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Erratum

Erratum and addendum to “A variational theory of hyperbolic Lagrangian coherent structures” [Physica D 240 (2011) 574–598]

Mohammad Farazmand [a,*], George Haller [b,a]

[a] Department of Mathematics and Statistics, McGill University, 817 Sherbrooke St. West, Montreal, QC H3A 2K6, Canada

[b] Department of Mechanical Engineering, McGill University, 817 Sherbrooke St. West, Montreal, QC H3A 2K6, Canada

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ABSTRACT

This brief note corrects a minor error in the statement of the main result in Haller (2011) [1] on a variational approach to Lagrangian coherent structures. We also show that the corrected formulation leads to a substantial simplification of LCS criteria for two-dimensional flows.

1. The set-up and notation

We first recall the set-up and the main definitions from [1]. Consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t), \qquad \mathbf{x} \in U \subset \mathbb{R}^n, \quad t \in [\alpha, \beta], \tag{1}$$

where U is an open, bounded subset of $\mathbb{R}^n$, $[\alpha, \beta] \subset \mathbb{R}$ is a bounded time interval and $\mathbf{v} : U \times [\alpha, \beta] \to \mathbb{R}^n$ is a class C^3 vector field in its arguments.

Let $\mathbf{x}(t, t_0, \mathbf{x}_0)$ denote a trajectory of (1) passing the point $\mathbf{x}_0 \in U$ at time $t_0 \in [\alpha, \beta]$. The flow map $\mathbf{F}_{t_0}^{t}(\mathbf{x}_0)$ takes an initial condition $\mathbf{x}_0$ to its position at time t, i.e.,

$$\mathbf{F}_{t_0}^{t} : U \to U,$$
$$\mathbf{x}_0 \mapsto \mathbf{x}(t, t_0, \mathbf{x}_0).$$

The Cauchy–Green strain tensor field associated with the flow map is defined as

$$\mathbf{C}_{t_0}^{t_0+T}(\mathbf{x}_0) = \left(\nabla \mathbf{F}_{t_0}^{t_0+T}(\mathbf{x}_0)\right)^* \nabla \mathbf{F}_{t_0}^{t_0+T}(\mathbf{x}_0), \tag{2}$$

where $\nabla \mathbf{F}_{t_0}^{t_0+T}$ is the Jacobian of the flow map, and the star refers to matrix transposition. The eigenvalues λ_i and corresponding unit eigenvectors $\boldsymbol{\xi}_i$ of the symmetric, positive definite tensor $\mathbf{C}_{t_0}^{t_0+T}$ are defined by the relations

$$\mathbf{C}_{t_0}^{t_0+T}(\mathbf{x}_0)\boldsymbol{\xi}_i(\mathbf{x}_0, t_0, T) = \lambda_i(\mathbf{x}_0, t_0, T)\boldsymbol{\xi}_i(\mathbf{x}_0, t_0, T),$$
$$|\xi_i(\mathbf{x}_0, t_0, T)| = 1, \quad i = 1, 2, \dots, n,$$
$$0 < \lambda_1(\mathbf{x}_0, t_0, T) \le \lambda_2(\mathbf{x}_0, t_0, T) \le \cdots \le \lambda_n(\mathbf{x}_0, t_0, T).$$

1.1. LCS as special material surfaces

Consider a smooth curve $\mathcal{M}(t_0)$ at time t_0, which is advected by the flow map into a time-evolving material surface $\mathcal{M}(t) = F_{t_0}^{t}(\mathcal{M}(t_0))$. For each point $\mathbf{x}_0 \in \mathcal{M}(t_0)$, we denote the tangent space of the initial material curve by $T_{\mathbf{x}_0}\mathcal{M}(t_0)$, and its normal space by $N_{\mathbf{x}_0}\mathcal{M}(t_0)$.

As argued in [1], over a time interval of length T, an initially normal unit perturbation $\mathbf{n}_0 \in N_{\mathbf{x}_0}\mathcal{M}(t_0)$ is advected by the linearized flow map into the vector $\nabla \mathbf{F}_{t_0}^{t_0+T}(\mathbf{x}_0)\mathbf{n}_0$. The surface-normal component of this advected vector is given by the *normal repulsion rate*

$$\rho_{t_0}^{t_0+T}(\mathbf{x}_0, \mathbf{n}_0) = \left\langle \mathbf{n}_t, \nabla \mathbf{F}_{t_0}^{t_0+T}(\mathbf{x}_0)\mathbf{n}_0 \right\rangle, \tag{3}$$

where $\mathbf{n}_t$ is a unit vector normal to $\mathcal{M}(t)$ at the point $\mathbf{x}_t$. The normal repulsion rate can be expressed in terms of the Cauchy–Green strain tensor as

$$\rho_{t_0}^{t_0+T}(\mathbf{x}_0, \mathbf{n}_0) = \frac{1}{\sqrt{\left\langle \mathbf{n}_0, \left[\mathbf{C}_{t_0}^{t_0+T}(\mathbf{x}_0)\right]^{-1} \mathbf{n}_0 \right\rangle}}. \tag{4}$$

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* Corresponding author.
E-mail address: faraz@math.mcgill.ca (M. Farazmand).

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Normally repelling material lines are defined in [1] as compact material lines on which $\rho_{t_0}^{t_0+T}(\mathbf{x}_0, \mathbf{n}_0) > 1$ holds, and the normal repulsion rate $\rho_{t_0}^{t_0+T}(\mathbf{x}_0, \mathbf{n}_0)$ dominates any growth of tangent vectors in $T_{\mathbf{x}_0}\mathcal{M}(t_0)$ under the linearized flow map $\nabla\mathbf{F}_{t_0}^{t_0+T}(\mathbf{x}_0)$.

As proved in [1], for large T values, normally repelling material surfaces are rare and tightly packed: if one exists at a point x_0, then it must be exponentially close to the subspace spanned by the first $n-1$ eigenvectors of the Cauchy–Green strain tensor. Out of these rare surfaces, the variational theory of LCS seeks those exceptional ones that repel nearby trajectories at locally the highest rate in the flow. To characterize such material surfaces, we recall the following definitions from [1]:

Definition 1 (*Hyperbolic Weak LCS*)**.** Consider a compact normally repelling material surface $\mathcal{M}(t) \in U$. We call $\mathcal{M}(t)$ a repelling weak LCS (WLCS) over $[t_0, t_0+T]$ if its normal repulsion rate admits stationary values along $\mathcal{M}(t_0)$ among all locally C^1-close material surfaces. We call $\mathcal{M}(t)$ an attracting WLCS over $[t_0, t_0+T]$ if it is a repelling WLCS over $[t_0, t_0+T]$ in backward time. Finally, we call $\mathcal{M}(t)$ a hyperbolic WLCS over $[t_0, t_0+T]$ if it is a repelling or attracting WLCS over the same time interval.

Definition 2 (*Hyperbolic LCS*)**.** Consider a compact, normally repelling material surface $\mathcal{M}(t) \in U$. We call $\mathcal{M}(t)$ a repelling LCS over $[t_0, t_0+T]$ if its normal repulsion rate admits a pointwise non-degenerate maximum along $\mathcal{M}(t_0)$ among all locally C^1-close material surfaces. We call $\mathcal{M}(t)$ an attracting LCS over $[t_0, t_0+T]$ if it is a repelling LCS over $[t_0, t_0+T]$ in backward time. Finally, we call $\mathcal{M}(t)$ a hyperbolic LCS over $[t_0, t_0+T]$ if it is a repelling or attracting LCS over the same time interval.

1.2. The existence theorem for LCS

We now re-state the main existence result, Theorem 7, from [1] with a slight correction.

Theorem 1 (*Sufficient and Necessary Conditions for WLCS and LCS*)**.** *Consider a compact material surface $\mathcal{M}(t) \subset U$ over the interval $[t_0, t_0+T]$. Then:*

(i) *$\mathcal{M}(t)$ is a repelling weak LCS (WLCS) over $[t_0, t_0+T]$ if and only if all the following hold for all $\mathbf{x}_0 \in \mathcal{M}(t_0)$:*
   1. $\lambda_{n-1}(\mathbf{x}_0, t_0, T) \neq \lambda_n(\mathbf{x}_0, t_0, T) > 1$;
   2. $\boldsymbol{\xi}_n(\mathbf{x}_0, t_0, T) \perp T_{\mathbf{x}_0}\mathcal{M}(t_0)$;
   3. $\langle \nabla\lambda_n(\mathbf{x}_0, t_0, T), \boldsymbol{\xi}_n(\mathbf{x}_0, t_0, T)\rangle = 0$.

(ii) *$\mathcal{M}(t)$ is a repelling LCS over $[t_0, t_0+T]$ if and only if:*
   1. *$\mathcal{M}(t)$ is a repelling WLCS over $[t_0, t_0+T]$;*
   2. *The matrix $\mathbf{L}(\mathbf{x}_0, t_0, T)$ is positive definite for all $\mathbf{x}_0 \in \mathcal{M}(t_0)$ with the definition given in* Box I.

Proof. This theorem is stated and proved in an almost identical form in [1]. The only difference in the present re-statement is the form of the diagonal terms $2\frac{\lambda_n-\lambda_1}{\lambda_1\lambda_n}$ of the matrix $\mathbf{L}$ defined in (5), which are listed incorrectly in [1] as $\frac{2\lambda_n-\lambda_1}{\lambda_1\lambda_n}$. Here we only outline the corrections to the proof, and refer the reader to [1] for the full proof.

The correct form of eq. 52 in [1] is

$$\begin{aligned}
&[C^{-1}_{ij,k} n^i_\epsilon n^j_\epsilon (x^k_\epsilon)' + 2C^{-1}_{ij}(n^i_\epsilon)' n^j_\epsilon]'_{\epsilon=0} = \cdots \\
&= \alpha^2 C^{-1}_{ij,kl}\xi^i_n\xi^j_n\xi^k_n\xi^l_n - 4\alpha\alpha_{,p}C^{-1}_{ij,k}e^i_p\xi^j_n\xi^k_n \\
&\quad + \mathbf{4}C^{-1}_{ij}\left(-\frac{1}{2}\alpha_{,p}\alpha_{,p}\xi^i_n - \alpha\xi^k_{n,p}\beta^k e^i_p\right)\xi^j_n + 2\alpha_{,p}\alpha_{,q}C^{-1}_{ij}e^i_p e^j_q \\
&= \alpha^2 C^{-1}_{ij,kl}\xi^i_n\xi^j_n\xi^k_n\xi^l_n - 4\alpha\alpha_{,p}C^{-1}_{ij,k}\xi^i_p\xi^j_n\xi^k_n \\
&\quad - \mathbf{2}\frac{\alpha_{,p}\alpha_{,p}}{\lambda_n} - \frac{\mathbf{4}\alpha}{\lambda_n}\xi^k_{n,p}\beta^k\xi^i_p\xi^i_n + 2\frac{\alpha_{,p}\alpha_{,p}}{\lambda_p} \\
&= \alpha^2 C^{-1}_{ij,kl}\xi^i_n\xi^j_n\xi^k_n\xi^l_n - 4\alpha\alpha_{,p}C^{-1}_{ij,k}\xi^i_p\xi^j_n\xi^k_n \\
&\quad + \alpha_{,p}\alpha_{,p}\left[\frac{2}{\lambda_p} - \frac{\mathbf{2}}{\lambda_n}\right],
\end{aligned}$$

where we have highlighted the corrected coefficients in boldface. As a result, in equations (31) and (53) of [1], the diagonal terms of the matrix $\mathbf{L}$ (except the first one) will change to

$$2\frac{\lambda_n - \lambda_p}{\lambda_p\lambda_n},$$

as stated in the present theorem. □

2. The case of two-dimensional flows

Following the argument of Tang et al. [2], we now show that the corrected condition (ii)/2 of Theorem 1 can be simplified significantly in the case of two-dimensional flows. We use the following result from [1]:

Lemma 1. *At each point of a WLCS the following identity holds:*

$$\nabla^2\mathbf{C}^{-1}[\boldsymbol{\xi}_n, \boldsymbol{\xi}_n, \boldsymbol{\xi}_n, \boldsymbol{\xi}_n] = -\frac{1}{\lambda_n^2}\left\langle\boldsymbol{\xi}_n, \nabla^2\lambda_n\boldsymbol{\xi}_n\right\rangle + 2\sum_{q=1}^{n-1}\frac{\lambda_n-\lambda_q}{\lambda_n\lambda_q}\left\langle\boldsymbol{\xi}_q, \nabla\boldsymbol{\xi}_n\boldsymbol{\xi}_n\right\rangle^2. \tag{6}$$

Proof. See Theorem 7 in [1] for a proof. □

Applying this lemma to two-dimensional flows, we obtain the following result.

Theorem 2 (*Sufficient and Necessary Conditions for WLCS and LCS in Two Dimensions*)**.** *Consider a compact material line $\mathcal{M}(t) \subset U$ over the interval $[t_0, t_0+T]$. Then:*

(i) *$\mathcal{M}(t)$ is a repelling weak LCS (WLCS) over $[t_0, t_0+T]$ if and only if all the following hold for all $\mathbf{x}_0 \in \mathcal{M}(t_0)$:*
   1. $\lambda_1(\mathbf{x}_0, t_0, T) \neq \lambda_2(\mathbf{x}_0, t_0, T) > 1$;
   2. $\boldsymbol{\xi}_2(\mathbf{x}_0, t_0, T) \perp T_{\mathbf{x}_0}\mathcal{M}(t_0)$;
   3. $\langle\nabla\lambda_2(\mathbf{x}_0, t_0, T), \boldsymbol{\xi}_2(\mathbf{x}_0, t_0, T)\rangle = 0$.

(ii) *$\mathcal{M}(t)$ is a repelling LCS over $[t_0, t_0+T]$ if and only if:*
   1. *$\mathcal{M}(t)$ is a repelling WLCS over $[t_0, t_0+T]$;*
   2. $\left\langle\boldsymbol{\xi}_2, \nabla^2\lambda_2\boldsymbol{\xi}_2\right\rangle < 0$.

Proof. We only need to show that condition (ii)/2 of the present theorem is equivalent to condition (ii)/2 of Theorem 1. By Sylvester's theorem, the matrix $\mathbf{L}(\mathbf{x}_0, t_0, T)$ defined in Theorem 1 is positive definite if and only if all the leading principal minors of $\mathbf{L}$ are positive. In the case of $n = 2$, this amounts to the two requirements

$$\nabla^2\mathbf{C}^{-1}[\boldsymbol{\xi}_2, \boldsymbol{\xi}_2, \boldsymbol{\xi}_2, \boldsymbol{\xi}_2] > 0, \tag{7a}$$

$$\det\mathbf{L} > 0. \tag{7b}$$

By Lemma 1, the inequality (7a) is equivalent to

$$\left\langle\boldsymbol{\xi}_1, \nabla\boldsymbol{\xi}_2\boldsymbol{\xi}_2\right\rangle^2 > \frac{\lambda_1}{2(\lambda_2-\lambda_1)\lambda_2}\left\langle\boldsymbol{\xi}_2, \nabla^2\lambda_2\boldsymbol{\xi}_2\right\rangle. \tag{8}$$

Again by Lemma 1 and a straightforward calculation, the inequality (7b) is equivalent to

$$-2\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2^3}\left\langle\boldsymbol{\xi}_2, \nabla^2\lambda_2\boldsymbol{\xi}_2\right\rangle > 0.$$

Since $0 < \lambda_1 \leq \lambda_2$, this last inequality is in turn equivalent to

$$\mathbf{L}(\mathbf{x}_0, t_0, T) = \begin{pmatrix} \nabla^2\mathbf{C}^{-1}[\boldsymbol{\xi}_n, \boldsymbol{\xi}_n, \boldsymbol{\xi}_n, \boldsymbol{\xi}_n] & 2\dfrac{\lambda_n - \lambda_1}{\lambda_1\lambda_n}\langle \boldsymbol{\xi}_1, \nabla\boldsymbol{\xi}_n\boldsymbol{\xi}_n\rangle & \cdots & 2\dfrac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}\lambda_n}\langle \boldsymbol{\xi}_{n-1}, \nabla\boldsymbol{\xi}_n\boldsymbol{\xi}_n\rangle \\ 2\dfrac{\lambda_n - \lambda_1}{\lambda_1\lambda_n}\langle \boldsymbol{\xi}_1, \nabla\boldsymbol{\xi}_n\boldsymbol{\xi}_n\rangle & 2\dfrac{\lambda_n - \lambda_1}{\lambda_1\lambda_n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2\dfrac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}\lambda_n}\langle \boldsymbol{\xi}_{n-1}, \nabla\boldsymbol{\xi}_n\boldsymbol{\xi}_n\rangle & 0 & \cdots & 2\dfrac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}\lambda_n} \end{pmatrix} \quad (5)$$

Box I.

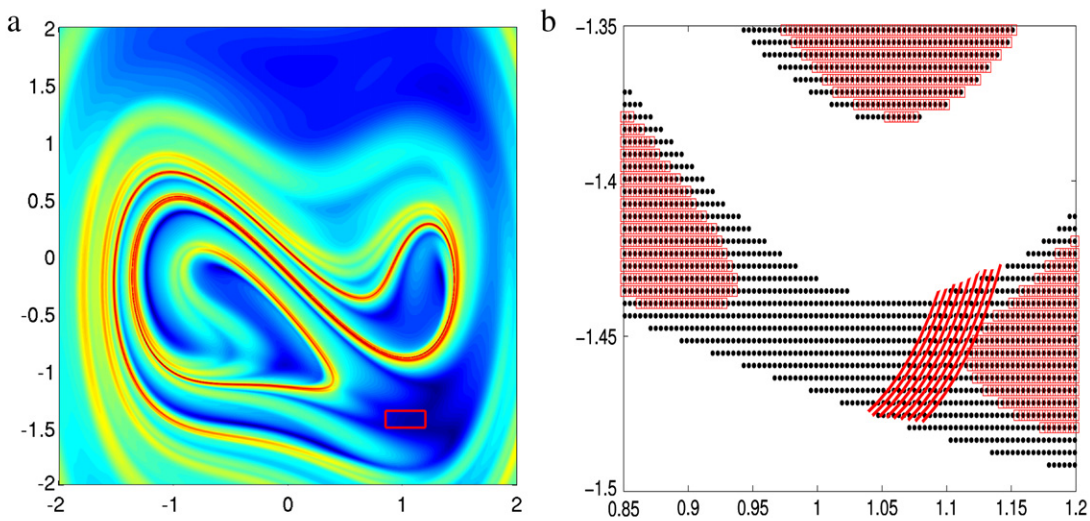

Fig. 1. (a) The forward-time FTLE field for the forced Duffing Eq. (10) with $\epsilon = 0.3$ and $\omega = 1$. The rectangle marks a trench of the FTLE field. (b) A close-up of the area marked by the rectangle in part (a). The sets $\mathcal{L}_w$ (black dots) and $\mathcal{L}$ (red squares) are shown. The red curves correspond to the material lines that qualify as LCS through the incorrect matrix $\mathbf{L}_w$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\left\langle \boldsymbol{\xi}_2, \nabla^2\lambda_2\boldsymbol{\xi}_2 \right\rangle < 0. \quad (9)$$

Now, since $\left\langle \boldsymbol{\xi}_1, \nabla\boldsymbol{\xi}_2\boldsymbol{\xi}_2 \right\rangle^2 \geq 0$, inequality (8) follows from inequality (9). Hence, $\mathbf{L}$ being positive definite is equivalent to (9). □

3. An example

In this section, we demonstrate that the incorrect form of the matrix $\mathbf{L}(\mathbf{x}_0, t_0, T)$ given in [1] may identify some non-hyperbolic material lines as hyperbolic LCS, while the corrected form (5) rules out such non-hyperbolic material lines.

Consider the periodically forced Duffing equation given by

$$\ddot{x} - x + x^3 = \epsilon \cos(\omega t) \quad (10)$$

with $\epsilon = 0.3$ and $\omega = 1$. The geometry of the stable and unstable manifolds of this system is well known (see, e.g., [3]). As shown in, e.g., [4], for large enough integration times T, compact subsets of the stable manifold are repelling LCS that are closely approximated by the ridges of the finite-time Lyapunov exponent (FTLE) field

$$\Lambda_{t_0}^{t_0+T}(\mathbf{x}_0) = \frac{1}{2|T|}\log\lambda_2(\mathbf{x}_0, t_0, T). \quad (11)$$

The FTLE field for the system (10) is shown in Fig. 1a for $T = 3\pi$.

Denote the incorrect form of the matrix $\mathbf{L}$ given in [1] by $\mathbf{L}_w$, and the correct form given in (5) above by $\mathbf{L}$. Furthermore, define $\mathcal{L}_w$ and $\mathcal{L}$ as the sets of points where $\mathbf{L}_w$ and $\mathbf{L}$ are positive definite, respectively.

Fig. 1b shows a close-up of a trench of the FTLE field together with the sets $\mathcal{L}$ (red squares) and $\mathcal{L}_w$ (black dots). Note that the set $\mathcal{L}$ appears as a subset of $\mathcal{L}_w$. The red curves in Fig. 1b are chosen everywhere orthogonal to the second eigenvector $\boldsymbol{\xi}_2$ of the Cauchy–Green strain tensor, and hence satisfy condition (i)/2 of Theorem 1. The remaining conditions of part (i) of the theorem are also satisfied at each point of these curves, with condition (i)/3 relaxed to the inequality $|\langle \nabla\lambda_2, \boldsymbol{\xi}_2\rangle| < 0.1$ to accommodate LCS of finite thickness (see Haller [1] and Farazmand and Haller [5]). Hence, if the incorrect matrix $\mathbf{L}_w$ is used in checking the condition (ii)/2 of Theorem 1, these curves qualify as repelling LCS.

However, a trench of the FTLE field indicates material lines that repel other material lines at locally the weakest rate, and hence cannot be LCS. The correct form (5) of the matrix $\mathbf{L}$ does indeed exclude such material lines from the set of admissible hyperbolic LCS since they do not intersect the set $\mathcal{L}$.

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