathcal{M}_\varepsilon^j$ simply disappears. A direct application of the KAM theory to the dynamics on $\mathcal{M}_\varepsilon^j$ is not possible either since the restricted symplectic form is noncanonical. As a result, the restricted Hamiltonian vector field cannot be written in a near-integrable form in any obvious way.

Nonetheless, the survival of the majority of the above tori can most likely be obtained by going through the main steps of the KAM construction in the context of our particular system. In fact, we expect much more of these tori to survive than in usual applications of the KAM theorem. The reason is that the order of possible resonances between the frequencies of unperturbed tori is $\mathcal{O}(1/\varepsilon^2)$. For our $C^r$ smooth Hamiltonian $H_R^j$ this means that the first resonant terms in its Fourier series that cannot be removed by successive changes of coordinates, have amplitudes of order $\mathcal{O}(\varepsilon^{2(r+1)})$. This follows from the usual estimates on the decay of Fourier amplitudes for $C^r$ functions (see, e.g., Lochak and Meunier [41]). Then applying an argument similar to Theorem 1 in Neishtadt [50], but replacing the $\mathcal{O}(e^{-c/\varepsilon^2})$ Fourier amplitude estimate for the analytic case with $\mathcal{O}(\varepsilon^{2(r+1)})$ for $C^r$ Hamiltonians, one should obtain that the relative measure of the destroyed tori is of order $\mathcal{O}(\varepsilon^{2(r+1)})$ instead of the usual $\mathcal{O}(\sqrt{\varepsilon})$.

## 8. Conclusions: chaos and diffusion near the equilibrium

In this paper we studied the structure of the phase space near resonant equilibria in a class of 3-DOF Hamiltonian systems. We proved that under general conditions on the quartic terms of the Taylor expansion of the Hamiltonian and for small detunings from the exact resonance, there exist two large families of quasiperiodic solutions on most energy surfaces. However, the two domains of quasiperiodic motions are separated by hyperbolic structures which can create sophisticated families of motions. In particular, we gave criteria for the existence of non-trivial multi-pulse heteroclinic and homoclinic connections between three-dimensional invariant spheres with two different time scales. These connections pass repeatedly near the plane $p_1 = q_1 = 0$ (i.e., $z_1 = \bar{z}_1 = 0$) before they asymptote to tori in forward and backward time.

Some of the results in this paper on the truncated normal form are extensions and improvements of those appearing in Haller and Wiggins [23]. In an upcoming paper (Haller and Wiggins [25]) we apply the results of our present study to low energy oscillations in a 3-DOF model of the classical water molecule. In that problem motions on the surviving 3-tori represent regular, quasiperiodic exchange of energy between neighborhoods of nonlinear normal modes, whereas the multi-pulse solutions connecting whiskered tori give a natural mechanism for *irregular, irreversible energy transfer* between these neighborhoods.

We conclude this paper by some comments on near-equilibrium diffusion in the class of resonant Hamiltonians we considered. Since $K_1$ is an integral at any order of truncation for the the normal form, the whiskered tori we constructed on the 4-spheres $E(K_1, h_2)$ (see (6.16)) within the level surfaces $\bar{H} = \text{const.}$ are isolated from each other, and there are no transition chains created by the transversal intersection of whiskers that would yield diffusion through different 4-spheres. In the full system (7.1) this isolation of the (possibly surviving) whiskered tori does not exist any more and the whiskers of tori originally lying on different level surfaces of $H_2 \equiv K_1$ do intersect generically (see Section 7.4). However, these secondary intersections are results of further splittings caused by the "tail" and are smaller than any power of $\varepsilon$ for $C^\infty$ Hamiltonians. This is not surprising since the speed of the drift through the tangles of these secondary intersections has to obey Nekhorosev's general estimates (Nekhorosev [51], Arnold et al. [5]) as established in Lochak [42] for perturbations of resonant linear oscillators (see also Benettin and Gallavotti [6] for the case of nonresonant oscillators). Although the tori we constructed are not amenable to the usual Melnikov-type methods that are used in the study of Arnold diffusion, the elements of the construction of a transition chain of tori would be the same in our case. That would, however, involve the study of exponentially small splitting directions. Related results exist for rapidly forced 1-DOF systems (see Holmes et al. [27], Delshams and Seara [14], Ellison et al. [16] and the references therein) which can be applied to normal forms of 2-DOF Hamiltonians (cf. Holmes et al. [27]). Important results concerning the splitting of separatrices in two classes of model problems appeared recently in Chierchia and Gallavotti [7] where the splitting distances turned out to be non-exponentially small in all directions. We also mention the paper of Churchill and Rod [10] which proves the existence of homoclinic and heteroclinic orbits in rapidly forced symmetric systems without control over the transversality of these orbits.

We emphasize that in our problem the possible exponentially slow diffusion created by intersecting whiskers *does not imply instability* for the equilibrium. In particular, throughout our study we assumed for convenience that the Hamiltonian in positive definite at the origin, hence *the resonant equilibrium is Lyapunov-stable.* So, as opposed to the usual instability associated with diffusion, in our problem diffusion means a "mixing" of solutions on a given five-dimensional energy sphere near the resonant stable equilibrium. This mixing is rather intense in directions tangent to the 4-spheres $E(K_1, h_2)$, because in these directions the splitting of whiskers is of order $\mathcal{O}(\varepsilon)$ for the original Hamiltonian (1.3). Numerical results visualizing the structures corresponding to this mixing will appear in Haller and Wiggins [25]. On the other hand, in directions vertical to the 4-spheres the possible diffusion is exponentially slow and it is questionable whether it is numerically observable at all. For this reason we believe that the energetical implications of this "vertical" diffusion are negligible compared to the observable chaos associated with "horizontal mixing" in the vicinities of the 4-spheres $E(K_1, h_2)$.

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## Appendix A. Calculation of the phase shifts

In this appendix we outline the calculations of the phase shifts defined in (5.6) for the 2-DOF subsystem (4.16). Since $\dot{\psi}_2 \equiv 0$ on the manifold $\mathcal{W}_0^0$, we immediately obtain $\Delta\psi_2^0(K_2) \equiv 0$. The remaining two phase shifts are given by the integral

$$\Delta\psi_2^{\pm 1}(K_2) = -\omega_3 \int_{-\infty}^{\infty} \left(d + a x_2^{h,\pm 1}(t; K_1, K_2)\right) dt, \tag{A.1}$$

where $x_2^{h,\pm 1}(t; K_1, K_2)$ is taken along the appropriate heteroclinic connection (cf. (6.3)).

From (4.16) (with $\varepsilon = 0$) and (4.3), we have

$$dt = \frac{dx_2}{\dot{x}_2} = \frac{dx_2}{2x_1\,(d + a x_2)},$$

from which we see that

$$-\omega_3 \int (d + a x_2)\, dt = -\frac{\omega_3}{2} \int \frac{dx_2}{x_1}. \tag{A.2}$$

Therefore, expressing $x_1$ as a function of $x_2$ on the heteroclinic connections, and substituting (A.2) into (A.1) yields

$$\Delta\psi_2^{\pm 1}(K_2) = -\omega_3 \int_{-d/a}^{\pm\sqrt{K_1 - \omega_3 K_2}} \frac{dx_2}{\sqrt{K_1 - \omega_3 K_2 - x_2^2}},$$

Computing these integrals, and using the identity $\sin^{-1} x + \cos^{-1} x = \pi/2$ gives

$$\Delta\psi_2^{+}(K_2) = -\omega_3 \pi + \omega_3 \cos^{-1} \frac{d}{a\sqrt{K_1 - \omega_3 K_2}},$$

$$\Delta\psi_2^{-}(K_2) = \omega_3 \cos^{-1} \frac{d}{a\sqrt{K_1 - \omega_3 K_2}}.$$

## Appendix B. Proof of Lemma 7.1

We start by recalling that the for $d = 0$ (zero detuning) the cubic normal form Hamiltonian $H_{01}$ is given by

$$H_{01}(x,K) = a(K_1 - \omega_3 K_2 - x_1^2 - x_2^2)x_2.$$

Regarding $K_1$ as a fixed parameter, this 2-DOF Hamiltonian is exactly of the form studied in Kummer [36], p. 89. (In Kummer's notation $H_{01} = K^{(0)}$, $K_1 = I$, $K_2 = J_3$, $x_1 = y$, $x_2 = x$, $\chi = \psi_2$, and his symplectic form has a sign opposite to ours.) Using Kummer's approach from that paper, we can factor the Hamiltonian by writing

$$H_{01} = C^{3/2}\lambda, \quad C = (K_1 - \omega_3 K_2)^{3/2}, \quad \lambda = H_{01}/J^{3/2}.$$

As we discussed in Section 7.1, the Hamiltonian $H_{01}$ admits action-angle variables $(J,\Theta)$ on two open domains of its phase space corresponding to the elliptic regions shown in Fig. 2. We can therefore write

$$H_{01}(x,K) = \bar{H}_0(J_1,J_2,J_3) = C^{3/2}\lambda. \tag{B.1}$$

Kummer realized that the functional relationship between the quantities $C$ and $\lambda$ and the action-angle variables is somewhat easier to handle than the actual action-angle transformation itself from $(x,\psi_2,K_2)$ to $(J_2 \equiv K_2, \Theta_2, J_3, \Theta_3)$.(Note that in his notation $C = J$ and $\lambda = \rho$.) This enables us to write the second nondegeneracy condition in (7.8) in terms of $C$ and $\lambda$, and verify them for specific parameter values. We then argue, following Kummer, that the determinant of the Hessian in the nondegeneracy condition involves analytic functions of $\lambda$, hence if it is nonzero at some specific value, then it may be zero only for a finite number of $\lambda$ values in any finite $\lambda$-interval. Our calculation will differ from Kummer's in that he verifies isoenergetic nondegeneracy for $H_{01}$ which turns out to be less computational, than verifying nondegeneracy in the sense of (7.8). Finally, we only have to note that the nondegeneracy established this way clearly remains true for sufficiently small values of the detuning $d$, for which the Hessian of $\bar{H}_0$ can be considered as a perturbation of the Hessian for $d = 0$.

Following this program, we introduce the quantity $\mu$ by letting

$$\mu = \frac{J_3}{C}.$$

Kummer shows that $\mu$ is an analytic function of $\lambda$, and writes

$$\mu = \frac{1}{2\pi}\Omega(\lambda).$$

Here $\Omega$ is a complicated analytic function defined in terms of a complex integral. Its derivative $\Omega'$ is also analytic (in Kummer's notation $\Omega' = \omega_1$). For our purposes, all we need to know about the function $\Omega$ are the following:

$$\Omega\Big(\pm\frac{2}{\sqrt{27}}\Big) = 0, \quad \Omega'\Big(\pm\frac{2}{\sqrt{27}}\Big) = \pi. \tag{B.2}$$

Using the definitions of $\mu$ and $C$, implicit differentiation yields the relations

$$D_{J_2}\mu = \frac{1}{C}, \quad D_{J_3}\mu = \mu\frac{\omega_3}{C}, \quad D_{J_3}^2\mu = 2\mu\frac{\omega_3^2}{C^2}, \quad \lambda' = \frac{d\lambda}{d\mu} = \frac{2\pi}{\Omega'}. \tag{B.3}$$

Using these relations together with (B.1) we obtain the following expressions for the second partial derivatives of $H_{01}$:

$$D_{J_2}^2 H_{01} = \tfrac{3}{4}\omega_3^2 C^{-1/2}\lambda - 3\omega_3 C^{1/2}\lambda' D_{J_2}\mu + C^{3/2}\lambda''(D_{J_2}\mu)^2 + C^{3/2}\lambda' D_{J_2}^2\mu,$$
$$D_{J_3}^2 H_{01} = C^{-1/2}\lambda',$$
$$D_{J_2}D_{J_3}H_{01} = -\tfrac{1}{2}\omega_3 C^{-1/2}\lambda' + C^{1/2}\lambda'' D_{J_2}\mu.$$

These expressions and (B.2) yield that the determinant of the Hessian of $H_{01}$ evaluated at $\lambda = \pm 2/\sqrt{27}$ is given by

$$\left(D_{J_2}^2\bar{H}_0 D_{J_3}^2\bar{H}_0 - (D_{J_2}D_{J_3}\bar{H}_0)^2\right)\Big|_{\lambda=\pm 2/\sqrt{27}} = \frac{\omega_3^2}{2\lambda}\left(\pm\frac{6}{\sqrt{27}} - 2\right) \neq 0,$$

which proves the lemma as we discussed above. □

## Appendix C. Persistence of invariant manifolds and invariant foliations

### *C.1. Normally hyperbolic invariant manifolds*

We first give a brief description of the invariant manifold results that we will need following the notation and formulation in Wiggins [63]. The general set-up is as follows. Consider a $C^r$, $r \geq 1$ vector field on $\mathbb{R}^n$,

$$\dot{x} = f(x), \tag{C.1}$$

with its flow denoted by $\phi_t(x)$. Suppose that (C.1) has an *overflowing invariant manifold*, $\bar{M} = M \cup \partial M$. By the term *overflowing invariant* we mean that the vector field is tangent to $M$ and points strictly outward on $\partial M$. Therefore, all trajectories starting on $\partial M$ leave $\bar{M}$.

Suppose we have the following continuous splitting of the tangent bundle of $\mathbb{R}^n$ restricted to $M$:

$$T\mathbb{R}^n|_M = TM \oplus N^s \oplus N^u,$$

with the associated projections

$$\Pi^u : T\mathbb{R}^n|_M \to N^u, \tag{C.2}$$
$$\Pi^s : T\mathbb{R}^n|_M \to N^s. \tag{C.3}$$

We note that $N \equiv N^s \oplus N^u$ is the normal bundle of $M$. We assume that the subbundles $TM \oplus N^u$ and $TM \oplus N^s$ are each invariant under $D\phi_t$ for all $t < 0$ (i.e., overflowing invariant). Moreover, we assume that for each $p \in M$ $N_p^u$ is $u$-dimensional and $N_p^s$ is $s$-dimensional; therefore $\bar{M}$ is $n-(s+u)$-dimensional.

Growth rates of vectors in these subbundles under the linearized dynamics are characterized by generalized Lyapunov type numbers defined as follows:

$$\lambda^u(p) = \limsup_{t\to\infty} \| \Pi^u D\phi_{-t}(p)|_{N_p^u} \|^{1/t},$$
$$\nu^s(p) = \limsup_{t\to\infty} \| \Pi^s D\phi_t\left(\phi_{-t}(p)\right)|_{N_p^s} \|^{1/t},$$
$$\sigma^s(p) = \limsup_{t\to\infty} \frac{\log \| D\phi_{-t}|_M(p) \|}{-\log \| \Pi^s D\phi_t\left(\phi_{-t}(p)\right)|_{N_p^s} \|}. \tag{C.4}$$

The manifold $\bar{M}$ is called *normally hyperbolic* if for any point $p \in \bar{M}$, $\lambda^u(p), \nu^s(p) < 1$ and $\sigma^s(p) < 1/r$ hold. We have the following persistence theorem for normally hyperbolic invariant manifolds and their unstable manifolds.

*Theorem C.1* (Fenichel [17]). Suppose $\dot{x} = f(x)$ is a $C^r$ vector field on $\mathbb{R}^n$, $r \geq 1$. Let $\bar{M} \equiv M \cup \partial M$ be a $C^r$, compact connected manifold with boundary overflowing invariant under the vector field $f(x)$. Suppose $\nu^s(p) < 1$, $\lambda^u(p) < 1$, and $\sigma^s(p) < \frac{1}{r}$ for all $p \in M$. Then there exists a $C^r$ overflowing invariant manifold $W^u(\bar{M})$ containing $\bar{M}$ and tangent to the embedding of $N^u$ into $\mathbb{R}^n$ along $\bar{M}$ with trajectories in $W^u(\bar{M})$ approaching $\bar{M}$ as $t \to -\infty$. Moreover, the unstable manifold is persistent under perturbation in the sense that for any $C^r$ vector field $f^{\text{pert}}(x)$ $\mathcal{O}(\varepsilon)$ $C^1$-close to $f(x)$, with $\varepsilon$ sufficiently small, there is a manifold $W^u\left(\bar{M}^{\text{pert}}\right)$ overflowing invariant under $f^{\text{pert}}(x)$ and $C^r$ diffeomorphic to $W^u\left(\bar{M}\right)$.

Next, we consider the existence of foliations of $W^u(\bar{M})$ by submanifolds corresponding to initial conditions that approach the same trajectory on $\bar{M}$ at the most rapid rate as $t \to -\infty$. First, we need to characterize growth rates with generalized Lyapunov type numbers. In addition to the three type numbers given above, we have the two additional type numbers

$$\sigma^{cu}(p) = \limsup_{t\to\infty} \| D\phi_t|_M\left(\phi_{-t}(p)\right) \|^{1/t} \| \Pi^u D\phi_{-t}(p)|_{N_p^u} \|^{1/t},$$

$$\sigma^{su}(p) = \limsup_{t\to\infty} \| \Pi^u D\phi_{-t}(p)|_{N_p^u} \|^{1/t} \| \Pi^s D\phi_t\left(\phi_{-t}(p)\right)|_{N_p^s} \|^{1/t} . \tag{C.5}$$

We have the following *unstable manifold foliation theorem.*

*Theorem C.2* ( Fenichel [18]). Suppose $\dot{x} = f(x)$ is a $C^r$ vector field on $\mathbb{R}^n$, $r \geq 1$. Let $\bar{M} \equiv M \cup \partial M$ be a $C^r$, compact connected manifold with boundary, overflowing invariant under the vector field $f(x)$. Suppose $\lambda^u(p) < 1$, $\sigma^{cu}(p) < 1$, and $\sigma^{su}(p) < 1$ for every $p \in \bar{M}_1$. Then there exists a $n-(s+u)$-parameter family $\mathcal{F}^u = \cup_{p\in M} f^u(p)$ of $u$-dimensional surfaces $f^u(p)$ (with boundary), such that the following hold:

(1) $\mathcal{F}^u$ is a negatively invariant family, i.e., $\phi_{-t}\left(f^u(p)\right) \subset f^u\left(\phi_{-t}(p)\right)$ for any $t \geq 0$ and $p \in M$.

(2) The $u$-dimensional surfaces $f^u(p)$ are $C^r$.

(3) $f^u(p)$ is tangent at $p$ to the embedding of $N_p^u$ into $\mathbb{R}^n$.

(4) There exist $C_u, \lambda_u > 0$ such that if $q \in f^u(p)$ then

$$\| \phi_{-t}(q) - \phi_{-t}(p) \| < C_u e^{-\lambda_u t},$$

for any $t \geq 0$.

(5) Suppose $q \in f^u(p)$ and $q' \in f^u(p')$. Then

$$\frac{\| \phi_{-t}(q) - \phi_{-t}(p) \|}{\| \phi_{-t}(q') - \phi_{-t}(p) \|} \to 0 \quad \text{as} \quad t \uparrow \infty,$$

unless $p = p'$.

(6) $f^u(p) \cap f^u(p') = \emptyset$, unless $p = p'$.

(7) If the hypotheses of the unstable manifold theorem hold, i.e., if additionally $\nu^s(p) < 1$ and $\sigma^s(p) < \frac{1}{r}$ for every $p \in \bar{M}$, then the $u$-dimensional surfaces $f^u(p)$ are $C^r$ with respect to the basepoint $p$.

(8) $\mathcal{F}^u = W^u_{\text{loc}}(M)$.

We note that since $\bar{M}$ is overflowing invariant it only makes sense to consider the unstable manifold of $\bar{M}$. If $\bar{M}$ were *inflowing invariant* then we would consider dynamics in the limit $t \to +\infty$ and the above theorems would be recast as *stable manifold* and *stable manifold foliation* theorems. If $\bar{M}$ is *invariant*, such as would be the case if $M$ were boundaryless or if the both the unperturbed and perturbed vector fields were tangent to the boundary, then Theorems C.1 and C.2 can be applied to the time-reversed vector field to conclude that the manifold persists along with its stable and unstable manifolds with stable and unstable foliations.

*C.2. Proof of Theorems 7.3 and 7.4*

We now apply Theorems C.1 and C.2 to the 3-DOF system (7.1) to prove statements (i)–(iii) of Theorem 7.3 and statements (i)–(iii) of Theorem 7.4. We do not construct the manifold $\mathcal{M}_\varepsilon^j$ as the perturbation of $\bar{\mathcal{M}}_\varepsilon^j$, but as a perturbation of the manifold of equilibria $\mathcal{M}_0^j$ of the cubic normalized Hamiltonian (4.3). We note that $\mathcal{M}_0^j$ is both overflowing and inflowing invariant, i.e., invariant. As a result we can consider its survival under perturbation without modifying the vectorfield on its boundary. Similarly, both the persistence of its stable and unstable manifolds can be studied directly.

We first compute the generalized Lyapunov type numbers defined in (C.4),(C.5). We sketch the calculations for $\mathcal{M}_0^2$, the calculations for $\mathcal{M}_0^1$ are identical.

Recall that the invariant manifold $\mathcal{M}_0^2$ is given by

$$\mathcal{M}_0^2 \equiv \left\{(x, K, \psi) \in \mathcal{P} \,|\, x_1 = \bar{x}_1^2(K) = \sqrt{K_1 - \omega_3 K_2 - d^2/a^2},\ x_2 = -d/a,\ (K_2, \psi_2) \in A\right\}.$$

Letting $(\delta x, \delta K, \delta\psi)$ denote variations in the corresponding variables, the linearization of (4.7) about $\mathcal{M}_0^2$ is given by

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{K} \\ \delta\dot{\psi} \end{pmatrix} = B \begin{pmatrix} \delta x \\ \delta K \\ \delta\psi \end{pmatrix},$$

where

$$B = \varepsilon \begin{pmatrix} -2a\bar{x}_1^2(K) & 4d & a & -a\omega_3 & 0 & 0 \\ 0 & 2a\bar{x}_1^2(K) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & -a\omega_3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $-2a\bar{x}_1^2(K)$ and $2a\bar{x}_1^2(K)$ are simple eigenvalues of $B$ whereas 0 is a repeated eigenvalue. Clearly, there are four independent eigenvectors corresponding to 0 since the manifold $\mathcal{M}_0^2$ of fixed points is four-dimensional. As a result, after a linear change of coordinates the exponential of the matrix $B$ can be written in the form

$$e^{Bt} = \begin{pmatrix} e^{\varepsilon A_0 t} & 0 \\ 0 & Id_{4\times 4} \end{pmatrix}, \quad e^{\varepsilon A_0 t} = \mathrm{diag}\left(e^{-\varepsilon 2a\bar{x}_1^2 t}, e^{\varepsilon 2a\bar{x}_1^2 t}\right). \tag{C.6}$$

Note that the matrix $e^{Bt}$ is a representation of the linearized flow operator $D\phi_t$ used in the previous subsection. Furthermore, since $\mathcal{M}_0^2$ is a manifold of fixed points, for any $p \in \mathcal{M}_0^2$ we have $\phi_{-t}(p) \equiv p$. Using these facts together with (C.6) gives

$$\| \Pi^u D\phi_{-t}(p)|_{N_p^u} \| = \| \Pi^s D\phi_t\left(\phi_{-t}(p)\right)|_{N_p^s} \| = e^{-2\varepsilon a\bar{x}_1^2 t}, \quad \| D\phi_{-t}|_M(p) \| = 2, \tag{C.7}$$

where we used the usual Euclidean matrix norm. Then a direct substitution of these expressions into (C.4)-(C.5) shows that for fixed $\varepsilon > 0$,

$$\begin{aligned} \lambda^u(p) &= \lambda^u = e^{-2|a|\varepsilon\sqrt{K_1 - \omega_3 K_2 - d^2/a^2}} < 1, \\ \nu^u(p) &= \nu^u = e^{-2|a|\varepsilon\sqrt{K_1 - \omega_3 K_2 - d^2/a^2}} < 1, \end{aligned}$$

$$\sigma^s(p) = \sigma^s = 0,$$
$$\sigma^{cu}(p) = \sigma^{cu} = e^{-2|a|\varepsilon\sqrt{K_1-\omega_3K_2-d^2/a^2}} < 1,$$
$$\sigma^{su}(p) = \sigma^{su} = e^{-4|a|\varepsilon\sqrt{K_1-\omega_3K_2-d^2/a^2}} < 1.$$

Since all the type numbers computed above satisfy the conditions of Theorems C.1 and C.2 we conclude that for $\varepsilon > 0$ fixed, all vectorfields sufficiently $C^1$-close to the truncated cubic normal form admit invariant manifolds and foliations with the properties described in Theorems C.1 and C.2. This, however, does not imply the existence of such manifolds and foliations for the full 3-DOF system (7.1). The reason is that in that system the perturbation is of fixed order for $\varepsilon > 0$ fixed. In other words, in our problem the "strength" of the normal hyperbolicity and the size of the perturbation are *not* independent. This "weak hyperbolicity" is a general phenomenon that arises when one applies normally hyperbolic invariant manifold theory to normal forms.

The problem of weak normal hyperbolicity can be dealt with as follows. We consider the following artificial two-parameter system:

$$\begin{aligned}
\dot{x}_1 &= \varepsilon[a(K_1 - \omega_3K_2 - x_1^2 - x_2^2) - 2(d + ax_2)x_2] + \mathcal{O}(\bar{\varepsilon}^2),\\
\dot{x}_2 &= \varepsilon 2x_1\,(d + ax_2) + \mathcal{O}(\bar{\varepsilon}^2),\\
\dot{\psi}_2 &= -\varepsilon\omega_3\,(d + ax_2) + \mathcal{O}(\bar{\varepsilon}^2),\\
\dot{K}_2 &= \mathcal{O}(\bar{\varepsilon}^2),\\
\dot{\psi}_1 &= 1 + \varepsilon\,(d + ax_2) + \mathcal{O}(\bar{\varepsilon}^2),\\
\dot{K}_1 &= \mathcal{O}(\bar{\varepsilon}^2),
\end{aligned} \tag{C.8}$$

where we view $\varepsilon$ as small and fixed, and $\bar{\varepsilon}$ as an independent perturbation parameter. In this way, for $\bar{\varepsilon}$ sufficiently small, (C.8) has a normally hyperbolic invariant manifold $\mathcal{M}^2_{\bar{\varepsilon}}$. We want to argue that this is true for $\bar{\varepsilon} \leq \varepsilon$. In order to make this conclusion we must compute the generalized Lyapunov type numbers for the *perturbed* normally hyperbolic invariant manifold $\mathcal{M}^2_{\bar{\varepsilon}}$ and show that these type numbers satisfy the hypotheses of Theorems C.1 and C.2 for $\bar{\varepsilon} \leq \varepsilon$. Of course, this cannot generally be done without knowing an explicit form for the perturbed invariant manifold. However, an indirect argument due to Kopell [35] will suffice.

In the proof of Theorem C.1, the perturbed manifold is constructed as the graph of a section of a $C^r$ transversal bundle of the unperturbed manifold. Thus, symbolically, we denote the perturbed manifold as

$$\mathcal{M}^2_{\bar{\varepsilon}} = \operatorname{graph}\mathcal{G}_{\bar{\varepsilon}}.$$

We carry out the argument for the generalized Lyapunov type number $\lambda^u(p)$. The argument for the remaining four type numbers is similar. Using the above notation, the perturbed generalized Lyapunov type number $\lambda^u(p)$ is given by

$$\lambda^u(\mathcal{G}_{\bar{\varepsilon}}(p)) = \limsup_{t\to\infty} \| \Pi^u_{\bar{\varepsilon}} D\phi_{-t}(p)|_{N^u_{\bar{\varepsilon},p}} \|^{1/t}.$$

We write $t = nT + r$, where $n \in \mathbb{N}$ and $r \in [0,T)$. We then have the inequalities

$$\begin{aligned}
\| \Pi^u_{\bar{\varepsilon}} D\phi_{-t}(p)|_{N^u_{\bar{\varepsilon},p}} \|^{1/t} &= \| \Pi^u_{\bar{\varepsilon}} D\phi_{-(nT+r)}(p)|_{N^u_{\bar{\varepsilon},p}} \|^{1/(nT+r)}\\
&\leq \| \Pi^u_{\bar{\varepsilon}} D\phi_{-nT}(p)|_{N^u_{\bar{\varepsilon},p}} \|^{1/(nT+r)} \| \Pi^u_{\bar{\varepsilon}} D\phi_{-r}(p)|_{N^u_{\bar{\varepsilon},p}} \|^{1/t}\\
&\leq \| \Pi^u_{\bar{\varepsilon}} D\phi_{-T}(p)|_{N^u_{\bar{\varepsilon},p}} \|^{1/(T+r/n)} \| \Pi^u_{\bar{\varepsilon}} D\phi_{-r}(p)|_{N^u_{\bar{\varepsilon},p}} \|^{1/t}.
\end{aligned}$$

Taking the limsup of this expression as $t \to \infty$ gives

$$\lambda^u(\mathcal{G}_{\bar{\varepsilon}}(p)) \leq \| \Pi^u_{\bar{\varepsilon}} D\phi_{-T}(p)|_{N^u_{\bar{\varepsilon},p}} \|^{1/T}.$$

The generalized Lyapunov type numbers are not typically differentiable (or even continuous) functions of parameters. Therefore the significance of this inequality is that although the left hand side of the inequality may not be continuous in $\bar{\varepsilon}$, the right hand side is $C^{r-1}$ in $\bar{\varepsilon}$. Thus we have

$$\lambda^u(\mathcal{G}_{\bar{\varepsilon}}(p)) \leq \| \Pi^u_{\bar{\varepsilon}=0} D\phi_{-T}(p)|_{N^u_{\bar{\varepsilon}=0,p}} \|^{1/T} + \mathcal{O}(\bar{\varepsilon}) = e^{-2|a|\varepsilon\sqrt{K_1-\omega_3K_2-d^2/a^2}} + \mathcal{O}(\bar{\varepsilon}).$$

Thus, for $\varepsilon$ small and fixed, it follows that for $\bar{\varepsilon}$ sufficiently small we have

$$\lambda^u(\mathcal{G}_{\bar{\varepsilon}}(p)) \leq e^{-2|a|\varepsilon\sqrt{K_1-\omega_3K_2-d^2/a^2}} + \mathcal{O}(\bar{\varepsilon}) \leq e^{-2|a|\varepsilon\sqrt{K_1-\omega_3K_2-d^2/a^2}} + \mathcal{O}(\varepsilon) < 1.$$

Hence, $\lambda^u(\mathcal{G}_{\bar{\varepsilon}}(p)) \leq 1$ for $\bar{\varepsilon} \leq \varepsilon$.

Thus, Theorems C.1 and C.2 can be applied to our setting and the statements (i)–(iii) in Theorems 7.3 and 7.4 are just restatements of these results in the context of our specific problem.

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