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Physica D 194 (2004) 297–319
PHYSICA D
www.elsevier.com/locate/physd

Inertial manifolds and completeness of eigenmodes for unsteady magnetic dynamos

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Accepted 16 March 2003
Communicated by U. Frisch

Abstract

We prove the emergence of asymptotic spatial patterns in magnetic dynamos generated by unsteady fluid flows. The patterns emerge because solutions of the dynamo equation converge exponentially to a time-dependent inertial manifold. This inertial manifold exists for general time-aperiodic velocity fields under a spectral gap condition on the associated Stokes operator. For time-periodic velocity fields, we show that the inertial manifold is spanned by Floquet eigenmodes that are analogous to the *strange eigenmodes* observed in the mixing of diffusive tracers. This result gives an affirmative answer to the long-standing question of completeness of Floquet solutions in time-periodic dynamo problems.

Keywords: Magnetic dynamos; Eigenmodes; Inertial manifold

1. Introduction

1.1. The dynamo equation

The magnetic fields of cosmic bodies, such as the Sun and the Earth, are generated and maintained against dissipation by dynamo action in the electrically conducting fluid inside these bodies. The governing equation for such magnetic fields is the dynamo equation

$$\mathbf{B}_t + (\mathbf{v} \cdot \nabla)\mathbf{B} = \eta\nabla^2\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{v}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x}), \tag{1}$$

where $\mathbf{B}(\mathbf{x}, t)$ is the magnetic field, η the magnetic diffusivity, and $\mathbf{v}(\mathbf{x}, t)$ the velocity field of the fluid that satisfies the incompressibility condition

$$\nabla \cdot \mathbf{v} = \mathbf{0} \tag{2}$$

on a three-dimensional spatial domain Ω (see, e.g. [2, Chapter 1]).

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0167-2789/$ – see front matter
doi:10.1016/j.physd.2003.03.002

The growth or decay of the field **B** is usually quantified by the exponent

$$\gamma(\eta) = \sup_{\mathbf{B}_0} \limsup_{t\to\infty} \frac{1}{2t} \ln E(t),$$

with the magnetic energy $E(t)$ defined as

$$E(t) = \tfrac{1}{2} \int_{\Omega} |\mathbf{B}|^2 \, \mathrm{d}V.$$

The velocity field **v** is a *kinematic dynamo* (or, simply, a *dynamo*) if $\gamma(\eta) > 0$ for some $\eta > 0$. In other words, **v** is a dynamo if it generates exponentially growing magnetic energy for a nonzero value of the magnetic diffusivity. If $\gamma(\eta)$ remains positive and bounded away from zero as $\eta \to 0$, then the dynamo is called *fast*, otherwise it is called *slow* (see [25] or [2]).

1.2. Eigenmodes in the dynamo equation

Special solutions to (1) are often sought in the form

$$\mathbf{B}(\mathbf{x}, t) = \mathrm{e}^{\lambda t}\, \mathbf{e}(\mathbf{x}, t), \tag{3}$$

with the expectation that a general solution can be represented as

$$\mathbf{B}(\mathbf{x}, t) = \sum_{k=1}^{\infty} c_k\, \mathrm{e}^{\lambda_k t}\, \mathbf{e}_k(\mathbf{x}, t). \tag{4}$$

If such an infinite expansion exists, then estimating the growth rate of the magnetic field simplifies to studying the exponents λ_k.

The above approach has been widely employed in experimental and numerical dynamo studies. For instance, Gailitis et al. [6] assume a solution

$$\mathbf{B}(t) = \mathbf{a}_1\, \mathrm{e}^{\lambda_1 t} \sin(2\pi f_1 t + \phi_1) + \mathbf{a}_2\, \mathrm{e}^{\lambda_2 t} \sin(2\pi f_2 t + \phi_2)$$

for the case of a helical flow generated by a propeller in a closed volume of molten sodium. Peffley et al. [18] postulate the infinite expansion (4), and estimate the leading eigenvalue λ_1 from their experiments. Dudley and James [5] show that a number of stationary velocity models lead to positive λ_k exponents. For a time-periodic version of the classic Ponomarenko dynamo (see [20]), Normand [16] assume a solution of the form (3) and determine λ numerically.

In earlier work, Otani [17] observed a *wild eigenmode* (a solution (3) with a spatially complicated $\mathbf{e}(\mathbf{x}, t)$) for a dynamo with the stretch-fold-shear (SFS) mechanism. Later, Kaiser [9] used the form (4) to prove that a kinematic dynamo cannot exist for a purely poloidal magnetic field. These and additional uses of the eigenmode ansatz are surveyed by Childress and Gilbert [2].

For a steady velocity field $\mathbf{v}(\mathbf{x})$ and for $\eta > 0$, the ansatz (4) is justifiable: the spectrum of the magnetic field consists of a countable number of eigenvalues λ_k, each of finite multiplicity. The corresponding eigenfunctions $\mathbf{e}_k(\mathbf{x})$ are known to be complete in L^2.

Most of the studies cited above, however, are concerned with dynamo action generated by time-periodic velocity fields. For such fields, the existence of *some* eigenmodes of the form (3) follows from the results in Yudovich [24], but the completeness of those eigenmodes has been an open question.

Without a completeness result on the eigenmodes, all prior work that assumes the ultimate prevalence of a single growing eigenmode is on loose mathematical ground. This is because in the absence of completeness, one cannot view a general solution as a finite linear combination of eigenmodes plus a small error term; that error term may very well be large, invalidating any conclusion obtained from the eigenmodes. This lack of completeness for time-periodic flows, Childress and Gilbert [2] observe, has also hindered the study of fast dynamo action with arbitrary time dependence.

1.3. Results

This paper fills the above theoretical gap by proving the completeness of eigenmodes of the form (3) for time-periodic velocity fields. We obtain this completeness result under conditions on the spectrum of a Stokes operator associated with the dynamo equation. These conditions hold for two- and three-dimensional geometries, such as rectangles and cubes, but typically place bounds on the magnitude of the velocity field $\mathbf{v}$ once the magnetic diffusivity η is fixed.

More specifically, we first prove the existence of an inertial manifold for (1) in the case when $\mathbf{v}(\mathbf{x}, t)$ is a bounded velocity field with *arbitrary time dependence*. This implies that solutions of the dynamo equation converge to those of a time-dependent linear system of ordinary differential equations (ODEs). A set of fundamental solutions to this ODE then serves as an asymptotically emerging set of aperiodic eigenmodes for the dynamo equation. Out of these aperiodic eigenmodes, the fastest growing one will dominate for fast dynamos.

If $\mathbf{v}(\mathbf{x}, t)$ is continuous and periodic in time, then so is the ODE on the inertial manifold. As a result, for arbitrary small $\varepsilon > 0$, classic finite-dimensional Floquet theory guarantees an asymptotic expansion

$$\mathbf{B}(\mathbf{x}, t) = \sum_{k=1}^{N} c_k \, \mathrm{e}^{\lambda_k t}[\mathbf{b}_k^0(\mathbf{x}, t) + t\mathbf{b}_k^1(\mathbf{x}, t) + \cdots + t^{l(k)}\mathbf{b}_k^{l(k)}(\mathbf{x}, t)] + \mathbf{R}(\mathbf{x}, t), \qquad \|\mathbf{R}(t)\|_{H^1} \leq \varepsilon \exp[-\nu t],$$

where $\mathbf{b}_k^j(\mathbf{x}, t)$ are functions that are T-periodic in time and only depend on the velocity field $\mathbf{v}(\mathbf{x}, t)$ and the domain Ω. The constants c_k depend on the initial condition $\mathbf{B}_0(\mathbf{x})$; the index $l(k) \geq 0$ is an integer-valued function of k; the integer $N \geq 1$ and the constant $\nu > 0$ both depend on ε. It follows that any fast dynamo action is necessarily confined to the inertial manifold.

If the Floquet exponents of the ODE on the inertial manifold are all simple, then $l(k) = 0$ for all k. In that case, the eigenmode with the largest λ_k prevails asymptotically. We show that such an eigenmodes generates the Floquet solution

$$\mathbf{B}_\infty(\mathbf{x}, t) = \mathrm{e}^{(\alpha+\mathrm{i}\beta)t} \, \mathbf{e}_0(\mathbf{x}, t), \tag{5}$$

where the complex function $\mathbf{e}_0(\mathbf{x}, t)$ is T-periodic in time, and the real constants α and β satisfy

$$\alpha = \frac{-\eta\overline{\|\nabla \mathbf{e}_0\|^2} + \mathrm{Re}\,\overline{\langle \mathbf{e}_0, (\mathbf{e}_0^* \cdot \nabla)\mathbf{v}\rangle}}{\overline{\|\mathbf{e}_0\|^2}}, \qquad \beta = \frac{\alpha\overline{\|\mathrm{Re}\,\mathbf{e}_0\|^2} + \eta\overline{\|\mathrm{Re}\,\nabla\mathbf{e}_0\|^2} - \overline{\langle \mathrm{Re}\,\mathbf{e}_0, (\mathrm{Re}\,\mathbf{e}_0 \cdot \nabla)\mathbf{v}\rangle}}{\overline{\langle \mathrm{Re}\,\mathbf{e}_0, \mathrm{Im}\,\mathbf{e}_0\rangle}},$$

with the star referring to complex conjugation, the overbar denoting averaging over one time-period, and $\langle\cdot, \cdot\rangle$ denoting L^2 inner product (for the precise definition, see (9)). As formula (5) shows, $\mathbf{B}_\infty(\mathbf{x}, t)$ is time-periodic if $2\pi/\rho$ and T are rationally dependent, and is quasiperiodic otherwise.

These results establish a close relationship between the *strange eigenmodes* observed by Pierrehumbert [19] for the time-periodic advection–diffusion equation, and the wild eigenmodes described by Otani [17] and Childress and Gilbert [2] for the dynamo equation. In particular, both types of complex recurrent patterns arise from convergence to Floquet eigenmodes on an inertial manifold (cf. [14]).

2. Notation and definitions

In this section we collect the main ingredients we need to state our main result.

2.1. Boundary conditions and function spaces

We assume the spatially periodic boundary conditions

$$\mathbf{v}(\mathbf{x}+\mathbf{L},t)=\mathbf{v}(\mathbf{x},t),\qquad \mathbf{B}(\mathbf{x}+\mathbf{L},t)=\mathbf{B}(\mathbf{x},t) \tag{6}$$

for Eq. (1). Here $\mathbf{L}=(lL_1,mL_2,nL_3)$ is an arbitrary vector with integers (l,m,n) and real numbers L_i. We shall use the notation $\Omega=[0,L_1]\times[0,L_2]\times[0,L_3]\subset\mathbb{R}^3$ for the basic three-dimensional cell over which $\mathbf{v}$ and $\mathbf{B}$ are defined. The measure (area or volume) of Ω is given by

$$\kappa=\mathrm{mes}(\Omega)=\int_\Omega \mathrm{d}V.$$

Appended with the boundary conditions, the dynamo equation (1) assumes the form

$$\mathbf{B}_t+(\mathbf{v}\cdot\nabla)\mathbf{B}=\eta\nabla^2\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{v},\qquad \nabla\cdot\mathbf{B}=0,\qquad \mathbf{B}(\mathbf{x}+\mathbf{L},t)=\mathbf{B}(\mathbf{x},t),\qquad \mathbf{B}(\mathbf{x},0)=\mathbf{B}_0(\mathbf{x}), \tag{7}$$

where $\mathbf{v}$ is a spatially $\mathbf{L}$-periodic incompressible velocity field.

Direct integration of (7) shows that $(\mathrm{d}/\mathrm{d}t)\int_\Omega \mathbf{B}\,\mathrm{d}V\equiv\mathbf{0}$, i.e., the spatial mean of the initial condition $\mathbf{B}_0(\mathbf{x})$ is preserved in time. As is customary in the case of periodic boundary conditions (cf. [2]), we shall assume

$$\langle\mathbf{B}_0\rangle=\frac{1}{\kappa}\int_\Omega \mathbf{B}\,\mathrm{d}V=0 \tag{8}$$

throughout most of this paper. The case of nonzero $\langle\mathbf{B}_0\rangle$ is discussed in Section 3.5.

In order to define an appropriate phase space for the evolution equation (7), we first recall the notation $H^n(\Omega)$ for the Sobolev space of scalar-valued square-integrable functions that admit n square-integrable distributional derivatives on Ω (see, e.g. [1]). To accommodate the spatially periodic boundary conditions (6), we also recall the notion of the space $H^n_{\mathrm{per}}(\Omega)$, which is composed of functions that are triply $\mathbf{L}$-periodic in the spatial variable $\mathbf{x}$, and are elements of $H^n(U)$ for any open bounded set $U\subset\Omega$. By definition, we have $L^2_{\mathrm{per}}(\Omega)=H^0_{\mathrm{per}}(\Omega)$.

We consider the evolution equation (7) defined for vector-valued functions $\mathbf{B}$ whose coordinate components are all in $H^1_{\mathrm{per}}(\Omega)$, and whose divergence and mean vanish on Ω. Specifically, in dealing with (7), we shall use the function spaces

$$\mathbf{V}(\Omega)=\left\{\mathbf{B}\in H^1_{\mathrm{per}}(\Omega)\times H^1_{\mathrm{per}}(\Omega)\times H^1_{\mathrm{per}}(\Omega)|\nabla\cdot\mathbf{B}=0,\int_\Omega \mathbf{B}\,\mathrm{d}V=\mathbf{0}\right\},$$
$$\mathbf{H}(\Omega)=\text{the closure of }\mathbf{V}(\Omega)\text{ in }L^2_{\mathrm{per}}(\Omega)\times L^2_{\mathrm{per}}(\Omega)\times L^2_{\mathrm{per}}(\Omega),$$

and the inner product

$$\langle\mathbf{B}_1,\mathbf{B}_2\rangle=\int_\Omega \mathbf{B}_1\cdot\mathbf{B}_2\,\mathrm{d}V \tag{9}$$

on $\mathbf{H}(\Omega)$. For notational simplicity, we also introduce

$$\mathbf{L}(\Omega)=L^2_{\mathrm{per}}(\Omega)\times L^2_{\mathrm{per}}(\Omega)\times L^2_{\mathrm{per}}(\Omega).$$

As discussed by Temam [22], the orthogonal complement of $\mathbf{H}(\Omega)$ in $\mathbf{L}(\Omega)$ can be written as

$$\mathbf{H}^{\perp}(\Omega) = \overline{\{\nabla\phi | \phi \in C^1(\bar{\Omega}, \mathbb{R})\}}, \tag{10}$$

where $\overline{\{\cdot\}}$ refers to the closure of $\{\cdot\}$ in $\mathbf{L}(\Omega)$.

2.2. The Stokes operator

In our analysis, it will be convenient to eliminate the second equation in (1) by projecting the first equation in (1) onto the space of divergence-free $\mathbf{B}$ fields. Below we survey the properties of the corresponding projected ∇^2 operator.

Let $P : \mathbf{L}(\Omega) \to \mathbf{H}(\Omega)$ denote the orthogonal projection from $\mathbf{L}(\Omega)$ to $\mathbf{H}(\Omega)$. Then the Stokes operator A is defined as

$$A = -P\nabla^2 \tag{11}$$

on the domain

$$D(A) = \mathbf{H}(\Omega) \cap (H^2_{\text{per}}(\Omega) \times H^2_{\text{per}}(\Omega) \times H^2_{\text{per}}(\Omega))$$

of the phase space $\mathbf{H}(\Omega)$.

Temam [22] shows that A is a self-adjoint positive operator with an inverse that is compact on $\mathbf{H}(\Omega)$ (see also [21]). As a result, A has an unbounded set of discrete eigenvalues $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots$ with the corresponding real eigenfunctions $\mathbf{e}_1(\mathbf{x}), \dots, \mathbf{e}_n(\mathbf{x}), \dots$ forming an orthonormal basis in $\mathbf{H}(\Omega)$.

For later use, we point out that the eigenvalues of A and $-\nabla^2$ coincide on the space $\mathbf{H}(\Omega)$. To see this, we select an eigenvalue λ of the A with the corresponding eigenvector $\mathbf{e}$. Then, by the definition of A, there exists $\nabla\phi \in \mathbf{H}^{\perp}(\Omega)$ such that $-\nabla^2\mathbf{e} = A\mathbf{e} + \nabla\phi$. Therefore, for any $\mathbf{b} \in \mathbf{H}(\Omega)$, we have

$$\langle -\nabla^2\mathbf{e}, \mathbf{b}\rangle = \langle -P\nabla^2\mathbf{e} + \nabla\phi, \mathbf{b}\rangle = \langle A\mathbf{e}, \mathbf{b}\rangle = \langle \lambda\mathbf{e}, \mathbf{b}\rangle,$$

implying that λ is an eigenvalue of $-\nabla^2$. Conversely, if $-\nabla^2\mathbf{e} = \lambda\mathbf{e} \in \mathbf{H}(\Omega)$, then $-P\nabla^2\mathbf{e} = \lambda P\mathbf{e} = \lambda\mathbf{e}$, therefore λ is an eigenvalue of A.

3. Main results

3.1. Invariant and inertial manifolds

We now eliminate the second equation from (7) by applying the projection P to both sides of the first equation in (7). We obtain the equivalent set of equations

$$\mathbf{B}_t = -\eta A\mathbf{B} - P[(\mathbf{v}\cdot\nabla)\mathbf{B}] + P[(\mathbf{B}\cdot\nabla)\mathbf{v}], \qquad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x}) \tag{12}$$

on the state space $\mathbf{H}(\Omega)$.

We aim to decompose (12) into a finite-dimensional and an infinite-dimensional invariant subsystem, with the former system describing the evolution of time-dependent eigenmode-type solutions, and with the latter system admitting exponentially decaying solutions.

For $\mathbf{v} = 0$, any finite-dimensional eigenspace of A and its orthogonal complement render the above decomposition of $\mathbf{H}(\Omega)$. Indeed,

$$\mathbf{H}(\Omega)^+_n = \text{span}\,\{\mathbf{e}_i\}^n_{i=1}, \qquad \mathbf{H}(\Omega)^-_n = \text{span}\,\{\mathbf{e}_i\}^{\infty}_{i=n}$$

are invariant subspaces for Eq. (12) for vanishing $\mathbf{v}$.

For $\mathbf{v} \neq 0$, the above eigenspaces are no longer invariant. To decompose the dynamo equation in this case, we seek time-dependent invariant subspaces as perturbations of the slightly smaller eigenspaces

$$\mathbf{V}(\Omega)_n^+ = \mathbf{H}(\Omega)_n^+ \cap \mathbf{V}(\Omega), \qquad \mathbf{V}(\Omega)_n^- = \mathbf{H}(\Omega)_n^- \cap \mathbf{V}(\Omega),$$

with the norm inherited from $\mathbf{V}(\Omega)$. A perturbative approach is appropriate, because the second and third terms on the right-hand side of (12) involve operators whose norm on $\mathbf{V}(\Omega)_n^-$ is much smaller than the norm of $-\eta A$, provided that n is selected large enough.

To state our results on the invariant decomposition of (12), we define the quantities

$$u_0 = \sup_{(\mathbf{x},t)\in\Omega\times\mathbb{R}} |\nabla\mathbf{v}(\mathbf{x},t)|, \qquad v_0 = \sup_{(\mathbf{x},t)\in\Omega\times\mathbb{R}} |\mathbf{v}(\mathbf{x},t)|, \qquad w_0 = v_0 + u_0\mu_1^{-1/2}.$$

We assume that w_0 is bounded.

By a time-dependent *invariant manifold* $\mathcal{M}(t)$ for the dynamo equation we mean a one-parameter family of manifolds $\{\mathcal{M}(t) \subset \mathbf{V}(\Omega)\}_{t\in\mathbb{R}}$ such that $\mathbf{B}(t) \in \mathcal{M}(t)$ for some t implies $\mathbf{B}(t+s) \in \mathcal{M}(t+s)$ for all $s \in \mathbb{R}$.

Theorem 3.1.

(i) *Suppose that for some integer N, the eigenvalues μ_N and μ_{N+1} of the Stokes operator A satisfy the gap condition*

$$\frac{2}{\sqrt{\mu_{N+1}} - \sqrt{\mu_N}} + \frac{1}{\sqrt{\mu_{N+1}}} < \frac{\eta}{w_0}. \tag{13}$$

Then the dynamo equation admits an N-dimensional linear invariant manifold $\mathcal{M}(t)$ and a codimension-N linear invariant manifold $\mathcal{N}(t)$ such that

$$\mathcal{M}(t) \oplus \mathcal{N}(t) = \mathbf{V}(\Omega) \tag{14}$$

for any t. The manifolds $\mathcal{M}(t)$ and $\mathcal{N}(t)$ depend continuously on t. Furthermore, if $\mathbf{v}(\mathbf{x},t)$ is periodic or quasiperiodic in time, then so are $\mathcal{M}(t)$ and $\mathcal{N}(t)$.

(ii) *Assume further that the stronger gap condition*

$$\frac{2}{\sqrt{\mu_{N+1}} - \sqrt{\mu_N}} + \frac{1}{\sqrt{\mu_{N+1}}} < \frac{\eta}{3w_0} \tag{15}$$

holds. Then $\mathcal{M}(t)$ is an inertial manifold: it is an N-dimensional invariant manifold that attracts all solutions of (12).

We prove this theorem in Appendices A and B.

Theorem 3.1 enables us to decompose the dynamo equations into a finite-dimensional system with components in $\mathcal{M}(t)$, and an infinite-dimensional system with exponentially decaying components in $\mathcal{N}(t)$. Because $\mathcal{M}(t)$ is a linear subspace and (12) is a linear equation, solutions on $\mathcal{M}(t)$ satisfy a finite-dimensional linear ODE.

Koksch and Siegmund [10–12] show the existence of inertial manifolds for a general class nonautonomous nonlinear evolution equations by assuming the existence of certain invariant projections for the linear semiflow. Our proof of Theorem 3.1 shows that such invariant projections do exist for the dynamo equation (see the operators $L^N(t)$ and $L^\infty(t)$ defined in Appendices A and B). As a result, the inertial manifolds we have constructed will survive under uniformly bounded nonlinear perturbations to Eq. (12) by the results of Koksch and Siegmund [11].

3.2. Completeness of Floquet eigenmodes

If the linear ODE governing the dynamics on $\mathcal{M}(t)$ is continuous and T-periodic in t, then its solutions are linear combinations of *Floquet solutions* of the form

$$\mathbf{B}(t) = e^{\lambda t}[\mathbf{f}_0(t) + t\mathbf{f}_1(t) + \cdots + t^l \mathbf{f}_l(t)], \tag{16}$$

where l is a nonnegative integer, λ a complex parameter, and $\mathbf{f}_k(t)$ are continuous T-periodic $\mathbf{H}(\Omega)$-valued functions.

The exponent λ is usually called the *Floquet exponent* corresponding to the *eigenmode* $\mathbf{f}_0(t)+t\mathbf{f}_1(t)+\cdots+t^l\mathbf{f}_l(t)$. The integer l is the dimension of the Jordan block corresponding to λ in the canonical form of the matrix $\mathbf{A}$ in the Floquet decomposition

$$\mathbf{B}(t) = e^{\mathbf{A}t}\,\mathbf{\Pi}(t)\mathbf{B}(0),$$

where $\mathbf{\Pi}(t)$ is a continuous, T-periodic matrix-valued function.

The above imply that an arbitrary solution of (12) attains an asymptotically valid Floquet decomposition while it converges to $\mathcal{M}(t)$. The theorem below states this result in precise terms.

Theorem 3.2. *Assume that the velocity field* $\mathbf{v}(\mathbf{x}, t)$ *is T-periodic and continuous in time. Assume further that for any positive integer* N^*, *there exists an integer* $N \geq N^*$ *such that the gap condition (15) is satisfied. Then, for any* $\varepsilon > 0$, *there exist an integer* $N(\varepsilon)$ *and eigenmodes*

$$\mathbf{e}_k(\mathbf{x}, t) = \mathbf{f}_k^0(\mathbf{x}, t) + t\mathbf{f}_k^1(\mathbf{x}, t) + \cdots + t^{l(k)}\mathbf{f}_k^{l(k)}(\mathbf{x}, t), \quad k = 1, \dots, N(\varepsilon)$$

with Floquet exponents λ_k, *such that any solution* $\mathbf{B}(\mathbf{x}, t)$ *of Eq. (12) can be written as*

$$\mathbf{B}(\mathbf{x}, t) = \sum_{k=1}^{N(\varepsilon)} c_k\, e^{\lambda_k t}\mathbf{e}_k(\mathbf{x}, t) + \mathbf{R}(\mathbf{x}, t), \tag{17}$$

$$\|\mathbf{R}(t)\|_{\mathbf{V}(\Omega)} \leq \varepsilon \exp[-(0.5\eta\mu_{N(\varepsilon)+1} - 4\pi\eta^{-1}\, e^{-1/2} w_0^2)t], \tag{18}$$

where the coefficients c_k *depend on the initial condition* $\mathbf{B}_0(\mathbf{x})$.

We prove this theorem in Appendix C.

The proofs of Theorems 3.1 and 3.2 are motivated by the work of Chow et al. [3] on Floquet solutions of one-dimensional parabolic PDEs. An alternative approach to infinite-dimensional Floquet theory is offered by Kuchment [13]. While evolution equations of the type (1) are formally covered by the results in [13], the completeness of Floquet solutions in the dynamo equation only follows from those results for $\mathbf{v}(\mathbf{x}, t) \equiv \mathbf{0}$ (see [14] for details).

3.3. Generic form of Floquet solutions

We now consider the generic case in which the magnetic field $\mathbf{B}(\mathbf{x}, t)$ converges to a simple Floquet solution, i.e., to one with $l = 0$ in (16). From now on, an overbar will refer to time-averaging over the interval $[0, T]$, i.e., we write

$$\bar{a} = \frac{1}{T}\int_0^T a(t)\,dt.$$

We have the following result on the relation between Floquet exponents and the corresponding Floquet eigenmodes.

Theorem 3.3. *For a generic, two-dimensional, time-periodic, incompressible velocity field defined on the spatial domain* Ω, *the magnetic field* $\mathbf{B}(\mathbf{x}, t)$ *converges to a Floquet solution of the form*

$$\mathbf{B}_\infty(\mathbf{x}, t) = e^{(\alpha+i\beta)t}\,\mathbf{e}_0(\mathbf{x}, t), \tag{19}$$

where $\mathbf{e}_0(\mathbf{x}, t)$ *and* $\nabla\mathbf{e}_0(\mathbf{x}, t)$ *are square-integrable complex functions for all* $t > 0$, *and*

$$\alpha = \frac{-\eta\overline{\|\nabla\mathbf{e}_0\|^2} + \mathrm{Re}\,\overline{\langle\mathbf{e}_0, (\mathbf{e}_0^*\cdot\nabla)\mathbf{v}\rangle}}{\overline{\|\mathbf{e}_0\|^2}}, \qquad \beta = \frac{\alpha\overline{\|\mathrm{Re}\,\mathbf{e}_0\|^2} + \eta\overline{\|\mathrm{Re}\,\nabla\mathbf{e}_0\|^2} - \overline{\langle\mathrm{Re}\,\mathbf{e}_0, (\mathrm{Re}\,\mathbf{e}_0\cdot\nabla)\mathbf{v}\rangle}}{\overline{\langle\mathrm{Re}\,\mathbf{e}_0, \mathrm{Im}\,\mathbf{e}_0\rangle}}. \tag{20}$$

We prove this theorem in Appendix D.

Eq. (19) shows that $\mathbf{B}_\infty$ is either time-periodic or *quasiperiodic*. The latter case occurs if $\mathrm{Im}(\lambda_0)$, the imaginary part of the exponent in (19), is nonzero and rationally independent of $2\pi/T$, where T is the time-period of $\mathbf{v}$. If $\mathrm{Im}(\lambda_0)$ and $2\pi/T$ are rationally dependent, then $\mathbf{B}_\infty$ is again time-periodic, but with a period equal to the maximum of $2\pi/\mathrm{Im}(\lambda_0)$ and T. Therefore, $2\pi/\mathrm{Im}(\lambda_0) > T$ results in a *subharmonic* Floquet solution.

3.4. Evaluating the gap conditions

Theorems 3.1 and 3.2 rely on spectral gap conditions for the Stokes operator A. Here we evaluate these conditions for two specific types of spatial domains.

3.4.1. Example: two-dimensional rectangular domains

Consider the two-dimensional square domain $\Omega = [0, 2\pi] \times [0, 2\pi]$. The eigenvectors of the operator $-\nabla^2$ on $\mathbf{L}^2(\Omega)$ have the form

$$\mathbf{e} = (e_1, e_2) = (\phi_{11}(lx)\phi_{12}(my), \phi_{21}(lx)\phi_{22}(my)),$$

where $\phi_{ij}(p)$ is either $\sin(p)$ or $\cos(p)$, and l, m are nonnegative integers. The eigenvalue corresponding to $\mathbf{e}$ on $\mathbf{L}^2(\Omega)$ can be written as

$$\mu(l, m) = l^2 + m^2.$$

When we restrict the operator $-\nabla^2$ to the function space $\mathbf{H}(\Omega)$, we have to enforce the divergence-free condition $\nabla\cdot\mathbf{e} = 0$ on the eigenvectors (22). This implies the relation

$$l\phi_{11}'(lx)\phi_{12}(my) + m\phi_{21}(lx)\phi_{22}'(my) = 0,$$

which then gives

$$al + bm = 0,$$

where $a, b = 1, 0,$ or -1. If $a = b = 0$, then $l = m$ and $\mu(l, m)\| = l^2$. Otherwise, $\mu(l, m) = l^2 + m^2 = (bm)^2 = 2l^2$.

Since all numbers of the form l^2 are eigenvalues, the largest spectral gap is this example is at most one. The largest spectral gap would be strictly smaller than one if there were eigenvalue of the form $2m^2$ within every adjacent eigenvalue pair $(l^2, (l+1)^2)$. We want to show that one can find arbitrary large such eigenvalue pairs with no other eigenvalue falling between them. In other words, we want to show that for any integer N, there exists an $l \geq N$ such that $\sqrt{2}m \notin [l, l+1]$ for all integers m.

Assuming the contrary, we see that there must exist an integer N such that for any integer $i \geq 0$, there exists another integer $m(i)$ such that $\sqrt{2}m(i) \in [N+i, N+i+1]$. Since $m(i) \neq m(i-1)$ and $m(i) \geq m(i-1)$, we obtain $m(i) \geq m(i-1)+1$ for $i = 1, 2, \ldots$. We then have

$$m(i) \geq m(0) + i \quad \text{for } i = 1, 2, \ldots.$$

Since $\sqrt{2}m(0) \in [N, N+1]$ and $\sqrt{2}m(i) \in [N+i, N+i+1]$, we obtain $[\sqrt{2}m(0), \sqrt{2}m(i)] \subset [N, N+i+1]$. Consequently, $\sqrt{2}m(i) - \sqrt{2}m(0) \leq N+i+1-N = i+1$, implying $\sqrt{2} \leq (i+1)/(m(i)-m(0)) \leq (i+1)/i$ for any i. This last inequality, however, fails for large enough i, which establishes a contradiction.

We conclude that for any integer N, there exists $n \geq N$ such that $\sqrt{\mu_{n+1}} - \sqrt{\mu_n} = \sqrt{(l+1)^2} - \sqrt{l^2} = 1$. Thus the gap condition (15) holds for an appropriate choice of $n > N$ if

$$\frac{1}{\sqrt{\mu_{n+1}}} < \frac{\eta}{3w_0} - 2.$$

Since $\mu_n \to \infty$ as $n \to \infty$, this last inequality will hold for large enough n if

$$w_0 < \tfrac{1}{6}\eta. \tag{21}$$

Consequently, for any integer $N > 0$ we can find another integer $n > N$, such that the gap condition (21) is satisfied for n.

3.4.2. Example: three-dimensional cubic domain

As another example, consider the three-dimensional cubic domain $\Omega = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$. The eigenvectors of the operator $-\nabla^2$ on $\mathbf{L}^2(\Omega)$ have the form

$$\mathbf{e} = (e_1, e_2, e_3) = (\phi_{11}(lx)\phi_{12}(my)\phi_{13}(nz), \phi_{21}(lx)\phi_{22}(my)\phi_{23}(nz), \phi_{31}(lx)\phi_{32}(my)\phi_{33}(nz)), \tag{22}$$

where $\phi_{ij}(p)$ is either $\sin(p)$ or $\cos(p)$, and l, m, n are nonnegative integers. The eigenvalue corresponding to $\mathbf{e}$ on $\mathbf{L}^2(\Omega)$ can be written as

$$\sigma(l, m, n) = l^2 + m^2 + n^2. \tag{23}$$

Enforcing the divergence-free condition $\nabla \cdot \mathbf{e} = 0$ on the eigenvectors (22) gives

$$l\phi'_{11}(lx)\phi_{12}(my)\phi_{13}(nz) + m\phi_{21}(lx)\phi'_{22}(my)\phi_{23}(nz) + n\phi_{31}(lx)\phi_{32}(my)\phi'_{33}(nz) = 0,$$

which then yields

$$al + bm + cn = 0, \tag{24}$$

where $a, b, c = 1, 0$ or -1. For instance, if

$$e_1 = \sin(lx)\cos(my)\sin(nz), \qquad e_2 = \cos(lx)\sin(my)\sin(nz), \qquad e_1 = \cos(lx)\cos(my)\cos(nz),$$

then (24) gives $l + m - n = 0$.

From (23) and (24) we find that the eigenvalues of $-\nabla^2$ on the space $\mathbf{H}(\Omega)$ are equal to either $\mu(l, m, n) = l^2+m^2$ if $c = 0$, or to

$$\mu(l, m, n) = l^2 + m^2 + n^2 = l^2 + m^2 + (cn)^2 = l^2 + m^2 + (al + bm)^2 = 2l^2 + 2m^2 + 2ablm, \quad \text{if } c \neq 0.$$

Using the above result, we plot the function $g(n) = 2/(\sqrt{\mu_{n+1}} - \sqrt{\mu_n}) + 1/\sqrt{\mu_{n+1}}$ in Fig. 1 for integers up to $n = 50$. The figure shows that $g(2) < 4$, therefore, if the velocity field $\mathbf{v}$ and diffusivity η satisfy

$$w_0 < \tfrac{1}{12}\eta,$$

then the gap condition (15) holds for $N = 2$. As a result, a two-dimensional inertial manifold exists for velocity fields small enough in norm.

3.5. Magnetic fields of nonzero mean

For the proofs of our main results, the zero mean condition (8) is essential. (A main tool used in our estimates, the Poincaré inequality, would not be valid otherwise.) In addition, the zero mean condition is preferred for physical

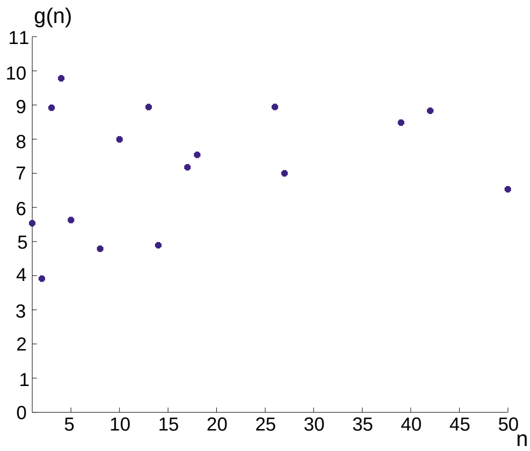

Fig. 1. The function $g(n)$ for $n = 1, \ldots, 50$.

reasons in the case of the periodic boundary conditions (cf. [2]). For completeness, however, we now discuss the implications of Theorems 3.1 and 3.2 for initial conditions of nonzero mean.

If $\langle \mathbf{B}_0 \rangle \neq \mathbf{0}$, then we let

$$\tilde{\mathbf{B}} = \mathbf{B} - \langle \mathbf{B}_0 \rangle \tag{25}$$

and observe that $\tilde{\mathbf{B}}$ satisfies

$$\tilde{\mathbf{B}}_t + (\mathbf{v} \cdot \nabla)\tilde{\mathbf{B}} = \eta \nabla^2 \tilde{\mathbf{B}} + (\tilde{\mathbf{B}} \cdot \nabla)\mathbf{v} + (\langle \mathbf{B}_0 \rangle \cdot \nabla)\mathbf{v}, \qquad \nabla \cdot \tilde{\mathbf{B}} = 0, \qquad \tilde{\mathbf{B}}(\mathbf{x}, t_0) = \mathbf{B}_0(\mathbf{x}) - \langle \mathbf{B}_0 \rangle \tag{26}$$

and that $\langle \tilde{\mathbf{B}} \rangle \equiv \mathbf{0}$ holds for all times. Thus the evolution of $\tilde{\mathbf{B}}$ can be understood by adding a source-type term to the right-hand side of the mean-zero dynamo equation (7).

Let $\hat{\mathbf{B}}$ be the solution of the dynamo equation with initial data $\mathbf{B}_0(\mathbf{x}) - \langle \mathbf{B}_0 \rangle$:

$$\hat{\mathbf{B}}_t + (\mathbf{v} \cdot \nabla)\hat{\mathbf{B}} = \eta \nabla^2 \hat{\mathbf{B}} + (\hat{\mathbf{B}} \cdot \nabla)\mathbf{v}, \qquad \nabla \cdot \hat{\mathbf{B}} = 0, \qquad \hat{\mathbf{B}}(\mathbf{x}, t_0) = \mathbf{B}_0(\mathbf{x}) - \langle \mathbf{B}_0 \rangle.$$

Also, let $\bar{\mathbf{B}}$ be the solution of the special initial value problem

$$\bar{\mathbf{B}}_t + (\mathbf{v} \cdot \nabla)\bar{\mathbf{B}} = \eta \nabla^2 \bar{\mathbf{B}} + (\bar{\mathbf{B}} \cdot \nabla)\mathbf{v} + (\langle \mathbf{B}_0 \rangle \cdot \nabla)\mathbf{v}, \qquad \nabla \cdot \bar{\mathbf{B}} = 0, \qquad \bar{\mathbf{B}}(\mathbf{x}, t_0) = \mathbf{0}. \tag{27}$$

We then have

$$\tilde{\mathbf{B}} = \hat{\mathbf{B}} + \bar{\mathbf{B}}, \tag{28}$$

where $\tilde{\mathbf{B}}(\mathbf{x}, t)$ is the solution of (26).

To describe the structure of $\bar{\mathbf{B}}$, we consider the three-parameter family of PDEs

$$\bar{\mathbf{B}}^j_t + (\mathbf{v} \cdot \nabla)\bar{\mathbf{B}}^j = \eta \nabla^2 \bar{\mathbf{B}}^j + (\bar{\mathbf{B}}^j \cdot \nabla)\mathbf{v} + (\mathbf{e}^j \cdot \nabla)\mathbf{v}, \qquad \nabla \cdot \bar{\mathbf{B}}^j = 0, \qquad \bar{\mathbf{B}}^j(\mathbf{x}, t_0) = \mathbf{0}, \tag{29}$$

where $\mathbf{e}^1 = (1, 0, 0)$, $\mathbf{e}^2 = (0, 1, 0)$, and $\mathbf{e}^3 = (0, 0, 1)$.

Multiplying the jth PDE in the family (29) by $\langle B_0^j \rangle$ (the jth coordinate component of the mean $\langle \mathbf{B}_0 \rangle$), then summing over j, we find that

$$\bar{\mathbf{B}} = \langle B_0^1 \rangle \bar{\mathbf{B}}^1 + \langle B_0^2 \rangle \bar{\mathbf{B}}^2 + \langle B_0^3 \rangle \bar{\mathbf{B}}^3 \tag{30}$$

is the solution of the special initial value problem (27). Thus $\bar{\mathbf{B}}$ always lies in a time-dependent finite-dimensional invariant subspace $\mathcal{E}(t)$ of dimension

$$M = \dim[\mathrm{span}\{\bar{\mathbf{B}}^1, \bar{\mathbf{B}}^2, \bar{\mathbf{B}}^3\}].$$

Here M is zero for mean-zero initial data, but varies between 1 and 3 otherwise.

From (25), (28) and (30) we conclude that the full magnetic field $\mathbf{B}(\mathbf{x}, t)$ is the sum of the mean $\langle \mathbf{B}_0 \rangle$ of the initial data $\mathbf{B}_0$, the solution $\bar{\mathbf{B}} = \Sigma_{j=1}^3 \langle B_0^j \rangle \bar{\mathbf{B}}^j$ obtained from (29) with zero initial data, and the solution $\hat{\mathbf{B}}(\mathbf{x}, t)$ of the dynamo equation with initial data $\mathbf{B}_0(\mathbf{x}) - \langle \mathbf{B}_0 \rangle$.

Now the solution component $\hat{\mathbf{B}}(\mathbf{x}, t)$ admits a complete Floquet expansion under the gap condition of Theorem 3.2, thus for any $\varepsilon > 0$, we have

$$\mathbf{B}(\mathbf{x}, t) = \sum_{j=1}^{3} \langle B_0^j \rangle [\mathbf{e}^j + \bar{\mathbf{B}}^j(\mathbf{x}, t)] + \sum_{k=1}^{N(\varepsilon)} \mathrm{e}^{\lambda_k t}\, \mathbf{e}_k(\mathbf{x}, t) + \mathbf{R}(\mathbf{x}, t),$$

$$\|\mathbf{R}(t)\|_{\mathbf{V}(\Omega)} \leq \varepsilon \exp[-(\eta \mu_{N(\varepsilon)+1} - 4\pi^2 w_0^2) t],$$

where the functions $\bar{\mathbf{B}}^j(\mathbf{x}, t)$, as solutions of (29), only depend on the velocity field $\mathbf{v}(\mathbf{x}, t)$ and the domain Ω. Thus magnetic fields of nonzero mean converge to an $N(\varepsilon) + M$-dimensional time-dependent inertial manifold.

4. Conclusions

In this paper, we have examined whether solutions of the time-periodic dynamo equation indeed admit an asymptotic Floquet decomposition, as often assumed. We have proved that they do, provided that the Stokes operator associated with the dynamo equation has large enough spectral gaps. The Floquet modes span a time-periodic inertial manifold $\mathcal{M}(t)$ to which all solutions of the dynamo equations converge. Instabilities associated with fast dynamo action are therefore always confined to $\mathcal{M}(t)$.

The divergence-free property of the magnetic field $\mathbf{B}(\mathbf{x}, t)$ forces the Stokes operator to have fewer eigenvalues on the space $\mathbf{H}(\Omega)$ than the Laplacian operator has on $L^2(\Omega)$. For this reason, our spectral gap conditions are less restrictive than analogous conditions for the time-periodic scalar advection–diffusion equation (see [14]).

We have shown how our gap conditions can be verified for two types of spatial domains: two-dimensional rectangles and three-dimensional cubes. For these types of domains, the spectral gap conditions translate to a smallness requirement on the velocity field. These requirements can be weakened by sharpening the general estimates in our proofs for specific geometries and velocity fields.

By the smoothing property of the parabolic dynamo equations (see [8]), any square-integrable initial data $\mathbf{B}_0(\mathbf{x}) = \mathbf{B}(\mathbf{x}, t_0)$ becomes a function in $\mathbf{H}(\Omega)$ *immediately* after the initial time t_0. Theorems 3.1 and 3.2 are therefore strong enough to apply for any realistic choice of the initial magnetic field.

Strictly speaking, we have proved the existence of the inertial manifold $\mathcal{M}(t)$ for general velocity fields that are aperiodic in time. In such cases, the solutions of the dynamo equation tend to a finite number of aperiodic eigenmodes that form a fundamental set of solutions for a linear system of ODEs on $\mathcal{M}(t)$.

We have also derived a general expression for Floquet eigenmodes on $\mathcal{M}(t)$ for the case of time-periodic velocity fields. Our results further underline the need to study eigenmodes of discrete dynamo maps that serve as models of a Poincaré map of a time-periodic flow (see [7]).

Due to mathematical and physical considerations, the magnetic field $\mathbf{B}$ is usually assumed to have zero spatial mean in the case of periodic boundary conditions. We have briefly discussed the case of nonzero spatial mean, concluding that solutions then converge to a larger inertial manifold that contains up to three eigenmodes that are not Floquet type.

Little is known analytically about the important $\eta \to 0$ limit of the weakest Floquet exponent $\lambda_0 = \alpha + i\beta$. Moffatt and Proctor [15] show that a topological constraint, the conservation of magnetic helicity, precludes a positive α for $\eta = 0$.

In the $\eta \to 0$ limit, our gap conditions require unbounded spectral gaps and hence break down. We still believe that the framework presented here should be helpful in proving the existence of fast dynamos: one hopes that a refinement of the estimates is Appendices A and B leads to a uniform bound on the dimension of the inertial manifold in the $\eta \to 0$ limit. In that case, any fast dynamo behavior is captured by a finite-dimensional ODE.

Acknowledgements

We would like to thank Andrew Gilbert for helpful discussions. We are also grateful to Peter Kuchment and John Mallet-Paret for detailed explanations on their work. We are indebted to Norbert Koksch for a close reading of the manuscript, and for pointing out an erroneous estimate in an earlier version. This research was supported by AFOSR grant no. F49620-00-1-0133. In addition, GH was partially supported by NSF grant no. DMS-01-02940.

Appendix A

Here we prove statement (i) of Theorem 3.1. We shall only construct the manifold $\mathcal{M}(t)$ explicitly, because the construction of $\mathcal{N}(t)$ is analogous.

A.1. *Some definitions*

We start by introducing the constants

$$\rho = \tfrac{1}{2}(\mu_{N+1} + \mu_N),$$

that measure the mean of adjacent eigenvalues of the Stokes operator A. These constants will simplify our notation in the upcoming estimates.

We let P_n^+ and P_n^- denote the orthogonal projections from $\mathbf{H}(\Omega)$ to $\mathbf{H}(\Omega)_n^+$ and to $\mathbf{H}(\Omega)_n^-$, respectively, and let

$$A_n^+ = A|_{\mathbf{H}(\Omega)_n^+}, \qquad A_n^- = A|_{\mathbf{H}(\Omega)_n^-},$$

denote the appropriate restrictions of A to $\mathbf{H}(\Omega)_n^+$ and $\mathbf{H}(\Omega)_n^-$.

We also recall that for any $\mathbf{B} = \sum_{i=1}^{\infty} a_i \mathbf{e}_i(\mathbf{x})$, the fractional power $A^{1/2}$ of A is defined as

$$A^{1/2}\mathbf{B} = \sum_{i=1}^{\infty} a_i \mu_i^{1/2} \mathbf{e}_i(\mathbf{x}),$$

with the domain

$$D(A^{1/2}) = \left\{ \mathbf{B} = \sum_{i=1}^{\infty} a_i \mathbf{e}_i(\mathbf{x}) \,\middle|\, \sum_{i=1}^{\infty} |a_i|_i^2 \mu_i < \infty \right\},$$

as discussed, for example, by Henry [8] and Sell and You [22, Section 3.7].

By (10), for any $\mathbf{B} \in D(A)$, we have

$$\nabla^2 \mathbf{B} = P\nabla^2 \mathbf{B} + \nabla\phi$$

for some $\nabla\phi \in \mathbf{H}^{\perp}(\Omega)$. We then obtain

$$\|\nabla \mathbf{B}\|_{L^2}^2 = -\int_{\Omega} \mathbf{B}\cdot\nabla^2\mathbf{B}\,\mathrm{d}A = \int_{\Omega} \mathbf{B}\cdot(A\mathbf{B}-\nabla\phi)\,\mathrm{d}A = \sum_{i=1}^{\infty} a_i^2\mu_i = \|A^{1/2}\mathbf{B}\|_{L^2}^2. \tag{A.1}$$

By a density argument, one can show that (A.1) also holds for any $\mathbf{B} = \sum_{i=1}^{\infty} a_i\mathbf{e}_i \in D(A^{1/2})$. As a result, we obtain that $A^{1/2}\mathbf{B}$ remains bounded in the L^2 norm precisely when $\mathbf{B}$ remains bounded in the H^1 norm. Therefore, $D(A^{1/2}) = \mathbf{V}(\Omega)$.

Finally, we define the function space

$$X_{\rho}^{-} = \left\{\mathbf{f} : (-\infty, 0] \to \mathbf{V}(\Omega) | \mathbf{f} \in C^0, \sup_{t\le 0} \mathrm{e}^{\eta\rho t}\|\mathbf{f}\|_{\mathbf{V}(\Omega)} < \infty\right\} \tag{A.2}$$

with the norm

$$\|\mathbf{f}\|_{X_{\rho}^{-}} = \sup_{t\le 0} \mathrm{e}^{\eta\rho t}\|\mathbf{f}\|_{\mathbf{V}(\Omega)}.$$

This complete metric space contains functions that grow slower in backward time than $\mathrm{e}^{-\eta\rho t}$ does. If nonempty, X_{ρ}^{-} is an invariant set for (12) by definition.

We want to construct an N-dimensional time-dependent invariant manifold $\mathcal{M}(t)$ for Eq. (12) with solutions that do not grow faster than $\mathrm{e}^{-\eta\rho t}$ in backward time. In other words, we want to solve (12) on the space X_{ρ}^{-} to obtain a finite-dimensional manifold of solutions that either grow, or decay slower to the zero solution than other solutions do.

A.2. Integral equation formulation

We introduce a phase parameter $\theta \in \mathbb{R}$ to account for solutions launched at an arbitrary initial time $t_0 = \theta$ and rewrite (12) as follows

$$\mathbf{U}_t = -\eta A\mathbf{U} - P[(\mathbf{v}(\theta+t)\cdot\nabla)\mathbf{U}] + P[(\mathbf{U}\cdot\nabla)\mathbf{v}(\theta+t)]. \tag{A.3}$$

(Recall that in the definition of X_{ρ}^{-}, the time variable t is restricted to nonpositive values.) The manifold $\mathcal{M}(t)$ will be constructed as the set of points through which the solutions of (A.3) do not grow faster than $\mathrm{e}^{-\eta\rho t}$ does as $t \to -\infty$.

For some fixed but yet unspecified integer N, we split $\mathbf{U}$ into two components: one in $\mathbf{H}(\Omega)_N^{+}$ and one in $\mathbf{H}(\Omega)_N^{-}$. In terms of these two solution components, (A.3) splits into a coupled set of two equations. Using the variation of constants formula, and taking the limit $t_0 \to -\infty$ for the initial time t_0 of the solution component in $\mathbf{H}(\Omega)_N^{-}$, one obtains the integral equation

$$\begin{aligned}\mathbf{U}(t) = {} & \mathrm{e}^{-\eta A_N^{+}t}\mathbf{p} + \int_0^t \mathrm{e}^{-\eta A_N^{+}(t-s)} P_N^{+}\{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s \\ & + \int_{-\infty}^t \mathrm{e}^{-\eta A_{N+1}^{-}(t-s)} P_{N+1}^{-}\{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s,\end{aligned} \tag{A.4}$$

with $\mathbf{p} = P_N^{+}\mathbf{U}(0)$. This integral equation is equivalent to the restriction of the dynamo equation to the set X_{ρ}^{-}.

A.3. Solving the integral equation

Defining the map F by the formula

$$\begin{aligned}F(\mathbf{U},\mathbf{p},\theta) = {} & \mathrm{e}^{-\eta A_N^{+}t}\mathbf{p} + \int_0^t \mathrm{e}^{-\eta A_N^{+}(t-s)} P_N^{+}\{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s \\ & + \int_{-\infty}^t \mathrm{e}^{-\eta A_{N+1}^{-}(t-s)} P_{N+1}^{-}\{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s,\end{aligned} \tag{A.5}$$

we observe that solutions of (A.4) are $(\mathbf{p}, \theta)$-dependent fixed points of $F(\cdot, \mathbf{p}, \theta)$. To prove that $F(\cdot, \mathbf{p}, \theta)$ has a unique fixed point, we shall show that $F(\cdot, \mathbf{p}, \theta)$ is a contraction mapping from the space X_ρ^- into itself.

To estimate the first term on the right-hand side of (A.5), note that

$$\|\mathrm{e}^{-\eta A_N^+ t}\mathbf{p}\|_{\mathbf{V}(\Omega)} \le \mathrm{e}^{-\eta\mu_N t}\|\mathbf{p}\|_{\mathbf{V}(\Omega)}. \tag{A.6}$$

To estimate the remaining two integral terms in $F(\mathbf{U}, \mathbf{p}, \theta)$, we shall use three ingredients. First, we recall that (see, e.g. [4, Chapter 2, Lemma 1.1])

$$\|A^{1/2}\,\mathrm{e}^{-\eta A_N^+ t}\|_{L^2} \le \mu_N^{1/2}\,\mathrm{e}^{-\eta\mu_N t}, \quad t \le 0. \tag{A.7}$$

Second, we recall from Sell and You [22, p. 94] the inequality

$$\|(\eta A)^{1/2}\,\mathrm{e}^{-\eta A_{N+1}^- t}\|_{L^2} \le \varphi_{N+1}(t) = \begin{cases} \dfrac{1}{\sqrt{2et}} & 0 < t \le \dfrac{1}{2\eta\mu_{N+1}}, \\ \sqrt{\eta\mu_{N+1}}\,\mathrm{e}^{-\eta\mu_{N+1}t} & \dfrac{1}{2\eta\mu_{N+1}} < t < \infty \end{cases} \tag{A.8}$$

and, for $\lambda < \eta\mu_{N+1}$, the relations

$$\int_0^\infty \varphi_{N+1}(t)\,\mathrm{d}t = \frac{2}{\sqrt{e\eta\mu_{N+1}}}, \qquad \int_0^\infty \varphi_{N+1}(t)\,\mathrm{e}^{\gamma t}\,\mathrm{d}t \le \frac{1}{\sqrt{\eta\mu_{N+1}}} + \frac{\sqrt{\eta\mu_{N+1}}}{\eta\mu_{N+1}-\gamma}. \tag{A.9}$$

Third, by (A.1), we have

$$\begin{aligned} &\left\|\int_0^t \mathrm{e}^{-\eta A_N^+(t-s)}P_N^+\{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s\right\|_{\mathbf{V}(\Omega)} \\ &\quad = \left\|A^{1/2}\int_0^t \mathrm{e}^{-\eta A_N^+(t-s)}P_N^+\{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s\right\|_{L^2}. \end{aligned} \tag{A.10}$$

Using (A.7) and (A.10), we estimate the second term in the definition of $F(\mathbf{U}, \mathbf{p}, \theta)$ as follows:

$$\begin{aligned} &\left\|\int_0^t \mathrm{e}^{-\eta A_N^+(t-s)}P_N^+\{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s\right\|_{\mathbf{V}(\Omega)} \\ &\quad \le \int_t^0 \mu_N^{1/2}\,\mathrm{e}^{\eta\mu_N(s-t)}\|-P((\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)) + P((\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s))\|_{L^2}\,\mathrm{d} \\ &\quad \le \int_t^0 \mu_N^{1/2}\,\mathrm{e}^{\eta\mu_N(s-t)}(v_0\|A^{1/2}\mathbf{U}(s)\|_{L^2} + u_0\|\mathbf{U}(s)\|_{L^2})\,\mathrm{d}s \ \ (\text{note that } \mu_1^{1/2}\|\mathbf{U}\|_{L^2} \le \|A^{1/2}\mathbf{U}\|_{L^2}) \\ &\quad \le w_0\int_t^0 \mu_N^{1/2}\,\mathrm{e}^{\eta\mu_N(s-t)}\|\mathbf{U}(s)\|_{\mathbf{V}(\Omega)}\,\mathrm{d}s \le w_0\mu_N^{1/2}\|\mathbf{U}\|_{X_\rho^-}\int_t^0 \mathrm{e}^{\eta\mu_N(s-t)-\eta\rho s}\,\mathrm{d}s \\ &\quad = \frac{w_0\mu_N^{1/2}}{\eta(\rho-\mu_N)}\|\mathbf{U}\|_{X_\rho^-}\,\mathrm{e}^{-\eta\mu_N t}(\mathrm{e}^{\eta(\mu_N-\rho)t}-1) \le \frac{w_0\mu_N^{1/2}}{\eta(\rho-\mu_N)}\|\mathbf{U}\|_{X_\rho^-}\,\mathrm{e}^{-\eta\rho t}. \end{aligned} \tag{A.11}$$

Using (A.8)–(A.10), it then follows that

$$\begin{aligned} &\left\|\int_{-\infty}^t \mathrm{e}^{-\eta A_{N+1}^-(t-s)}P_{N+1}^-\{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s\right\|_{\mathbf{V}(\Omega)} \\ &\quad \le \int_{-\infty}^t \eta^{-1/2}\varphi_{N+1}(t-s)\|-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\|_{L^2}\,\mathrm{d}s \\ &\quad \le w_0\int_{-\infty}^t \eta^{-1/2}\varphi_{N+1}(t-s)\|\mathbf{U}(s)\|_{\mathbf{V}(\Omega)}\,\mathrm{d}s \le w_0\|\mathbf{U}\|_{X_\rho^-}\int_{-\infty}^t \eta^{-1/2}\varphi_{N+1}(t-s)\,\mathrm{e}^{-\eta\rho s}\,\mathrm{d}s \\ &\quad = w_0\eta^{-1/2}\|\mathbf{U}\|_{X_\rho^-}\,\mathrm{e}^{-\eta\rho t}\int_0^\infty \varphi_{N+1}(s)\,\mathrm{e}^{\eta\rho s}\,\mathrm{d}s \le w_0\eta^{-1}\|\mathbf{U}\|_{X_\rho^-}\left(\frac{1}{\sqrt{\mu_{N+1}}} + \frac{\sqrt{\mu_{N+1}}}{\mu_{N+1}-\rho}\right)\mathrm{e}^{-\eta\rho t}. \end{aligned} \tag{A.12}$$

The estimates (A.6), (A.11) and (A.12) together imply that $F(\mathbf{U}, \mathbf{p}, \theta)$ is bounded, and hence $F(\mathbf{U}, \mathbf{p}, \theta)$ indeed maps into X_ρ^-.

Next, we want to show that F defines a contraction mapping on X_ρ^-. From (A.11) and (A.12) we see that for any $\mathbf{U}_1, \mathbf{U}_2 \in X_\rho^-$,

$$\|F(\mathbf{U}_1, \mathbf{p}, \theta) - F(\mathbf{U}_2, \mathbf{p}, \theta)\|_{X_\rho^-} \le \left(\frac{2w_0(\sqrt{\mu_N} + \sqrt{\mu_{N+1}})}{\eta(\mu_{N+1} - \mu_N)} + \frac{w_0}{\eta\sqrt{\mu_{N+1}}} \right) \|\mathbf{U}_1 - \mathbf{U}_2\|_{X_\rho^-}. \tag{A.13}$$

But (A.13) and condition (13) together establish that F is a contraction mapping on X_ρ^-.

As a contraction mapping on a complete metric space, F has a unique fixed point $\mathbf{U}(t; \mathbf{p}, \theta)$ for any θ and $\mathbf{p}$, implying a unique solution for (A.4) in X_ρ^-. Denote

$$K(w_0, \eta, \mu_{N+1}, \mu_N) = \frac{2w_0(\sqrt{\mu_N} + \sqrt{\mu_{N+1}})}{\eta(\mu_{N+1} - \mu_N)} + \frac{w_0}{\eta\sqrt{\mu_{N+1}}}.$$

Then for such a fixed point $\mathbf{U}(t; \mathbf{p}, \theta)$, the estimates (A.6) and (A.13) give

$$\begin{aligned} \|\mathbf{U}\|_{X_\rho^-} &= \|F(\mathbf{U}, \mathbf{p}, \theta)\|_{X_\rho^-} \le \|F(\mathbf{U}, \mathbf{p}, \theta) - F(0, \mathbf{p}, \theta)\|_{X_\rho^-} + \|F(0, \mathbf{p}, \theta)\|_{X_\rho^-} \\ &\le K(w_0, \eta, \mu_{N+1}, \mu_N)\|\mathbf{U}\|_{X_\rho^-} + \|\mathbf{p}\|_{\mathbf{V}(\Omega)}, \end{aligned}$$

which in turn gives

$$\|\mathbf{U}\|_{X_\rho^-} \le \frac{1}{1 - K(w_0, \eta, \mu_{N+1}, \mu_N)} \|\mathbf{p}\|_{\mathbf{V}(\Omega)}. \tag{A.14}$$

Then, based on (A.12) and (A.14), the linear operator

$$\begin{aligned} L^N(t+\theta)\mathbf{p} &= \int_{-\infty}^{t} \mathrm{e}^{-\eta A_{N+1}^-(t-s)} P_{N+1}^- \{-P[(\mathbf{v}(\theta+s)\cdot\nabla)\mathbf{U}(s)] + P[(\mathbf{U}(s)\cdot\nabla)\mathbf{v}(\theta+s)]\}\,\mathrm{d}s \\ &= \int_{-\infty}^{0} \mathrm{e}^{\eta A_{N+1}^-\tau} P_{N+1}^- \{-P[(\mathbf{v}(\theta+t+\tau)\cdot\nabla)\mathbf{U}(t+\tau)] + P[(\mathbf{U}(t+\tau)\cdot\nabla)\mathbf{v}(\theta+t+\tau)]\}\,\mathrm{d}\tau \end{aligned} \tag{A.15}$$

satisfies the estimate

$$\|L^N(t+\theta)\mathbf{p}\|_{\mathbf{V}(\Omega)} \le K(w_0, \eta, \mu_{N+1}, \mu_N)\|\mathbf{U}\|_{X_\rho^-} \le \frac{K(w_0, \eta, \mu_{N+1}, \mu_N)}{1 - K(w_0, \eta, \mu_{N+1}, \mu_N)} \|\mathbf{p}\|_{\mathbf{V}(\Omega)},$$

which implies

$$\|L^N(t+\theta)\|_{B(\mathbf{V}(\Omega), \mathbf{V}(\Omega))} \le \frac{K(w_0, \eta, \mu_{N+1}, \mu_N)}{1 - K(w_0, \eta, \mu_{N+1}, \mu_N)}. \tag{A.16}$$

In addition, if $\mathbf{v}(\mathbf{x}, t)$ is T-periodic in t, then so is $\mathbf{U}(s; \mathbf{p}, \theta)$ in θ by (A.4). Hence, by (A.15), $L(t+\theta)$ is T-periodic in θ, and so is the set

$$\mathcal{M}(t+\theta) = \{\mathbf{p} + L^N(t+\theta)\mathbf{p} | \mathbf{p} \in \mathbf{V}(\Omega)_N^+\}. \tag{A.17}$$

A.4. Invariance of $\mathcal{M}(t)$

To show that $\mathcal{M}(t)$ is an invariant manifold for Eq. (12), it suffices to show that

$$\mathbf{U}(t; \mathbf{p}, \theta) = P_N^+ \mathbf{U}(t; \mathbf{p}, \theta) + L^N(t+\theta) P_N^+ \mathbf{U}(t; \mathbf{p}, \theta). \tag{A.18}$$

For simplicity, we now fix $\theta = 0$ in our argument. The same argument carries through for arbitrary fixed θ.

Using the variable change $\tau = s - t$, we obtain from (A.4) that

$$\mathbf{U}(t;\mathbf{p},0) = \mathrm{e}^{-\eta A_N^+ t}\mathbf{p} + \int_0^t \mathrm{e}^{-\eta A_N^+(t-s)} P_N^+\{-P[(\mathbf{v}(s)\cdot\nabla)\mathbf{U}(s;\mathbf{p},0)] + P[(\mathbf{U}(s;\mathbf{p},0)\cdot\nabla)\mathbf{v}(s)]\}\,\mathrm{d}s + \int_{-\infty}^0 \mathrm{e}^{\eta A_{N+1}^- \tau} P_{N+1}^-\{-P[(\mathbf{v}(t+\tau)\cdot\nabla)\mathbf{U}(t+\tau;\mathbf{p},0)] + P[(\mathbf{U}(t+\tau;\mathbf{p},0)\cdot\nabla)\mathbf{v}(t+\tau)]\}\,\mathrm{d}\tau. \tag{A.19}$$

Replacing t by $t+\tau$ in Eq. (A.4) gives

$$\mathbf{U}(t+\tau;\mathbf{p},0) = \mathrm{e}^{-\eta A_N^+(t+\tau)}\mathbf{p} + \int_0^{t+\tau} \mathrm{e}^{-\eta A_N^+(t+\tau-s)} P_N^+\{-P[(\mathbf{v}(s)\cdot\nabla)\mathbf{U}(s;\mathbf{p},0)] + P[(\mathbf{U}(s;\mathbf{p},0)\cdot\nabla)\mathbf{v}(s)]\}\,\mathrm{d}s + \int_{-\infty}^{t+\tau} \mathrm{e}^{-\eta A_{N+1}^-(t+\tau-s)} P_{N+1}^-\{-P[(\mathbf{v}(s)\cdot\nabla)\mathbf{U}(s;\mathbf{p},0)] + P[(\mathbf{U}(s;\mathbf{p},0)\cdot\nabla)\mathbf{v}(s)]\}\,\mathrm{d}s.$$

We pass to the new variable $r = s - t$ of integration to find that

$$\mathbf{U}(t+\tau;\mathbf{p},0) = \mathrm{e}^{-\eta A_N^+\tau} P_N^+\mathbf{U}(t;\mathbf{p},0) + \int_0^{\tau} \mathrm{e}^{-\eta A_N^+(\tau-r)} P_N^+\{-P[(\mathbf{v}(t+r)\cdot\nabla)\mathbf{U}(t+r;\mathbf{p},0)] + P[(\mathbf{U}(t+r;\mathbf{p},0)\cdot\nabla)\mathbf{v}(t+r)]\}\,\mathrm{d}r + \int_{-\infty}^{\tau} \mathrm{e}^{-\eta A_{N+1}^-(\tau-r)} P_{N+1}^-\{-P[(\mathbf{v}(t+r)\cdot\nabla)\mathbf{U}(t+r;\mathbf{p},0)] + P[(\mathbf{U}(t+r;\mathbf{p},0)\cdot\nabla)\mathbf{v}(t+r)]\}\,\mathrm{d}r,$$

which implies that

$$\mathbf{U}(t+\tau;\mathbf{p},0) = \mathbf{U}(\tau;P_N^+\mathbf{U}(t;\mathbf{p},0),t).$$

It follows, therefore, from (A.19) that

$$\mathbf{U}(t;\mathbf{p},0) = P_N^+\mathbf{U}(t;\mathbf{p},0) + \int_{-\infty}^0 \mathrm{e}^{\eta A_{N+1}^-\tau} P_{N+1}^-\{-P[(\mathbf{v}(t+\tau)\cdot\nabla)\mathbf{U}(\tau;P_N^+\mathbf{U}(t;\mathbf{p},0),t)] + P[(\mathbf{U}(\tau;P_N^+\mathbf{U}(t;\mathbf{p},0),t)\cdot\nabla)\mathbf{v}(t+\tau)]\}\,\mathrm{d}\tau = P_N^+\mathbf{U}(t;\mathbf{p},0) + L^N(t)P_N^+\mathbf{U}(t;\mathbf{p},0),$$

thus (A.18) indeed holds. Note that the uniqueness of the fixed point of the map (A.5) has been crucial in showing the invariance of $\mathcal{M}(t)$.

To complete the proof of statement (i) of Theorem 3.1, it remains to show that the direct sum decomposition (14) holds. This is relegated to Appendix B.

Appendix B

Here we complete the proof of statement (i) of Theorem 3.1, and also prove statement (ii). We shall introduce new coordinates that align with the manifolds $\mathcal{M}(t)$ and $\mathcal{N}(t)$, then show that the coordinate in the direction of $\mathcal{N}(t)$ decays to zero exponentially along solutions of the dynamo equation.

B.1. Preliminary estimates and lemmas

We begin by noting that the manifold $\mathcal{N}(t)$ admits a representation analogous to that of $\mathcal{M}(t)$ given in (A.17):

$$\mathcal{N}(t+\theta) = \{\mathbf{q} + L^{\infty}(t+\theta)\mathbf{q} | \mathbf{q} \in \mathbf{V}(\Omega)^{-}_{N+1}\}, \tag{B.1}$$

where $L^{\infty}(t+\theta) : \mathbf{V}(\Omega)^{-}_{N+1} \to \mathbf{V}(\Omega)^{+}_{N}$ is a bounded linear operator that depends continuously on t, and satisfies the estimate (cf. (A.16))

$$\|L^{\infty}(t+\theta)\|_{B(\mathbf{V}(\Omega),\mathbf{V}(\Omega))} \leq \frac{K(w_0, \eta, \mu_{N+1}, \mu_N)}{1 - K(w_0, \eta, \mu_{N+1}, \mu_N)}. \tag{B.2}$$

We now fix $\theta = 0$ for simplicity; the arguments below are similar for $\theta \neq 0$.

With the help of $L^{\infty}(t)$, we define the linear operator $\Lambda_N(t) : \mathbf{V}(\Omega) \to \mathbf{V}(\Omega)$ by letting

$$\Lambda_N(t)\mathbf{B} = L^N(t)\mathbf{p} + L^{\infty}(t)\mathbf{q},$$

with $\mathbf{p} = P^{+}_{N}\mathbf{B} \in \mathbf{V}(\Omega)^{+}_{N}$ and $\mathbf{q} = P^{-}_{N+1}\mathbf{B} \in \mathbf{V}(\Omega)^{-}_{N+1}$. If the stronger gap condition (15) is satisfied, then estimates (A.16) and (B.2) guarantee the boundedness of $\Lambda_N(t)$. More specifically, we have the bound

$$\|\Lambda_N(t)\|_{B(\mathbf{V}(\Omega),\mathbf{V}(\Omega))} \leq \frac{2K(w_0, \eta, \mu_{N+1}, \mu_N)}{1 - K(w_0, \eta, \mu_{N+1}, \mu_N)} < 1. \tag{B.3}$$

In addition, $\Lambda_N(t)$ is continuous and T-periodic in t whenever $\mathbf{v}(\mathbf{x}, t)$ is T-periodic, because $L^N(t)$ and $L^{\infty}(t)$ have similar properties.

As a direct consequence of (B.3), the operator

$$\Phi_N(t) = I + \Lambda_N(t) \tag{B.4}$$

has the following properties.

Lemma B.1. *Suppose that the gap condition* (15) *is satisfied. Then*

1. *The inverse* $\Phi_N^{-1}(t)$ *of the operator* $\Phi_N(t)$ *is a bounded linear operator for all* $t \in \mathbb{R}$.
2. $\Phi_N^{-1}(t)$ *is continuous and T-periodic in t if* $\mathbf{v}(\mathbf{x}, t)$ *is T-periodic.*
3. $\|\Phi_N(t)\|_{B(\mathbf{V}(\Omega),\mathbf{V}(\Omega))} \leq 2$ *and* $\|\Phi_N^{-1}(t)\|_{B(\mathbf{V}(\Omega),\mathbf{V}(\Omega))} \leq C$ *for all* $t \in \mathbb{R}$, *where the constant* $C > 0$ *is independent of t.*

We aim to decompose Eq. (12) into coordinate components aligned with $\mathcal{M}(t)$ and $\mathcal{N}(t)$. The following lemma shows that such a decomposition is possible, and completes the proof of statement (i) of Theorem 3.1.

Lemma B.2. *For each* $t \in \mathbb{R}$, *we have the direct sum*

$$\mathbf{V}(\Omega) = \mathcal{M}(t) \oplus \mathcal{N}(t). \tag{B.5}$$

Proof. Since

$$[I + \Lambda_N(t)]^{-1}\mathbf{B} = P^{+}_{N}[I + \Lambda_N(t)]^{-1}\mathbf{B} + P^{-}_{N+1}[I + \Lambda_N(t)]^{-1}\mathbf{B},$$

we have

$$\begin{aligned}\mathbf{B} &= [I+\Lambda_N(t)]\{P_N^+[I+\Lambda_N(t)]^{-1}\mathbf{B} + P_{N+1}^-[I+\Lambda_N(t)]^{-1}\mathbf{B}\} \\ &= P_N^+[I+\Lambda_N(t)]^{-1}\mathbf{B} + P_{N+1}^-[I+\Lambda_N(t)]^{-1}\mathbf{B} + L^N(t)P_N^+[I+\Lambda_N(t)]^{-1}\mathbf{B} \\ &\quad + L^\infty(t)P_{N+1}^-[I+\Lambda_N(t)^{-1}]\mathbf{B}. \end{aligned} \tag{B.6}$$

Observe that

$$P_N^+[I+\Lambda_N(t)]^{-1}\mathbf{B} + L^N(t)P_N^+[I+\Lambda_N(t)]^{-1}\mathbf{B} \in \mathcal{M}(t),$$
$$P_{N+1}^-[I+\Lambda_N(t)]^{-1}\mathbf{B} + L^\infty(t)P_{N+1}^-[I+\Lambda_N(t)^{-1}]\mathbf{B} \in \mathcal{N}(t).$$

Thus, by (B.6), any $\mathbf{B}$ can be written as the sum of two vectors, one in $\mathcal{M}(t)$ and one in $\mathcal{N}(t)$.

As a result, to prove (B.5), it suffices to show that $\mathcal{M}(t)\cap\mathcal{N}(t) = \{0\}$. Let $\mathbf{B}\in\mathcal{M}(t)\cap\mathcal{N}(t)$. Then there exist $\mathbf{p}\in\mathbf{V}(\Omega)_N^+$ and $\mathbf{q}\in\mathbf{V}(\Omega)_{N+1}^-$ such that

$$\mathbf{B} = \mathbf{p} + \mathbf{0} + L^N(t)\mathbf{p} + L^\infty(t)\mathbf{0} = \mathbf{q} + \mathbf{0} + L^N(t)\mathbf{0} + L^\infty(t)\mathbf{q},$$

which gives

$$(I+\Lambda_N(t))\mathbf{p} = (I+\Lambda_N(t))\mathbf{q}.$$

Hence we have $\mathbf{p}=\mathbf{q}$, which is only possible if $\mathbf{p}=\mathbf{q}=\mathbf{0}$ by Lemma B.1. □

B.2. Decay of solutions to $\mathcal{M}(t)$

As mentioned earlier, we shall establish the decay of all solutions by writing $\mathbf{B}$ in terms of coordinates aligned with $\mathcal{M}(t)$ and $\mathcal{N}(t)$, and then by proving that the latter coordinate component decays exponentially. The reader may find Fig. 2 helpful in interpreting the new coordinates we introduce. In our forthcoming argument, we set $\theta = 0$ for simplicity; a similar argument is valid for the case $\theta \neq 0$.

Projecting an arbitrary initial condition $\Phi_N^{-1}(0)\mathbf{B}_0 \in \mathbf{V}(\Omega)$ onto the subspaces $\mathbf{V}(\Omega)_N^+$ and $\mathbf{V}(\Omega)_{N+1}^-$,we obtain the vectors $\mathbf{p}_0 = P_N^+\Phi_N^{-1}(0)\mathbf{B}_0$ and $\mathbf{q}_0 = P_{N+1}^-\Phi_N^{-1}(0)\mathbf{B}_0$, respectively, with the map Φ_N defined in (B.4). As a result, we have

$$\mathbf{B}_0 = [I+\Lambda_N(0)](\mathbf{p}_0+\mathbf{q}_0) = \mathbf{p}_0 + L^N(0)\mathbf{p}_0 + \mathbf{q}_0 + L^\infty(0)\mathbf{q}_0.$$

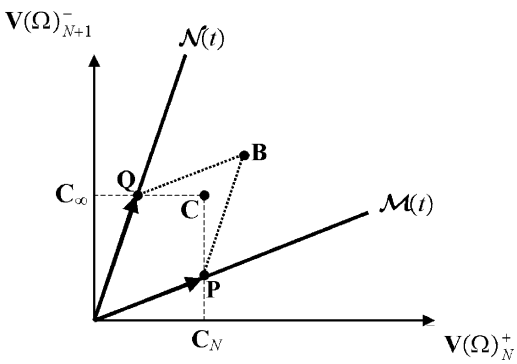

Fig. 2. The definition of the coordinates $\mathbf{C}_N$ and $\mathbf{C}_\infty$.

Let $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ be the solutions of Eq. (12) with the initial conditions $\mathbf{p}_0+L^N(0)\mathbf{p}_0$ and $\mathbf{q}_0+L^\infty(0)\mathbf{q}_0$, respectively. It then follows from (A.18) that

$$\begin{aligned}\mathbf{B}(t) = \mathbf{P}(t) + \mathbf{Q}(t) &= P_N^+\mathbf{P}(t) + L^N(t)P_N^+\mathbf{P}(t) + P_{N+1}^-\mathbf{Q}(t) + L^\infty(t)P_{N+1}^-\mathbf{Q}(t)\\ &= [I + \Lambda_N(t)][P_N^+\mathbf{P}(t) + P_{N+1}^-\mathbf{Q}(t)].\end{aligned}$$

Using the above observation, we now introduce the coordinates in which we shall study the decay of solutions to $\mathcal{M}(t)$. We let

$$\begin{aligned}&\mathbf{C}(t) = \Phi_N^{-1}(t)\mathbf{B}(t) = P_N^+\mathbf{P}(t) + P_{N+1}^-\mathbf{Q}(t), \qquad \mathbf{C}_N(t) = P_N^+\mathbf{C}(t) = P_N^+\mathbf{P}(t),\\ &\mathbf{C}_\infty(t) = P_{N+1}^-\mathbf{C}(t) = P_{N+1}^-\mathbf{Q}(t).\end{aligned} \tag{B.7}$$

The geometry of these coordinates is shown schematically in Fig. 2.

To show the decay of solutions to $\mathcal{M}(t)$, it suffices to show that $\mathbf{Q}(t)$ decays to zero. In turn, since

$$\mathbf{Q}(t) = [I + \Lambda_N(t)][P_{N+1}^-\mathbf{Q}(t)] = [I + \Lambda_N(t)]\mathbf{C}_\infty(t),$$

it suffices to show that $\mathbf{C}_\infty$ decays to zero.

We start by noting that $\mathbf{P} = P_N^+\mathbf{P}(t) + L^N(t)P_N^+\mathbf{P}(t)$ is a solution of (12), therefore

$$\begin{aligned}&[P_N^+\mathbf{P}(t) + L^N(t)P_N^+\mathbf{P}(t)]_t + P\{(\mathbf{v}\cdot\nabla)[P_N^+\mathbf{P}(t) + L^N(t)P_N^+\mathbf{P}(t)]\}\\ &\quad = -\eta A[P_N^+\mathbf{P}(t) + L^N(t)P_N^+\mathbf{P}(t)] + P\{([P_N^+\mathbf{P}(t) + L^N(t)P_N^+\mathbf{P}(t)]\cdot\nabla)\mathbf{v}\}.\end{aligned}$$

Applying the projection P_N^+ to both sides of this last equation, and noting that

$$P_N^+L^N(t)P_N^+\mathbf{P}(t) \equiv \mathbf{0},$$

we obtain a finite-dimensional homogeneous linear system of ODEs

$$\mathbf{C}_{Nt} + P_N^+P[(\mathbf{v}\cdot\nabla)\Phi_N(t)\mathbf{C}_N] = -\eta A_N^+\mathbf{C}_N + P_N^+P[(\Phi_N(t)\mathbf{C}_N\cdot\nabla)\mathbf{v}] \tag{B.8}$$

for the dynamics on $\mathcal{M}(t)$. A similar argument that uses P_{N+1}^- leads to the equation

$$\mathbf{C}_{\infty t} + P_{N+1}^-P[(\mathbf{v}\cdot\nabla)\Phi_N(t)\mathbf{C}_\infty] = -\eta A_{N+1}^-\mathbf{C}_\infty + P_{N+1}^-[P(\Phi_N(t)\mathbf{C}_\infty\cdot\nabla)\mathbf{v}] \tag{B.9}$$

for the dynamics on the manifold $\mathcal{N}(t)$.

To show that the coordinate $\mathbf{C}_\infty$ decays to zero along solutions of (12), we first estimate $\varphi_{N+1}(t)$ defined in (A.8). For this, we observe that

$$\max_{t\geq 0} t^\delta\,\mathrm{e}^{-bt} = \left(\frac{\delta}{b}\right)^\delta \mathrm{e}^{-\delta}$$

for any $\delta, b > 0$. It then follows that for $0 < t \leq 1/2\eta\mu_{N+1}$,

$$\varphi_{N+1}(t) = \frac{1}{\sqrt{2et}} \leq \frac{1}{\sqrt{et}}\,\mathrm{e}^{-\eta\mu_{N+1}t/2}\,\mathrm{e}^{\eta\mu_{N+1}t/2} \leq \frac{\mathrm{e}^{-1/4}}{\sqrt{t}}\,\mathrm{e}^{-\eta\mu_{N+1}t/2}$$

and for $1/2\eta\mu_{N+1} < t < \infty$,

$$\varphi_{N+1}(t) = \sqrt{\eta\mu_{N+1}}\,\mathrm{e}^{-\eta\mu_{N+1}t} = \sqrt{\eta\mu_{N+1}}t^{-1/2}\,\mathrm{e}^{-\eta\mu_{N+1}t/2}t^{1/2}\,\mathrm{e}^{-\eta\mu_{N+1}t/2} \leq \frac{\mathrm{e}^{-1/2}}{\sqrt{t}}\,\mathrm{e}^{-\eta\mu_{N+1}t/2}.$$

Therefore, we deduce from (A.8), (B.2) and (B.9) that

$$\|\mathbf{C}_\infty(t)\|_{\mathbf{V}(\Omega)} \le \|e^{-\eta A^-_{N+1}t}\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)} + \left\|\int_0^t e^{-\eta A^-_{N+1}(t-s)} P^-_{N+1}\{P(\mathbf{v}(s)\cdot\nabla)[I+L^\infty(s)]\mathbf{C}_\infty(s)\}\,ds\right\|_{\mathbf{V}(\Omega)}$$
$$+ \left\|\int_0^t e^{-\eta A^-_{N+1}(t-s)} P^-_{N+1}P\{([I+L^\infty(s)]\mathbf{C}_\infty(s)\cdot\nabla)\mathbf{v}(s)\}\,ds\right\|_{\mathbf{V}(\Omega)} \le e^{-\eta\mu_{N+1}t}\|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)}$$
$$+ \int_0^t \eta^{-1/2}\varphi_{N+1}(t-s)\|P^-_{N+1}\{P(\mathbf{v}(s)\cdot\nabla)[I+L^\infty(s)]\mathbf{C}_\infty(s)\}\|_{L^2}\,ds$$
$$+ \int_0^t \eta^{-1/2}\varphi_{N+1}(t-s)\|P^-_{N+1}P\{([I+L^\infty(s)]\mathbf{C}_\infty(s)\cdot\nabla)\mathbf{v}(s)\}\|_{L^2}\,ds \le e^{-\eta\mu_{N+1}t}\|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)}$$
$$+ v_0\int_0^t \eta^{-1/2}\varphi_{N+1}(t-s)\|\nabla[I+L^\infty(s)]\mathbf{C}_\infty(s)\|_{L^2}\,ds$$
$$+ u_0\int_0^t \eta^{-1/2}\varphi_{N+1}(t-s)\|[I+L^\infty(s)]\mathbf{C}_\infty(s)\|_{L^2}\,ds.$$

This implies

$$\|\mathbf{C}_\infty(t)\|_{\mathbf{V}(\Omega)} \le e^{-\eta\mu_{N+1}t}\|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)} + 2v_0\int_0^t \eta^{-1/2}\varphi_{N+1}(t-s)\|\mathbf{C}_\infty(s)\|_{\mathbf{V}(\Omega)}\,ds$$
$$+ 2\mu_1^{-1/2}u_0\int_0^t \eta^{-1/2}\varphi_{N+1}(t-s)\|\mathbf{C}_\infty(s)\|_{\mathbf{V}(\Omega)}\,ds \le e^{-\eta\mu_{N+1}t}\|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)} + 2w_0\eta^{-1/2}e^{-1/4}$$
$$\times\int_0^t (t-s)^{-1/2}e^{-\eta\mu_{N+1}(t-s)/2}\|\mathbf{C}_\infty(s)\|_{\mathbf{V}(\Omega)}\,ds,$$

leading to

$$\|\mathbf{C}_\infty(t)\|_{\mathbf{V}(\Omega)}e^{\eta\mu_{N+1}t/2} \le \|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)} + 2w_0\eta^{-1/2}e^{-1/4}\int_0^t (t-s)^{-1/2}e^{\eta\mu_{N+1}s/2}\|\mathbf{C}_\infty(s)\|_{\mathbf{V}(\Omega)}\,ds. \tag{B.10}$$

To integrate this last inequality, we recall a modified form of the classic Gronwall inequality (see, e.g., Henry [8, Lemma 7.1.1]). Suppose that $a, b \ge 0$, $\delta > 0$, and the function $\phi(t)$ is nonnegative and locally integrable on $[0, +\infty)$, satisfying

$$\phi(t) \le a + b\int_a^t (t-s)^{\delta-1}\phi(s)\,ds, \quad 0 \le t < +\infty.$$

Then

$$\phi(t) \le aE_\delta(\theta t), \quad 0 \le t < +\infty, \tag{B.11}$$

where

$$\theta = (b\Gamma(\delta))^{1/\delta}, \qquad E_\delta(z) = \sum_{n=0}^\infty \frac{z^{n\delta}}{\Gamma(n\delta+1)} \simeq \frac{e^z}{\delta} \quad \text{as } z\to+\infty.$$

The generalized Gronwall inequality (B.11) applied to (B.10) gives

$$\|\mathbf{C}_\infty(t)\|_{\mathbf{V}(\Omega)}e^{\eta\mu_{N+1}t/2} \le \|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)}E_{1/2}((2w_0\eta^{-1/2}e^{-1/4}\sqrt{\pi})^2 t)$$
$$\le C\|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)}\exp(4\pi\eta^{-1}e^{-1/2}w_0^2 t), \tag{B.12}$$

or, equivalently,

$$\|\mathbf{C}_\infty(t)\|_{\mathbf{V}(\Omega)} \leq C\|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)} \exp[-(0.5\eta\mu_{N+1} - 4\pi\eta^{-1}\,\mathrm{e}^{-1/2}w_0^2)t], \tag{B.13}$$

where C denotes a positive constant. Under the gap condition (15), we have

$$\frac{3}{\sqrt{\mu_{N+1}}} \leq \frac{2}{\sqrt{\mu_{N+1}} - \sqrt{\mu_N}} + \frac{1}{\sqrt{\mu_{N+1}}} < \frac{\eta}{3w_0}$$

and then $\eta\mu_{N+1}/2 > 40w_0^2/\eta > 4\pi\eta^{-1}\,\mathrm{e}^{-1/2}w_0^2$. Hence (B.13) shows the exponential decay of the coordinate component transverse to $\mathcal{M}(t)$, completing the proof of statement (ii) of Theorem 3.1.

Appendix C

Here we prove Theorem 3.2.

By the assumptions of Theorem 3.2, for any small $\varepsilon > 0$, we can select an integer $N(\varepsilon)$ such that the gap condition (15) holds and

$$\|\mathbf{C}_\infty(0)\|_{\mathbf{V}(\Omega)} = \|P_{N+1}^{-}\Phi_N^{-1}(0)\mathbf{B}_0\|_{\mathbf{V}(\Omega)} \leq \frac{\varepsilon}{2C}, \tag{C.1}$$

where the coordinate $\mathbf{C}_\infty$ and the mapping $\Phi_{N(\varepsilon)}$ are defined in (B.7) and (B.4), respectively, and C is the constant appearing in the estimate (B.13). The inequality (C.1) follows because the remainder term of $N(\varepsilon)$th-order expansion of $\Phi_N^{-1}(0)\mathbf{B}_0$ with respect to the eigenfunctions of A can be made arbitrarily small for large enough $N(\varepsilon)$.

For the above choice of $N(\varepsilon)$, Theorem 3.1 gives the existence of an $N(\varepsilon)$-dimensional linear inertial manifold $\mathcal{M}(t)$. Restricting the dynamo equation to this manifold yields an $N(\varepsilon)$-dimensional, homogenous set of linear ODEs, which we wrote out in Eq. (B.8). The coefficient matrix of this set of equations is continuous in t by assumption.

Invoking the classical Floquet theory for ordinary differential equations (see, e.g. [24]), we obtain that solutions of the ODE on $\mathcal{M}(t)$ are linear combinations of Floquet solutions of the form

$$\mathbf{f}_k(\mathbf{x},t) = \mathbf{f}_k^0(\mathbf{x},t) + t\mathbf{f}_k^1(\mathbf{x},t) + \cdots + t^{l(k)}\mathbf{f}_k^{l(k)}(\mathbf{x},t), \quad k = 1,\ldots,N(\varepsilon),$$

where $\mathbf{f}_k^j(\mathbf{x},t)$ are T-periodic in time. This means that a solution on $\mathcal{M}(t)$ can be written as

$$\mathbf{C}_{N(\varepsilon)} = \sum_{k=1}^{N(\varepsilon)} c_k\,\mathrm{e}^{\lambda_k t}\mathbf{f}_k(\mathbf{x},t),$$

where λ_k denotes the Floquet exponent corresponding to $\mathbf{f}_k(\mathbf{x},t)$.

We then obtain that

$$\mathbf{e}_k(\mathbf{x},t) = \Phi_{N(\varepsilon)}(t)\mathbf{f}_k(\mathbf{x},t) = \Phi_{N(\varepsilon)}(t)\mathbf{f}_k^0(\mathbf{x},t) + t\Phi_{N(\varepsilon)}(t)\mathbf{f}_k^1(\mathbf{x},t) + \cdots + t^{l(k)}\Phi_{N(\varepsilon)}(t)\mathbf{f}_k^{l(k)}(\mathbf{x},t),$$
$$k = 1,\ldots,N(\varepsilon),$$

are eigenmodes of Eq. (12) with eigenvalues λ_k, and that the full solution of (12) can be written as

$$\mathbf{B}(\mathbf{x},t) = \Phi_{N(\varepsilon)}(t)(\mathbf{C}_N + \mathbf{C}_\infty) = \sum_{k=1}^{N(\varepsilon)} c_k\,\mathrm{e}^{\lambda_k t}\mathbf{e}_k(\mathbf{x},t) + \Phi_{N(\varepsilon)}(t)\mathbf{C}_\infty,$$

which proves (17). The estimate (18) then follows from (B.13).

Appendix D

Here we prove Theorem 3.3.

In the generic case, the Floquet exponents associated with the linear flow on the inertial manifold $\mathcal{M}(t)$ are not repeated. In that case, substitution of the simple Floquet solution $e^{\lambda_0 t}\mathbf{e}_0(\mathbf{x}, t)$ into the dynamo equation (1) gives

$$\partial_t \mathbf{e}_0 + \lambda_0 \mathbf{e}_0 + (\mathbf{v}\cdot\nabla)\mathbf{e}_0 = \eta\nabla^2\mathbf{e}_0 + (\mathbf{e}_0\cdot\nabla)\mathbf{v}. \tag{D.1}$$

Multiplying this equation by $\mathbf{e}_0^*$ (the complex conjugate of $\mathbf{e}_0$) leads to

$$\mathbf{e}_0^*\cdot\partial_t\mathbf{e}_0 + \lambda_0|\mathbf{e}_0|^2 + \mathbf{e}_0^*\cdot(\mathbf{v}\cdot\nabla)\mathbf{e}_0 = \eta\mathbf{e}_0^*\cdot\nabla^2\mathbf{e}_0 + \mathbf{e}_0^*\cdot(\mathbf{e}_0\cdot\nabla)\mathbf{v}.$$

We add this last equation to its complex conjugate, and integrate over the domain Ω to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}_0\|^2 &= -2\,\mathrm{Re}\,\lambda_0\|\mathbf{e}_0\|^2 - \int_\Omega \mathbf{v}\cdot\nabla(\|\mathbf{e}_0\|^2)\,\mathrm{d}V + 2\eta\int_\Omega \mathbf{e}_0^*\cdot\nabla^2\mathbf{e}_0\,\mathrm{d}V + 2\,\mathrm{Re}\int_\Omega \mathbf{e}_0^*\cdot(\mathbf{e}_0\cdot\nabla)\mathbf{v}\,\mathrm{d}V \\ &= -2\,\mathrm{Re}\,\lambda_0\|\mathbf{e}_0\|^2 - 2\eta\|\nabla\mathbf{e}_0\|^2 + 2\,\mathrm{Re}\int_\Omega \mathbf{e}_0^*\cdot(\mathbf{e}_0\cdot\nabla)\mathbf{v}\,\mathrm{d}V,\end{aligned} \tag{D.2}$$

where we used the incompressibility of $\mathbf{v}$ as well as the boundary conditions on $\mathbf{v}$ and $\mathbf{e}_0$.

By the T-periodicity of the function $\mathbf{e}_0$, integration of (D.2) with respect to t over $[0, T]$ gives

$$\alpha = \mathrm{Re}\,\lambda_0 = \frac{-\eta\overline{\|\nabla\mathbf{e}_0\|^2} + \mathrm{Re}\,\overline{\langle\mathbf{e}_0, (\mathbf{e}_0^*\cdot\nabla)\mathbf{v}\rangle}}{\overline{\|\mathbf{e}_0\|^2}}, \tag{D.3}$$

which proves the first formula in (20).

Next, we split $\mathbf{e}_0$ and λ_0 into real and imaginary parts by letting

$$\mathbf{e}_0(\mathbf{x}, t) = \mathbf{g}(\mathbf{x}, t) + \mathrm{i}\mathbf{h}(\mathbf{x}, t), \qquad \lambda_0 = \alpha + \mathrm{i}\beta,$$

where α and β are real constants, and $\mathbf{g}$ and $\mathbf{h}$ are real vector-valued functions that satisfy the boundary conditions, and are T-periodic in t. Substitution into (D.1) then gives a complex equation whose real part is

$$\mathbf{g}_t + (\mathbf{v}\cdot\nabla)\mathbf{g} = -\alpha\mathbf{g} + \beta\mathbf{h} + \eta\nabla^2\mathbf{g} + (\mathbf{g}\cdot\nabla)\mathbf{v}.$$

Multiplying this equation by $\mathbf{g}$, integrating over the domain Ω, then averaging in time as before leads to

$$\beta = \frac{\alpha\overline{\|\mathbf{g}\|^2} + \eta\overline{\|\nabla\mathbf{g}\|^2} - \overline{\langle\mathbf{g}, (\mathbf{g}\cdot\nabla)\mathbf{v}\rangle}}{\overline{\langle\mathbf{g}, \mathbf{h}\rangle}}, \tag{D.4}$$

which proves the second formula in (20).

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