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GYROSCOPIC STABILITY AND ITS LOSS IN SYSTEMS WITH TWO ESSENTIAL COORDINATES

GYÖRGY HALLER

Department of Applied Mechanics, Technical University of Budapest, H-1521 Budapest, Hungary

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Abstract—Cyclic systems with two essential coordinates are studied. The particular case is investigated, when a steady motion of the system is neutrally stable in linear approximation due to gyroscopic effects on an originally unstable equilibrium. Conditions are given to ensure the preservation of stability for the non-linear conservative system. In addition, a universal loss of stability is proved as a result of non-total dissipation or acceleration acting on the system.

INTRODUCTION

A great number of mechanical systems are characterized by the property that they have some generalized coordinates which contribute only with their derivatives to the expression of the energy of the system, and the generalized forces corresponding to them are zero. Systems of this type are usually called cyclic, or Routh-systems, because the coordinates in question are mostly angle coordinates, and the motion can most conveniently be described by Routh equations [1, 2].

The usage of Routh equations makes it possible to reduce the dimension of the phase space by eliminating the cyclic coordinates, which is equivalent to fixing the generalized momenta canonically conjugated to them [3]. In the reduced phase space a steady motion of the system will appear as an equilibrium. In most cases the stability of this equilibrium, i.e. the stability of the steady motion, is uniquely determined by the properties of the Routh function [1, 2]. There exists, however, a case when the equilibrium has a "static" instability of even order in the absence of gyroscopic effects. In this case the energy cannot be used as a Lyapunov function to prove the possible stability of the equilibrium, which possesses pure imaginary pairs of eigenvalues in linear approximation.

There are several works dealing with this stability problem, which is often mentioned as gyroscopic stabilization (e.g. [4]). A lot of instability criteria have been established (e.g. [4–8]), but theorems concerning stability state only the possibility of stabilization if gyroscopic forces are large enough [4, 9]. However, since stability is always neutral in these cases, strictly speaking, the results mentioned are valid for the linearized systems only.

This trend is basically due to the fact that the usual approach to gyroscopic stability, using Lyapunov's Direct Method, fails to be applicable since the equilibrium is not an extreme point of the total mechanical energy. Moreover, it is simply not likely to find any proper Lyapunov functions because of the expectably sophisticated phase space structure, which was revealed by the Kolmogorov–Arnold–Moser theory (KAM) in the 1960s [3, 10, 11].

Direct applications of the KAM theory to practical examples seem to be limited to given conservative systems (see [3, 11, 12], etc.) but, to our best knowledge, no criterion has been established to capture gyroscopic stability in general.

The first goal of this paper is to give a sufficient condition for the stability of gyroscopically stabilized steady motions. We will focus on the case when our system has two essential coordinates. The reason for this initial assumption is two-fold. Firstly, this is the maximum number of essential freedom for which Lyapunov stability can be proved by means of the KAM theory. Secondly, none of the known examples of possible gyroscopic stabilization exceeds this limit (see, e.g. [11–14]), because in practical mechanical systems it is not desirable to increase complexity by a construction with more than two essential coordinates.

Contributed by P. B. Hagedorn.

The second goal of the paper is to examine the type of instability occurring under the effect of dissipative or accelerating forces on the gyroscopically stabilized equilibrium. The destabilizing effect of positive or negative total dissipative perturbation is known from quite a long time ago [9]. Here we give the same result for the case of non-total dissipation or acceleration, and determine the type of instability arising in systems with two essential coordinates.

Finally the results are demonstrated on a mechanical system with two essential and one cyclic coordinate. Within the frame of the example we examine the stability of the steady motion of a freely rotating flexible shaft–disc system.

1. NOTATIONS AND DEFINITIONS

Let us consider a conservative mechanical system with n degrees-of-freedom, where $n \geq 3$. We assume that the Lagrangian of the system does not depend explicitly on $n-2$ of the generalized coordinates, which will be called cyclic coordinates, and can be written in the form of a vector

$$\mathbf{c} = \mathrm{col}(c_1 \;\; c_2 \, \ldots \, c_{n-2}), \quad \mathbf{c} \in \mathbf{R}^{n-2}.$$

The remainder of the coordinates will be called essential coordinates, and can be written in the vector

$$\mathbf{q} = \mathrm{col}(q_1 \;\; q_2), \quad \mathbf{q} \in \mathbf{R}^2.$$

Our system can most conveniently be described by introducing the vector of generalized momenta canonically conjugated to the cyclic coordinates:

$$\mathbf{p} = (p_1 \;\; p_2 \, \ldots \, p_{n-2}) = \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{c}})}{\partial \dot{\mathbf{c}}}, \quad \mathbf{p}^{\mathrm{T}} \in \mathbf{R}^{n-2}, \tag{1.1}$$

where $L = T - \Pi$ is the Lagrangian of the system with kinetic energy $T(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{c}})$ and potential energy $\Pi(\mathbf{q})$. Since our system is conservative (holonomic, scleronomic under the action of time-independent potential forces), the kinetic energy takes the form

$$T = \tfrac{1}{2}\,\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{A}_1(\mathbf{q})\dot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{A}_2(\mathbf{q})\dot{\mathbf{c}} + \tfrac{1}{2}\dot{\mathbf{c}}^{\mathrm{T}}\mathbf{A}_3(\mathbf{q})\dot{\mathbf{c}}, \tag{1.2}$$

where

$$\mathbf{A}_1 \in \mathbf{R}^{2\times 2}, \quad \mathbf{A}_2 \in \mathbf{R}^{2\times(n-2)}, \quad \mathbf{A}_3 \in \mathbf{R}^{(n-2)\times(n-2)}.$$

In (1.2) $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are assumed to be smooth in $\mathbf{q}$, $\mathbf{A}_1$ and $\mathbf{A}_3$ are non-singular. The potential energy Π is also supposed to be smooth in $\mathbf{q}$.

Expressions (1.1) and (1.2) imply

$$\dot{\mathbf{c}}^{\mathrm{T}} = (\mathbf{p} - \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{A}_2)\,\mathbf{A}_3^{-1}. \tag{1.3}$$

We can introduce the Routh function

$$R(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{p}) = L - \mathbf{p}\dot{\mathbf{c}}, \tag{1.4}$$

after substituting (1.3) into L. Straightforward calculations show that the Routh function (1.4) can be written in the form

$$R = R_2 + R_1 - R_0, \tag{1.5}$$

where

$$R_2 = \tfrac{1}{2}\,\dot{\mathbf{q}}^{\mathrm{T}}(\mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_3^{-1}\mathbf{A}_2^{\mathrm{T}})\dot{\mathbf{q}}, \quad R_1 = \mathbf{p}\mathbf{A}_3^{-1}\mathbf{A}_2^{\mathrm{T}}\dot{\mathbf{q}}, \quad R_0 = \tfrac{1}{2}\mathbf{p}\mathbf{A}_3^{-1}\mathbf{p}^{\mathrm{T}} + \Pi,$$

and the subscripts of R refer to the highest powers of $\dot{q}_i$ ($i = 1, 2$) in the expressions. We note that R_0 is usually called Routh potential. The Routh equations of motion are the following:

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{\mathbf{q}}} - \frac{\partial R}{\partial \mathbf{q}} = 0,$$
$$\dot{\mathbf{c}}^T = (\mathbf{p} - \dot{\mathbf{q}}^T \mathbf{A}_2)\mathbf{A}_3^{-1},$$
$$\dot{\mathbf{p}} = 0. \tag{1.6}$$

In the sequel we will focus on the steady motions of system (1.6), which are defined by

$$\mathbf{q} \equiv \mathbf{q}^0 = \text{const.}, \quad \dot{\mathbf{c}} \equiv \dot{\mathbf{c}}^0 = \text{const.} \tag{1.7}$$

It is well known that $\mathbf{q}^0$ can be determined as a function of $\mathbf{p}$ from the following equation:

$$\frac{\partial R_0}{\partial \mathbf{q}} = \frac{1}{2}\mathbf{p}\frac{\partial \mathbf{A}_3^{-1}}{\partial \mathbf{q}}\mathbf{p}^T + \frac{\partial \Pi}{\partial \mathbf{q}} = 0,$$

because the Routh potential has vanishing derivatives on steady motions [1, 2].

Fixing a value of $\mathbf{p}$ and linearizing (1.6) along a steady motion defined by (1.7), we are given a system of two equations for the essential motion:

$$\mathbf{M}\ddot{\tilde{\mathbf{q}}} + \mathbf{G}\dot{\tilde{\mathbf{q}}} + \mathbf{S}\tilde{\mathbf{q}} = \mathbf{0}, \quad \tilde{\mathbf{q}} \in \mathbf{R}^2, \tag{1.8}$$

where $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^0$, $\mathbf{M}, \mathbf{G}, \mathbf{S} \in \mathbf{R}^{2\times 2}$, $\mathbf{M}$ is positive definite symmetrical, $\mathbf{S}$ is symmetrical, $\mathbf{G}$ is antisymmetrical, and

$$\mathbf{M} = (\mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_3^{-1}\mathbf{A}_2^T)|_{\mathbf{q}^0}, \quad \mathbf{S} = \left(\frac{1}{2}\mathbf{p}\frac{d^2\mathbf{A}_3^{-1}}{d\mathbf{q}^2}\mathbf{p}^T + \frac{d^2\Pi}{d\mathbf{q}^2}\right)\Bigg|_{\mathbf{q}^0},$$

$$\mathbf{G} = \left[\mathbf{p}\frac{d\mathbf{A}_3^{-1}}{d\mathbf{q}}\mathbf{A}_2^T + \mathbf{p}\mathbf{A}_3^{-1}\frac{d\mathbf{A}_2^T}{d\mathbf{q}} - \mathbf{A}_2\left(\frac{d\mathbf{A}_3^{-1}}{d\mathbf{q}}\right)^T\mathbf{p}^T - \frac{d\mathbf{A}_2}{d\mathbf{q}}\mathbf{A}_3^{-1}\mathbf{p}^T\right]\Bigg|_{\mathbf{q}^0}.$$

It is well known from [1, 4, 15], etc. that

Case 1: If $\mathbf{S}$ is positive definite then the steady motion of (1.6) is stable.
Case 2: If $\mathbf{S}$ is indefinite then the steady motion of (1.6) is unstable.
Case 3: If $\mathbf{S}$ is negative definite then the steady motion of (1.6) may be stable.

Case 3 is called gyroscopic stabilization, and a detailed study of this case can be found in [4]. According to the results, for given $\mathbf{M}$ and $\mathbf{S}$, there exists a $\mathbf{G}$ for which (1.8) will possess two pairs of pure imaginary eigenvalues. Let us denote these eigenvalues by $\pm i\omega_1$ and $\pm i\omega_2$, where ω_1 and ω_2 are called basic frequencies. If the basic frequencies are rationally commensurable then the solutions of (1.8) will be periodic. If, however, the basic frequencies are rationally independent then the solutions will be quasiperiodic, and any of them will constitute an everywhere dense subset of a two-dimensional invariant torus. Clearly, this latter case occurs with probability 1 in general.

If we want to prove the stability of the steady motion of (1.6) in case 3, there are two basic steps to take. First, we have to guarantee the neutral stability of the linearized system (1.8). Secondly, we have to show the persistence of this stability after taking the non-linear terms of (1.6) into account. Naturally, the conditions for stability will depend on the fixed value of $\dot{\mathbf{c}}^0$ providing us the critical values of the cyclic velocities at which stability fails.

In the next section we will make preparations to answer the question involved in the second step on the basis of the Normal Form theory (cf. [3, 10, 16]).

2. FOUR-DIMENSIONAL NORMAL FORM CALCULATION

If we introduce the new variables

$$x_j \doteq \tilde{q}_j = q_j - q_j^0, \quad x_{j+2} \doteq \dot{\tilde{q}}_{j+2} = \dot{q}_{j+2}, \quad j = 1, 2,$$

system (1.6) can be rewritten in the form

$$\dot{\mathbf{x}} = \mathbf{N}\mathbf{x} + \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{N} = \begin{pmatrix} 0 & \mathbf{I}_2 \\ -\mathbf{M}^{-1}\mathbf{S} & -\mathbf{M}^{-1}\mathbf{G} \end{pmatrix}, \quad \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

and $\mathbf{f} = \mathrm{col}(0\ 0\ f_3\ f_4)$ is analytical in a neighbourhood of the origin.

In case 3 N possesses two imaginary pairs of eigenvalues, which are assumed to be different to make N diagonalizable. Using a matrix $\mathbf{T} \in \mathbf{C}^{2\times 2}$, which contains the eigenvectors of N, we can transform system (2.1) to its eigenbasis by means of the linear transformation

$$\mathbf{z} = \mathrm{col}(z_1\ \bar{z}_1\ z_2\ \bar{z}_2) = \mathbf{T}^{-1}\mathbf{x}, \quad z_j \in \mathbf{C}, \quad j = 1, 2,$$

to obtain the equations

$$\dot{z}_j = i\omega_j z_j + h^j(z_1, \bar{z}_1, z_2, \bar{z}_2), \quad j = 1, 2, \tag{2.2}$$

where h^j is a convergent power series from order two with complex coefficients.

To make (2.2) more tractable, we transform it to its normal form (see [10] or [16] for a detailed introduction to Normal Form theory). We use an analytical transformation of the form

$$z_j = \mathrm{w}_j + \Phi^j(\mathrm{w}_1, \bar{\mathrm{w}}_1, \mathrm{w}_2, \bar{\mathrm{w}}_2), \quad j = 1, 2, \tag{2.3}$$

where Φ^j are convergent complex series of order two.

Lemma 2.1. Let us suppose that for arbitrary non-zero r_1, r_2 integers

$$r_1\omega_1 + r_2\omega_2 \neq 0, \quad 0 < |r_1| + |r_2| \leq 4. \tag{2.4}$$

Then transformation (2.3) transforms system (2.2) to the following normal form:

$$\dot{\mathrm{w}}_j = i\omega_j \mathrm{w}_j + k^j(\mathrm{w}_1, \bar{\mathrm{w}}_1, \mathrm{w}_2, \bar{\mathrm{w}}_2), \quad j = 1, 2, \tag{2.5a}$$

or

$$\begin{aligned} \dot{\mathrm{w}}_1 &= i\omega_1 \mathrm{w}_1 + ia_{11}\mathrm{w}_1\mathrm{w}_2\bar{\mathrm{w}}_2 + ia_{12}\mathrm{w}_1^2\bar{\mathrm{w}}_1 + O(|(\mathrm{w}_1\ \bar{\mathrm{w}}_1\ \mathrm{w}_2\ \bar{\mathrm{w}}_2)|^4), \\ \dot{\mathrm{w}}_2 &= i\omega_2 \mathrm{w}_2 + ia_{21}\mathrm{w}_2^2\bar{\mathrm{w}}_2 + ia_{22}\mathrm{w}_1\bar{\mathrm{w}}_1\mathrm{w}_2 + O(|\mathrm{w}_1\ \bar{\mathrm{w}}_1\ \mathrm{w}_2\ \bar{\mathrm{w}}_2)|^4), \end{aligned} \tag{2.5b}$$

$$a_{lm} \in \mathbf{R}, \quad l, m = 1, 2.$$

Proof. Since N possesses two pairs of pure imaginary eigenvalues, (2.4) implies that the normal form of (2.2) (see [9] or [10]) takes the form

$$\dot{\mathrm{w}}_1 = i\omega_1 \mathrm{w}_1 + \alpha_{11}\mathrm{w}_1\mathrm{w}_2\bar{\mathrm{w}}_2 + \alpha_{12}\mathrm{w}_1^2\bar{\mathrm{w}}_1 + O(|(\mathrm{w}_1\ \bar{\mathrm{w}}_1\ \mathrm{w}_2\ \bar{\mathrm{w}}_2)|^4),$$

$$\dot{\mathrm{w}}_2 = i\omega_2 \mathrm{w}_2 + \alpha_{21}\mathrm{w}_2^2\bar{\mathrm{w}}_2 + \alpha_{22}\mathrm{w}_1\bar{\mathrm{w}}_1\mathrm{w}_2 + O(|(\mathrm{w}_1\ \bar{\mathrm{w}}_1\ \mathrm{w}_2\ \bar{\mathrm{w}}_2)|^4),$$

$$\alpha_{lm} \in \mathbf{C}, \quad l, m = 1, 2.$$

(Condition (2.4) ensures the removability of the second and third order terms of (2.2) with the exception of those indicated above.)

We only have to show that the coefficients α_{lm} are pure imaginary, i.e. the normal form is general transcendental (cf. [10]). First we show that the whole system (1.6) is invariant for the reversal of time. Let us reverse the time by $\tau = -t$ and consider a $(\mathbf{q}_r, \dot{\mathbf{q}}_r, \dot{\mathbf{c}}_r)$ reversal of some $(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{c}})$ solution of (1.6). Obviously

$$\mathbf{q}(t) = \mathbf{q}_r(-\tau), \quad \dot{\mathbf{q}}(t) = -\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{q}_r(-\tau), \quad \dot{\mathbf{c}}(t) = -\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{c}_r(-\tau),$$

so from (1.6) we have

$$\mathbf{p}_r = -\mathbf{p} = \text{const.},$$

for the $\mathbf{p}_r$ generalized momentum which is determined by the initial values of the reversed motion. Thus, on the basis of (1.5), $R_r = R$ holds for the reversed Routh function, which immediately implies the reversibility of system (1.6), since

$$\frac{\mathrm{d}}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}\tau}.$$

This fact causes systems (2.1) and (2.2) to be invariant for the transformation $\tau = -t$, $\mathbf{p}_r = -\mathbf{p}$. Then, applying the ideas of [9] for reversible systems, and making use of $\mathbf{p}_r = -\mathbf{p}$, we obtain the statement of the lemma (cf. [9] p. 37).

The following lemma helps to determine the coefficients of the formal power series Φ^j and k^j.

Lemma 2.2. The unknown coefficients of (2.3) and (2.5) are determined by the following two equations:

$$i\omega_l\left(\mathrm{w}_l\frac{\partial\Phi^l}{\partial \mathrm{w}_l} - \Phi^l - \bar{\mathrm{w}}_l\frac{\partial\Phi^l}{\partial\bar{\mathrm{w}}_l}\right) + i\omega_m\left(\mathrm{w}_m\frac{\partial\Phi^l}{\partial\mathrm{w}_m} - \bar{\mathrm{w}}_m\frac{\partial\Phi^l}{\partial\bar{\mathrm{w}}_m}\right)$$

$$= h^l(\mathrm{w}_1 + \Phi^1, \bar{\mathrm{w}}_1 + \overline{\Phi^1}, \mathrm{w}_2 + \Phi^2, \bar{\mathrm{w}}_2 + \overline{\Phi^2}) - k^l(\mathrm{w}_1, \bar{\mathrm{w}}_1, \mathrm{w}_2, \bar{\mathrm{w}}_2)\left(1 + \frac{\partial\Phi^l}{\partial\mathrm{w}_l}\right)$$

$$- k^m(\mathrm{w}_1, \bar{\mathrm{w}}_1, \mathrm{w}_2, \bar{\mathrm{w}}_2)\frac{\partial\Phi^l}{\partial\mathrm{w}_m} - \overline{k^l}(\mathrm{w}_1, \bar{\mathrm{w}}_1, \mathrm{w}_2, \bar{\mathrm{w}}_2)\frac{\partial\Phi^l}{\partial\bar{\mathrm{w}}_m} - \overline{k^m}(\mathrm{w}_1, \bar{\mathrm{w}}_1, \mathrm{w}_2, \bar{\mathrm{w}}_2)\frac{\partial\Phi^l}{\partial\mathrm{w}_m},$$

$$\binom{l}{m} = \binom{1}{2}, \binom{2}{1}. \tag{2.5}$$

Proof. The proof is similar to that of [10] for the one-dimensional normal form used in the study of Hopf bifurcations. After differentiating (2.3) with respect to t, substituting it into (2.2) and comparing with (2.3), we can easily obtain equation (2.5). Then the method of undetermined coefficients provides the second order coefficients of Φ^l as functions of the coefficients of h^m up to order three. At the same time, the unknown coefficients of k^l can be gained from (2.5) as functions of the second order terms of Φ^m ($l, m = 1, 2$).

On the basis of lemma 2.2 the a_{lm} coefficients in (2.5) have been calculated and the results can be found in the Appendix.

3. SUFFICIENT CONDITION FOR STABILITY

In this section we formulate a sufficient condition for the steady motion of (1.6) to be stable in Lyapunov sense with respect to essential perturbation in Case 3.

From this point we will use the notation

$$\mathbf{B}\cdot\cdot\mathbf{C} \doteq \sum_{i,j=1}^{n} b_{ij}c_{ij}$$

for the double dot product of matrices $\mathbf{B}\in\mathbf{R}^{n\times m}$ and $\mathbf{C}\in\mathbf{R}^{n\times m}$.

Theorem 3.1. Let us suppose that for systems (1.6) and (1.8) the following are satisfied:

(i) S is negative definite

(ii) $\dfrac{\det \mathbf{G}}{\det \mathbf{M}} > -\mathbf{M}^{-1}\cdot\cdot\mathbf{S} + 2\sqrt{\dfrac{\det \mathbf{S}}{\det \mathbf{M}}}$

(iii) $\dfrac{\det \mathbf{G}}{\det \mathbf{M}} \neq -\mathbf{M}^{-1}\cdot\cdot\mathbf{S} + \dfrac{r^2+1}{r}\sqrt{\dfrac{\det \mathbf{S}}{\det \mathbf{M}}}, \quad r = 2, 3$

(iv) $\det \mathbf{A} \neq 0$, where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is defined by the coefficients of h^p (see (2.2)) in the Appendix.

Then the steady motion (1.7) of (1.6) is stable in Lyapunov sense.

Proof. Simple calculations show that the characteristic equation of (1.8) has the form

$$\lambda^4 + \left(\frac{\det \mathbf{G}}{\det \mathbf{M}} + \mathbf{M}^{-1}\cdot\cdot\mathbf{S}\right)\lambda^2 + \frac{\det \mathbf{S}}{\det \mathbf{M}} = 0. \tag{3.1}$$

We know that $\det \mathbf{S} > 0$, since S is negative definite, and $\det \mathbf{M} > 0$, since M is positive definite ($\det \mathbf{G} > 0$ because G is antisymmetrical and of even dimension). Applying the Routh–Hurwitz criterion to (3.1) we obtain that the necessary condition of stability for (1.8) is

$$\frac{\det \mathbf{G}}{\det \mathbf{M}} + \mathbf{M}^{-1}\cdot\cdot\mathbf{S} > 0,$$

which is guaranteed in (ii). Furthermore, (ii) also implies (1.8) to have two pure imaginary pairs of eigenvalues, $\pm i\omega_1$ and $\pm i\omega_2$, making (1.8) stable.

Now, we have to prove that stability holds for the non-linear system (1.6) using the basic theorem of the KAM theory.

First, it requires system (1.6) to have a normal form (2.5), i.e. using lemma 2.1, ω_1 and ω_2 are assumed to be not strongly resonant. More precisely,

$$r_1\omega_1 + r_2\omega_2 \neq 0$$

should hold for every r_1, r_2 non-zero integers satisfying the inequality

$$0 < |r_1| + |r_2| \leq 4.$$

Since $\omega_1 > 0$ and $\omega_2 > 0$, the possible resonant cases are

$$\omega_1 = r\omega_2, \quad r = 1, 2, 3. \tag{3.2}$$

Solving (3.1) with $\lambda = i\omega$ we get

$$\omega_{1,2} = \sqrt{\frac{1}{2}\left(\frac{\det \mathbf{G}}{\det \mathbf{M}} + \mathbf{M}^{-1}\cdot\cdot\mathbf{S}\right) \pm \sqrt{\frac{1}{4}\left(\frac{\det \mathbf{G}}{\det \mathbf{M}} + \mathbf{M}^{-1}\cdot\cdot\mathbf{S}\right)^2 - \frac{\det \mathbf{S}}{\det \mathbf{M}}}}. \tag{3.3}$$

Substituting (3.3) into (3.2), after short calculations, we obtain

$$\frac{\det \mathbf{G}}{\det \mathbf{M}} + \mathbf{M}^{-1}\cdot\cdot\mathbf{S} \neq \frac{r^2+1}{r}\sqrt{\frac{\det \mathbf{S}}{\det \mathbf{M}}}, \quad r = 1, 2, 3,$$

as a non-resonance condition which holds on the basis of (ii) and (iii).

In addition to the non-resonance condition, according to [10], a non-degeneracy condition must be satisfied which requires the normal form (2.5b) to be non-degenerate in the sense

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0,$$

but this is exactly what is assumed in (iv).

Consequently, on the basis of the KAM theory, we have proved that the invariant tori of the linear system (1.8) will be preserved in majority (with some deformation) if we consider the whole non-linear system (1.6). Here "majority" means that if we shrink a measurable V set to the origin then the measure of initial values in V, for which the solutions run on tori will be tending to the measure of V.

Our system is conservative, therefore there are three-dimensional level manifolds in the four-dimensional phase space which involve orbits with the same energy. As it is pointed out in [3], the two-dimensional preserved tori divide the three-dimensional level manifolds of the energy. Because of this, any solutions starting from the interior of a torus cannot escape from it, which completes the proof of Lyapunov stability.

Remark 3.2. Condition (iv) for non-degeneracy in theorem 3.1 is often considered to be of minor importance because, apart from constructed examples, it is fulfilled with probability 1 for given mechanical systems. In other words, for realistic mechanical systems degeneracy of this kind may occur for a set of parameters which has zero measure in the parameter space. In addition, the basic theorem of the KAM theory is a sufficient one, and the lack of condition (iv) does not imply instability.

Remark 3.3. Clearly, if (i)–(iv) in theorem 3.1 are satisfied, stability will also hold for such ω_1 and ω_2 values which are "almost" strongly resonant. However, the diameter of the domain of stability tends to zero while the ratio of ω_1 and ω_2 is tending to 1, 2 or 3. Therefore, in cases close to internal resonances system (1.6) may produce an unstable-like behaviour for lack of a tangible domain of stability.

4. HIGH AND LOW FREQUENCY LOSSES OF STABILITY

In this section we will investigate the case when a gyroscopically stabilized system is exposed to the effect of dissipative or accelerating forces. The destabilizing impact of these perturbations is well known if the matrices corresponding to these forces are definite (e.g. [1]).

Now we study semi-definite dissipative or accelerated systems with two essential coordinates. In addition to this, a universal loss of stability will be shown regardless of the definiteness or semi-definiteness of the perturbation.

First we deal with positive dissipation, i.e. we suppose that system (1.6) is under the effect of dissipative forces. We assume that the dissipative forces are of Rayleigh-type, i.e. they can be derived from a Rayleigh function D of the form

$$D = \tfrac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\varepsilon\mathbf{B}(\mathbf{q})\dot{\mathbf{q}}, \tag{4.1}$$

where $\varepsilon > 0$, $\mathbf{B} \in \mathbf{R}^{2\times 2}$, and $\mathbf{B}$ is continuous in $\mathbf{q}$. Moreover, $\mathbf{B}$ is supposed to be symmetrical and positive semi-definite or positive definite.

Now equation (1.6) is modified by D to the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial \dot{\mathbf{q}}} - \frac{\partial R}{\partial \mathbf{q}} = -\frac{\partial D}{\partial \dot{\mathbf{q}}},$$

$$\dot{\mathbf{c}}^{\mathrm{T}} = (\mathbf{p} - \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{A}_2)\mathbf{A}_3^{-1}.$$

$$\dot{\mathbf{p}} = 0. \tag{4.2}$$

After the same reduction of the phase space as in Section 1, we have the same equations for the equilibria of the reduced system (steady motions of the original system (1.6)).

Linearizing (4.2) along a steady motion defined by (1.7) we are given the following system of linear differential equations (cf. [1, 4]):

$$\mathbf{M}\ddot{\tilde{\mathbf{q}}} + (\tilde{\mathbf{K}} + \mathbf{G})\dot{\tilde{\mathbf{q}}} + \mathbf{S}\tilde{\mathbf{q}} = 0, \tag{4.3}$$

where **M**, **G**, **S** and $\tilde{\mathbf{q}}$ are the same as in (1.8), $\tilde{\mathbf{K}}$ is symmetrical, either positive definite or positive semi-definite, and

$$\tilde{\mathbf{K}} = \varepsilon\mathbf{B}(\mathbf{q}^0) = \varepsilon\mathbf{K}, \quad \varepsilon > 0,$$

where **K** is obviously of the same type as $\tilde{\mathbf{K}}$.

Straightforward calculations show the characteristic equation of (4.3) to be of the form

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where

$$a_3 = \mathbf{M}^{-1}\cdot\cdot\tilde{\mathbf{K}}, \quad a_2 = \mathbf{M}^{-1}\cdot\cdot\mathbf{S} + \frac{\det\tilde{\mathbf{K}} + \det\mathbf{G}}{\det\mathbf{M}}, \quad a_1 = \frac{\mathbf{K}^*\cdot\cdot\mathbf{S}}{\det\mathbf{M}}, \quad a_0 = \frac{\det\mathbf{S}}{\det\mathbf{M}}. \tag{4.4}$$

In (4.4) $\tilde{\mathbf{K}}^*$ denotes the associate matrix of $\tilde{\mathbf{K}}$, i.e.

$$\tilde{\mathbf{K}}^* = \begin{pmatrix} \tilde{k}_{22} & -\tilde{k}_{12} \\ -k_{21} & \tilde{k}_{11} \end{pmatrix}.$$

Lemma 4.1. Let us suppose that $\mathbf{A}, \mathbf{B} \in \mathbf{R}^{2\times 2}$, where $\mathbf{A} \neq 0$ is positive semi-definite or positive definite, and **B** is definite. Then

(i) $\mathbf{A}^*\cdot\cdot\mathbf{B} > 0$, if **B** is positive definite,

(ii) $\mathbf{A}^*\cdot\cdot\mathbf{B} < 0$, if **B** is negative definite.

Proof. Let us consider first the case when **B** is negative definite. Then, using the conditions of the lemma we have

$$a_{11} \geq 0, \quad a_{22} \geq 0, \quad a_{11} + a_{22} > 0, \tag{4.5}$$

and

$$b_{11} < 0, \quad b_{22} < 0, \tag{4.6}$$

thus from (4.5) and (4.6) we obtain

$$a_{11}b_{22} + a_{22}b_{11} < 0. \tag{4.7}$$

Using the negative definiteness of **B** we can write

$$\det\mathbf{B} = b_{11}b_{22} - b_{12}^2 > 0,$$

implying

$$|b_{12}| < \sqrt{b_{11}b_{22}}. \tag{4.8}$$

Similar reasoning leads to

$$|a_{12}| \leq \sqrt{a_{11}a_{22}}. \tag{4.9}$$

Multiplying (4.8) and (4.9) we are given

$$|a_{12}b_{12}| < \sqrt{a_{11}b_{22}a_{22}b_{11}}. \tag{4.10}$$

Using

$$\sqrt{a_{11}b_{22}a_{22}b_{11}} \leqslant \frac{a_{11}|b_{22}| + a_{22}|b_{11}|}{2} = \tfrac{1}{2}|a_{11}b_{22} + a_{22}b_{11}|,$$

from (4.10) we get

$$|a_{11}b_{22} + a_{22}b_{11}| > 2|a_{12}b_{12}|. \tag{4.11}$$

From (4.7) and (4.11) it follows that

$$a_{11}b_{22} + a_{22}b_{11} - 2a_{12}b_{12} = \mathbf{A}^{*}\cdot\cdot\mathbf{B} < 0,$$

so we have proved (i). But then (ii) follows from (i) because a positive definite matrix can always be written as a product of a negative definite matrix and -1.

From this point on we assume that

(H1) $\mathbf{S}$ is negative definite in (4.3),

$$\text{(H2)}\ \frac{\det \mathbf{G}}{\det \mathbf{M}} > -\mathbf{M}^{-1}\cdot\cdot\mathbf{S} + 2\sqrt{\frac{\det \mathbf{S}}{\det \mathbf{M}}}.$$

According to the proof of theorem 3.1, (H1) and (H2) imply system (4.3) has two different pairs of pure imaginary eigenvalues for $\varepsilon = 0$. We assume that $\varepsilon > 0$ is small, therefore the eigenvalues of (4.3) can be written in the form

$$\text{(H3)}\ \lambda_{1,2}(\varepsilon) = \alpha_1(\varepsilon) \pm i\omega_1(\varepsilon),$$

$$\lambda_{3,4}(\varepsilon) = \alpha_2(\varepsilon) \pm i\omega_2(\varepsilon), \quad \omega_1, \omega_2 > 0.$$

Finally, without loss of generality we can assume that

$$\text{(H4)}\ \omega_1(0) < \omega_2(0).$$

Theorem 4.2. Under hypotheses (H1)–(H4) the following are valid for the dissipative system (4.2) with $\varepsilon > 0$:
(i) the $\mathbf{q} = \mathbf{q}^0$ steady solution of (4.3) is unstable,
(ii) $\alpha_2 < 0 < \alpha_1$,
(iii) $|\alpha_1| < |\alpha_2|$.

Proof. Since

$$\mathbf{M}^{-1}\cdot\cdot\tilde{\mathbf{K}} = \frac{1}{\det \mathbf{M}}\mathbf{M}^{*}\cdot\cdot\tilde{\mathbf{K}} = \frac{1}{\det \mathbf{M}}\tilde{\mathbf{K}}^{*}\cdot\cdot\mathbf{M},$$

and $\det \mathbf{M}, \det \mathbf{G} > 0$, $\det \tilde{\mathbf{K}} \geq 0$, all the coefficients of (4.4) are positive with the exception of a_1 (see lemma 4.1), from which (i) immediately follows on the basis of the Routh–Hurwitz criterion. (Note that the proof of (i) requires (H1) and (H2) only. Actually, this result is known for higher dimensional systems as well (cf. e.g. [17]).)

Using the definition of $\tilde{\mathbf{K}}$ we can write (4.4) in the form

$$\lambda^4 + \varepsilon\frac{1}{\det \mathbf{M}}\mathbf{K}^{*}\cdot\cdot\mathbf{M}\lambda^3 + \left(\mathbf{M}^{-1}\cdot\cdot\mathbf{S} + \frac{\varepsilon^2 \det \mathbf{K} + \det \mathbf{G}}{\det \mathbf{M}}\right)\lambda^2$$

$$+ \varepsilon\frac{1}{\det \mathbf{M}}\mathbf{K}^{*}\cdot\cdot\mathbf{S}\lambda + \frac{\det \mathbf{S}}{\det \mathbf{M}} = 0. \tag{4.12}$$

Differentiating (4.12) with respect to ε, after some ordering, we obtain

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon} = -\frac{\mathbf{K}^{*}\cdot\cdot\mathbf{M}\lambda^3 + 2\varepsilon \det \mathbf{K}\lambda^2 + \mathbf{K}^{*}\cdot\cdot\mathbf{S}\lambda}{4\det \mathbf{M}\lambda^3 + 3\varepsilon\mathbf{K}^{*}\cdot\cdot\mathbf{M}\lambda^2 + 2(\mathbf{M}^{*}\cdot\cdot\mathbf{S} + \varepsilon^2\det \mathbf{K} + \det \mathbf{G})\lambda + \varepsilon\mathbf{K}^{*}\cdot\cdot\mathbf{S}}. \tag{4.13}$$

Taking (4.13) at $\varepsilon = 0+$ and substituting $\lambda(0) = i\omega(0) \neq 0$ and $\tilde{\mathbf{K}} = \varepsilon\mathbf{K}$, we are given the expression

$$\left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0+} = \frac{1}{2\det\mathbf{M}}\,\frac{\omega^2(0)\mathbf{K}^{*}\cdot\cdot\mathbf{M} - \mathbf{K}^{*}\cdot\cdot\mathbf{S}}{\dfrac{\det\mathbf{G}}{\det\mathbf{M}} + \mathbf{M}^{-1}\cdot\cdot\mathbf{S} - 2\omega^2(0)}. \tag{4.14}$$

Using lemma 4.1 we can see that the numerator of (4.14) is always positive, thus the sign of (4.14) depends on the sign of its denominator.

It is clear from (3.4) that the denominator of (4.14) could be zero only at the strong internal resonance $\omega = \omega_1(0) = \omega_2(0)$, but this is excluded by (H2). Therefore,

$$\left|\frac{d\lambda}{d\varepsilon}\Big|_{\varepsilon=0+}\right| < \infty$$

and

$$\left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0+} = \left.\frac{d\alpha}{d\varepsilon}\right|_{\varepsilon=0+} \begin{cases} >0, & \text{if } \omega < \omega^* \\ <0, & \text{if } \omega > \omega^* \end{cases} \tag{4.15}$$

where

$$\omega^* = \sqrt{\frac{1}{2}\left(\frac{\det\mathbf{G}}{\det\mathbf{M}} + \mathbf{M}^{-1}\cdot\cdot\mathbf{S}\right)}.$$

It follows from (4.15) that $\omega_1 < \omega^* < \omega_2$, and $\alpha_2 < 0 < \alpha_1$ for small ε, so we have proved (ii). But in (4.4) $a_3 > 0$, implying $\alpha_1 + \alpha_2 < 0$, which proves (iii).

Now we examine the effect of accelerating perturbation on a gyroscopically stabilized system of the form (1.6). We assume that the accelerating forces can be derived from an acceleration potential

$$D = \tfrac{1}{2}\dot{\mathbf{q}}^{T}\varepsilon\mathbf{B}(\mathbf{q})\dot{\mathbf{q}}, \quad \varepsilon < 0. \tag{4.16}$$

It follows from (4.16) that expressions (4.1)–(4.4) are still satisfied but with $\varepsilon < 0$.

Theorem 4.3. Under hypotheses (H1)–(H4) the following are valid for the accelerated system (4.2) with $\varepsilon < 0$:

(i) the $\mathbf{q} = \mathbf{q}^0$ steady solution of (4.3) is unstable,

(ii) $\alpha_2 > 0 > \alpha_1$,

(iii) $|\alpha_1| < |\alpha_2|$.

Proof. The theorem is a direct consequence of theorem 4.2 because $d\lambda/d\varepsilon$ is smooth in ε at $\varepsilon = 0$.

Universal losses of stability can be seen in Fig. 1 for the dissipative and for the accelerated system, i.e. for positive or negative ε respectively.

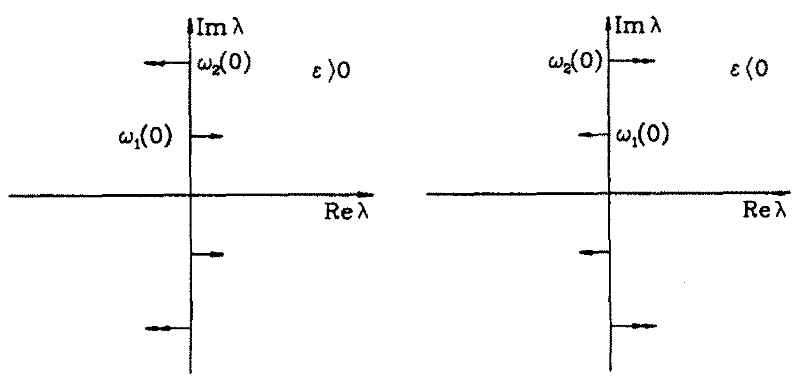

Fig. 1. Low and high frequency loss of gyroscopic stability for system (4.2).

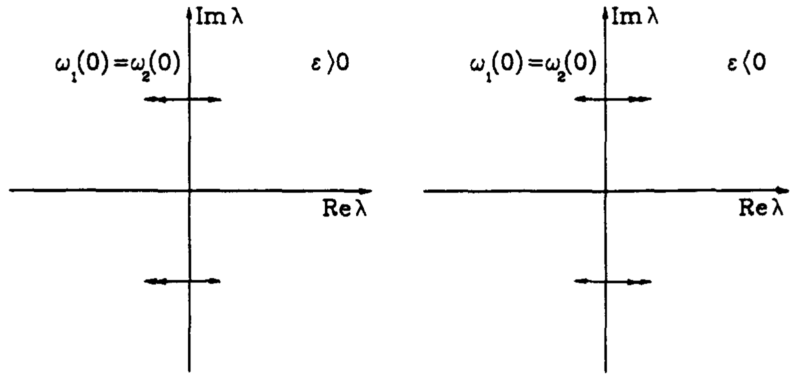

Fig. 2. Change of eigenvalues under perturbation at the strong internal resonance $\omega_1 = \omega_2$.

In case of the strong internal resonance $\omega_1 = \omega_2$ (see the proof of theorem 3.1) the displacement of eigenvalues from the imaginary axis is described by Fig. 2 for both the dissipative and the accelerated system.

It seems instructive to compare the results of theorem 4.2 and theorem 4.3. Dissipative forces cause a weak instability at lower frequencies, accelerating forces give rise to a stronger unstable behaviour at higher frequencies.

The majority of computer simulation programs and solving algorithms are not able to cope with the difficulty caused by very weak instability, and stabilize the motion showing endless bounded oscillations in the neighbourhood of the origin. This may suggest the presence of an attractor close to the origin in the non-linear system. Therefore, one must be careful when making a conclusion about attractors, because this suspection is based on bad simulation data, and the local non-existence of an attracting torus is simple to show in most cases. The real effect of dissipation is a slow, but definite way to unstable behaviour for practical values of damping.

5. AN EXAMPLE

We would like to demonstrate the results of the previous sections on the stability problem of a freely rotating flexible shaft–disc system. Our goal is to investigate the stability of the model above the critical angular velocity.

The physical model (cf. [13, 18]) is described in Fig. 3, which shows the disc from the direction of the theoretical axis of rotation. The disc, with mass m and radius R, is fixed on the shaft in its point D with excentricity e between its centre of gravity S and point D. For $\dot{c} = 0$, D falls in the line of the bearings, i.e. $D \equiv O$. For $\dot{c} \neq 0$ the shaft is flexibly bent which is modelized by a spring with stiffness s between O and D.

This system has two essential coordinates, q_1 and q_2, and one cyclic coordinate c. The equations of motion take the form

$$m\ddot{q}_1 - mq_1\left(\frac{p + J\dot{q}_2}{mq_1^2 + J}\right)^2 + s(q_1 - e\cos q_2) = 0,$$

$$\frac{mq_1^2 J}{mq_1^2 + J}\ddot{q}_2 + \frac{2mq_1\dot{q}_1 J(p + J\dot{q}_2)}{(mq_1^2 + J)^2} + seq_1 \sin q_2 = 0,$$

$$\dot{c} = \frac{p + J\dot{q}_2}{mq_1^2 + J},$$

$$\dot{p} = 0. \tag{5.1}$$

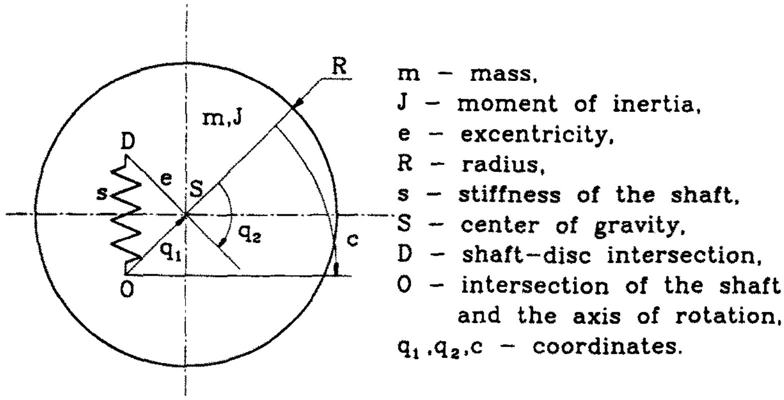

Fig. 3. Sketch of the mechanical model.

Introducing the critical angular velocity $\alpha = \sqrt{s/m}$ the steady motions of (5.1) above α are defined by

$$\dot{c} = \omega, \quad q_1^0 = \frac{\alpha^2 e}{\omega^2 - \alpha^2}, \quad q_2^0 = \pi. \tag{5.2}$$

Linearizing the first two equations of (5.1) along (5.2) we get a linear system of the form (1.8) with

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & \dfrac{mq_1^{0^2} J}{mq_1^{0^2} + J} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & -\dfrac{2mpq_1^{0^2} J}{(mq_1^{0^2} + J)^2} \\ \dfrac{2mpq_1^{0^2} J}{(mq_1^{0^2} + J)^2} & 0 \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} S + \dfrac{p^2(3mq_1^{0^2} - J)}{(mq_1^{0^2} + J)^2} & 0 \\ 0 & -seq_1^0 \end{pmatrix}.$$

Now we apply theorem 3.1 to system (5.1). From (i) of theorem 3.1 we get the condition

$$\omega > \omega_s, \tag{5.3}$$

where

$$\omega_s = \alpha \sqrt{1 + \sqrt[3]{\frac{4e^2}{R^2}\left(1 + \sqrt{1 - \frac{e^2}{2R^2}}\right)} + \sqrt[3]{\frac{4e^2}{R^2}\left(1 - \sqrt{1 - \frac{e^2}{2R^2}}\right)}}.$$

From condition (ii) of theorem 3.1 we obtain

$$\omega_s > \alpha \sqrt[4]{\frac{R^2 + e^2}{R^2}}. \tag{5.4}$$

It is easy to show that the validity of (5.3) implies (5.4) making (5.4) ignorable.

From condition (iii) of theorem 3.1, after some computation and ordering, we obtain the non-resonance conditions

$$(g_j - 1)y^4 + 4g_j y^3 + (4g_j + (3 - g_j)z)y^2 + 2z(2 - g_j)y + \tfrac{1}{4}g_j z^2 \neq 0, \quad j = 1, 2$$

with

$$g_1 = \frac{16}{25}, \quad g_2 = \frac{9}{25}, \quad z = \frac{2e^2}{R^2}, \quad y = \frac{\omega^2}{\alpha^2} - 1,$$

where the different values of g_j belong to the 1 : 2 and 1 : 3 resonances of the basic frequencies respectively. With the help of the sign rule of Descartes it can be shown that both internal

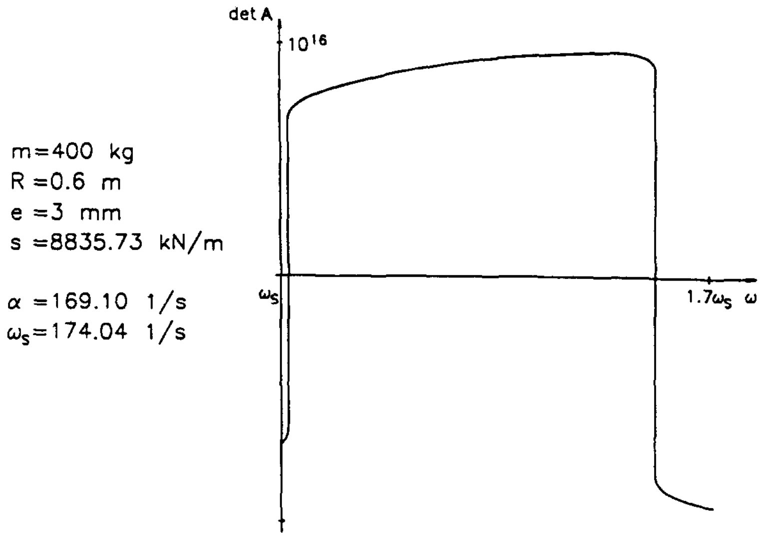

Fig. 4. The zeros of det A(ω).

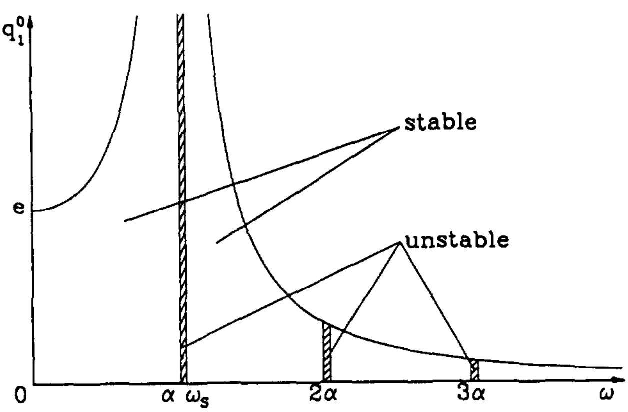

Fig. 5. Stability chart for system (Fig. 3).

resonances do occur for any choice of parameters, and the resonant angular velocities can be estimated as follows

$$c_1 \approx 3\alpha, \quad c_2 \approx 2\alpha. \tag{5.5}$$

Finally, if we check condition (iv) of theorem 3.1 by computer we will find that for fixed parameters and varying ω above α det A, as a function of ω, will have two zeros only (see Fig. 4). Hence, in accordance with remark 3.2, the degenerate angular velocities can be ignored.

In the view of (5.3), (5.5) and remark 3.3 a practical stability chart for system (5.1) can be seen in Fig. 5. The three bands of instability in Fig. 5 are of the same saddle type, but not of the same origin. The band at α results from the "linear instability" of the model and has been detected on the basis of the linearized system with **M**, **G** and **S**. The other two bands are due to possible "non-linear instability" on the basis of theorem 3.1. Here the instability of the rotor always means bounded motions in q_1, $\dot{q}_1$ and $\dot{q}_2$, but unbounded in q_2, as it can be concluded from the expression of the total energy of the system.

The results have been verified by computer simulation which showed deformed invariant tori in the neighbourhood of the origin of the reduced system for non-critical values of ω (see Fig. 6)

The effect of small dissipative perturbation (viscous damping on the spring) can be seen in Fig. 7. Here, as it was pointed out at the end of the last section, the numerical simulation

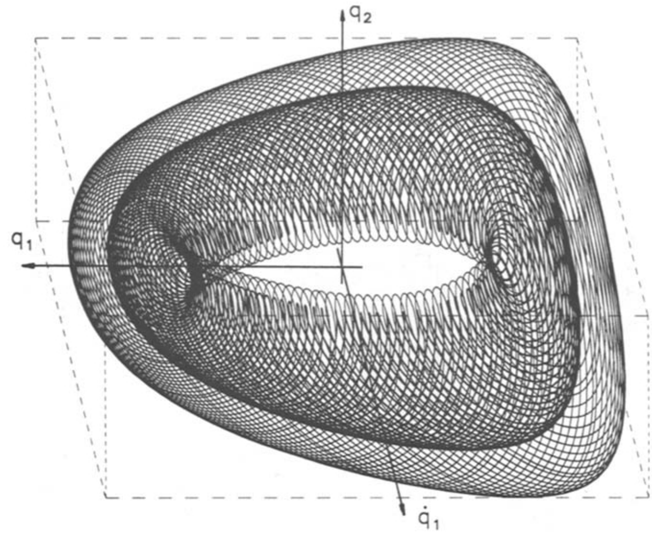

Fig. 6. Projection of two deformed invariant tori with the same energy onto the coordinate space $(q_1, \dot{q}_1, q_2)$. ($q_{1\max} = -q_{1\min} = 0.0018$ m, $\dot{q}_{1\max} = -\dot{q}_{1\min} = 0.2623$ m/s, $q_{2\max} = -q_{2\min} = 0.9810$, energy = 1895.56 kJ.)

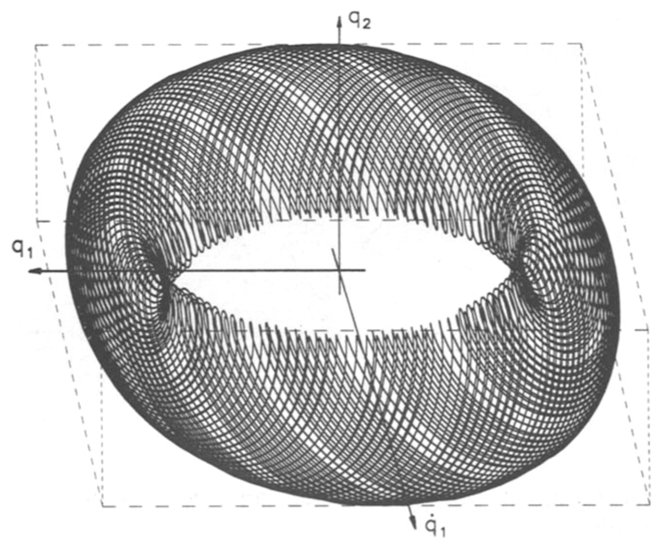

Fig. 7. Stable-like motion of (5.1) in case of small dissipative perturbation ($q_{1\max} = -\dot{q}_{1\min} =$ 0.0002 m, $\dot{q}_{1\max} = -\dot{q}_{1\min} = 0.0020$ m/s, $q_{2\max} = -q_{2\min} = 0.0071$, initial energy = 1895.56 kJ, $k = 10$ Ns/m).

shows steady oscillations near the unstable origin. In spite of this there is no attractor round the origin, as it can be proved via energetical considerations.

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APPENDIX

Here we give the necessary normal form coefficients for system (2.1) as functions of the coefficients of the standard form (2.2), in which $\mathbf{h}$ has the form

$$h^p = \sum_{j+k+l+m=2}^{\infty} h^p_{jklm} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m, \quad p = 1, 2,$$

where j, k, l and m are non-negative integers.

$$a_{11} = h^1_{1011} - i\Bigg(\frac{h^1_{1100}\bar{h}^1_{0011} + h^1_{0011}\bar{h}^2_{0110} - h^1_{0011}(2h^1_{2000} - h^2_{1010})}{\omega_1} + \frac{|h^1_{0101}|^2}{2\omega_1 + \omega_2} + \frac{|h^1_{0110}|^2}{2\omega_1 - \omega_2} + \frac{2h^1_{0002}\bar{h}^2_{0101}}{2\omega_2 + \omega_1} - \frac{2h^1_{0020}h^2_{1001}}{2\omega_2 - \omega_1} + \frac{h^1_{1001}\bar{h}^2_{0011} - h^1_{1010}h^2_{0011}}{\omega_2}\Bigg),$$

$$a_{12} = h^1_{2100} + i\Bigg(\frac{h^1_{1100}h^1_{2000} + 2h^1_{0200}\bar{h}^1_{2000} - |h^1_{1100}|^2}{\omega_1} + \frac{h^1_{0101}\bar{h}^2_{2000} - h^1_{0110}h^2_{2000}}{2\omega_1 - \omega_2} + \frac{h^1_{1010}h^2_{1100} - h^1_{1001}\bar{h}^2_{1100}}{\omega_2}\Bigg),$$

$$a_{21} = h^2_{0021} + i\Bigg(\frac{h^2_{0011}h^2_{0020} + 2h^2_{0002}\bar{h}^2_{0020} - |h^2_{0011}|^2}{\omega_2} + \frac{h^2_{0101}\bar{h}^1_{0020} - h^2_{1001}h^1_{0020}}{2\omega_2 - \omega_1} + \frac{h^2_{1010}h^1_{0011} - h^2_{0110}\bar{h}^1_{0011}}{\omega_1}\Bigg),$$

$$a_{22} = h^2_{1110} - i\Bigg(\frac{h^2_{0011}\bar{h}^2_{1100} + h^2_{1100}\bar{h}^1_{1001} - h^2_{1100}(2h^2_{0020} - h^1_{1010})}{\omega_2} + \frac{|h^2_{0101}|^2}{2\omega_2 + \omega_1} + \frac{|h^2_{1001}|^2}{2\omega_2 - \omega_1} + \frac{2h^2_{0200}\bar{h}^1_{0101}}{2\omega_1 + \omega_2} - \frac{2h^2_{2000}h^1_{0110}}{2\omega_1 - \omega_2} + \frac{h^2_{0110}\bar{h}^1_{1100} - h^2_{1010}h^1_{1100}}{\omega_1}\Bigg).$$