J. Fluid Mech. (2014), *vol.* 755, R3, doi:10.1017/jfm.2014.441

JFM RAPIDS
journals.cambridge.org/rapids

Addendum to 'Coherent Lagrangian vortices: the black holes of turbulence'

G. Haller[1,†] and F. J. Beron-Vera[2]

[1]Institute of Mechanical Systems, ETH Zurich, 8092 Zurich, Switzerland

[2]Department of Atmospheric Sciences, RSMAS, University of Miami, Miami, FL 33149, USA

(Received 22 May 2014; revised 27 June 2014; accepted 24 July 2014; first published online 14 August 2014)

In Haller & Beron-Vera (*J. Fluid Mech.*, vol. 731, 2013, R4) we developed a variational principle for the detection of coherent Lagrangian vortex boundaries. The solutions of this variational principle turn out to be closed null geodesics of the Lorentzian metric induced by a generalized Green–Lagrange strain tensor family. This metric interpretation implies a mathematical analogy between coherent Lagrangian vortex boundaries and photon spheres in general relativity. Here, we give an improved discussion of this analogy.

Key words: geophysical and astrophysical flows, geostrophic turbulence, ocean circulation

1. The main results of Haller & Beron-Vera (2013)

We consider a two-dimensional velocity field $v(x, t)$, with x labelling the location within a two-dimensional region U of interest and with t referring to time. Fluid trajectories generated by $v(x, t)$ are denoted $x(t; t_0, x_0)$, with x_0 referring to the initial position of the trajectory at time t_0. These trajectories solve the differential equation

$$\dot{x} = v(x, t) \tag{1.1}$$

and generate the flow map

$$F_{t_0}^{t}(x_0) := x(t; t_0, x_0), \tag{1.2}$$

which takes an initial position x_0 at time t_0 to its current position at time t.

The right Cauchy–Green strain tensor field associated with the flow map is defined as $C_{t_0}^{t}(x_0) = \nabla F_{t_0}^{t}(x_0)^{\top} \nabla F_{t_0}^{t}(x_0)$, with eigenvalues $\lambda_i(x_0)$ and eigenvectors $\xi_i(x_0)$ satisfying

$$C_{t_0}^{t}\xi_i = \lambda_i \xi_i, \quad |\xi_i| = 1, \, i = 1, 2; \, 0 < \lambda_1 \leqslant \lambda_2, \, \xi_1 \perp \xi_2. \tag{1.3}$$

In Haller & Beron-Vera (2013) we sought the time t_0 positions of Lagrangian vortex boundaries as closed stationary curves of the averaged Lagrangian strain. Such curves

† Email address for correspondence: georgehaller@ethz.ch

turned out to coincide with the zero-energy solutions of a one-parameter family of variational problems defined as

$$\delta\mathcal{E}_\lambda(\gamma)=0, \quad \mathcal{E}_\lambda(\gamma)=\oint_\gamma \langle r'(s), \boldsymbol{E}_\lambda(r(s))r'(s)\rangle\,\mathrm{d}s, \;\; \lambda\in\mathbb{R}^+. \tag{1.4}$$

Here, the strain energy functional $\mathcal{E}_\lambda(\gamma)$ is defined through the generalized Green–Lagrange strain tensor family

$$\boldsymbol{E}_\lambda(x_0)=\tfrac{1}{2}[\boldsymbol{C}_{t_0}^t(x_0)-\lambda^2\boldsymbol{I}], \quad \text{where } \boldsymbol{I} \text{ is the identity matrix.} \tag{1.5}$$

Consider the flow domain

$$U_\lambda=\{x_0\in U \mid \lambda_1(x_0)<\lambda^2<\lambda_2(x_0)\}, \tag{1.6}$$

where the tensor field $\boldsymbol{E}_\lambda$ has two non-zero eigenvalues of opposite sign. Then the quadratic function

$$g_\lambda(u, u)=\langle u, \boldsymbol{E}_\lambda u\rangle \tag{1.7}$$

defines a Lorentzian metric (Beem, Ehrlich & Kevin 1996) on U_λ, with signature $(-, +)$ inherited from the eigenvalue configuration of $\boldsymbol{E}_\lambda$. The zero-energy solutions of (1.4) are therefore precisely the closed null geodesics of the Lorentzian metric g_λ, which satisfy one of the two differential equations

$$r'(s)=\eta_\lambda^\pm(r(s)), \quad \eta_\lambda^\pm(r)=\sqrt{\frac{\lambda_2(r)-\lambda^2}{\lambda_2(r)-\lambda_1(r)}}\,\xi_1(r)\pm\sqrt{\frac{\lambda^2-\lambda_1(r)}{\lambda_2(r)-\lambda_1(r)}}\,\xi_2(r). \tag{1.8a,b}$$

In Haller & Beron-Vera (2013) we concluded that closed orbits of (1.8) (termed closed *λ lines*) must necessarily encircle metric singularities of g_λ. Such singularities occur at points x_0 where $\lambda_1(r)=\lambda_2(r)$ holds for the eigenvalues of the Cauchy–Green strain tensor.

The Lorentzian metric interpretation discussed above implies a geometric analogy between coherent Lagrangian vortex boundaries and photon spheres in cosmology. Below, we give more detail on this analogy, followed by an improved version of its summary with a more relevant reference.

2. More on the analogy with photon spheres

In the vicinity of any λ line in U_λ, the vector fields ξ_i define a curvilinear coordinate system with pointwise orthogonal coordinate lines. Direct substitution of ξ_i into the metric (1.7) gives

$$g_\lambda(\xi_i, \xi_i)=\tfrac{1}{2}(\lambda_i-\lambda^2), \tag{2.1}$$

showing that $g_\lambda(\xi_1, \xi_1)<0$ and $g_\lambda(\xi_2, \xi_2)>0$ everywhere in U_λ. This shows that the ξ_1 trajectories form the time-like coordinates and the ξ_2 trajectories form the space-like coordinates of the metric g_λ in U_λ (Beem *et al.* 1996). We further note that $g_\lambda(\eta_\lambda^\pm, \eta_\lambda^\pm)=0$, and hence a λ line is nowhere space-like in the language of Lorentzian geometry. Given that our closed null geodesics are nowhere space-like hypersurfaces built out of null geodesics, they are photon surfaces by the general definition of Claudel, Virbhadra & Ellis (2001).

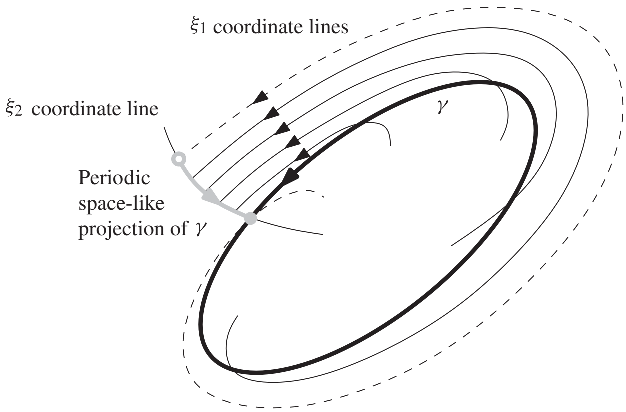

FIGURE 1. A closed null geodesic γ of the Lorentizan metric g_γ has a periodic space-like projection along the time-like coordinate lines. This space-like projection has a jump discontinuity due to the low dimensionality of the space–time, but evolves periodically along with the underlying closed null geodesic. This is in contrast to general λ lines which have aperiodic space-like projections.

Next, we note that

$$\left\langle \eta_\lambda^{\pm}(r), \xi_1(r) \right\rangle = \sqrt{\frac{\lambda_2(r) - \lambda^2}{\lambda_2(r) - \lambda_1(r)}} \in (0, 1), \quad r \in U_\lambda. \tag{2.2}$$

Consequently, trajectories of the $\xi_1(r)$ line field (ξ_1 coordinate lines) intersect any closed λ line γ transversely, with an angle of intersection that is always less than $\pi/2$, as sketched qualitatively in figure 1. In the same figure, we also show a representative trajectory of the $\xi_2(r)$ line field (a ξ_2 coordinate line), which is pointwise orthogonal to the ξ_1 coordinate lines by construction. We conclude that the projection of γ onto a space-like submanifold along the time-like coordinates results in a periodic (albeit discontinuous) space-like orbit (cf. figure 1).

In summary, a closed orbit γ of the $\eta_\lambda^{\pm}(r)$ vector field is a photon surface of the (U_λ, g_λ) space–time. Geodesics forming this photon surface have periodically moving projections on the space-like coordinates, with the projection taken along the time-like coordinate lines.

The orbit γ, therefore, satisfies a plausible extension of the Claudel–Virbhadra–Ellis definition of a photon sphere (Claudel *et al.* 2001) from symmetric higher-dimensional space–times to non-symmetric two-dimensional space–times. This extension relaxes the requirement for a rotational symmetry and smoothness of the projected spatially periodic orbits. Both of these features are unattainable in curved two-dimensional space–times, and hence can reasonably be waived.

3. Revised wording of the cosmological analogy

In view of the above discussion, the wording of the mathematical analogy between coherent Lagrangian eddy boundaries and photon spheres in cosmology (Haller & Beron-Vera 2013, p. 731, para. 2) should be revised as follows.

> The closed λ lines we have been seeking are therefore closed null geodesics of g_λ. Trajectories spanning these geodesics project along the

local time-like coordinates (trajectories of the ξ_1 vector field) onto periodic trajectories on the local space-like coordinates (trajectories of the ξ_2 vector field). In cosmology, the projection of some families of null geodesics from space–time onto the space-like variables also produces closed trajectories. Such families of null geodesics are often referred to as photon spheres, as their spatial projections trap photons orbiting around black holes (Claudel *et al.* 2001). In the cosmological context, null geodesics are tangent to light cones, which in our case are formed by the two vectors $\eta_\lambda^\pm(x_0)$ at x_0 (figure 2 of Haller & Beron-Vera 2013).

Acknowledgements

We acknowledge comments by A. Prain and V. Faraoni which made us realize the need for this clarifying note. We are grateful to M. Farazmand for his numerical work in connection with figure 1. F.J.B.-V. acknowledges support by NSF grant CMG0825547 and by a grant from the BP/Gulf of Mexico Research Initiative.

References

Beem, J. K., Ehrlich, P. L. & Kevin, L. E. 1996 *Global Lorentzian Geometry*. CRC Press.

Claudel, C.-M., Virbhadra, K. S. & Ellis, G. F. R. 2001 The geometry of photon surfaces. *J. Math. Phys.* **42**, 818–838.

Haller, G. & Beron-Vera, F. J. 2013 Coherent Lagrangian vortices: the black holes of turbulence. *J. Fluid Mech.* **731**, R4.