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# Nonlinear normal modes and spectral submanifolds: existence, uniqueness and use in model reduction

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**Abstract** We propose a unified approach to nonlinear modal analysis in dissipative oscillatory systems. This approach eliminates conflicting definitions, covers both autonomous and time-dependent systems and provides exact mathematical existence, uniqueness and robustness results. In this setting, a nonlinear normal mode (NNM) is a set filled with small-amplitude recurrent motions: a fixed point, a periodic orbit or the closure of a quasiperiodic orbit. In contrast, a spectral submanifold (SSM) is an invariant manifold asymptotic to a NNM, serving as the smoothest nonlinear continuation of a spectral subspace of the linearized system along the NNM. The existence and uniqueness of SSMs turns out to depend on a spectral quotient computed from the real part of the spectrum of the linearized system. This quotient may well be large even for small dissipation; thus, the inclusion of damping is essential for firm conclusions about NNMs, SSMs and the reduced-order models they yield.

**Keywords** Nonlinear normal modes · Invariant manifolds · Model reduction

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## 1 Introduction

Decomposing nonlinear oscillations in analogy with linear modal analysis has been an exciting perspective for several decades in multiple disciplines. In the engineering mechanics literature, this approach was initiated by Rosenberg [38], who defines a nonlinear normal mode in a conservative system as a synchronous periodic oscillation that reaches its maximum in all modal coordinates at the same time. Shaw and Pierre [39] offer an elegant alternative, envisioning nonlinear normal modes as invariant manifolds that are locally graphs over two-dimensional modal subspaces of the linearized system. These definitions have subsequently been relaxed and generalized to different settings, as surveyed by the recent reviews of Avramov and Mikhlin [3,4], Kerschen [25] and Renson et al. [37].

In conservative autonomous systems, a relationship between the above two views on nonlinear normal modes is established by the subcenter-manifold theorem of Lyapunov [17]. In its strongest version due to Kelley [23], this theorem guarantees that unique and analytic invariant manifolds tangent to two-dimensional modal subspaces of the linearized system at an elliptic fixed point persist in an analytic nonlinear system under appropriate nonresonance conditions. These persisting manifolds are in turn filled with periodic orbits. Roughly speaking, therefore, conservative Shaw–Pierre-type normal modes are just surfaces composed of Rosenberg-type normal modes, if one relaxes Rosenberg's synchrony requirement, as is routinely done in the literature.

A similar relationship, however, is absent between the two normal mode concepts for non-conservative or non-autonomous systems. In such settings, periodic orbits become rare and isolated in the phase space. At the same time, either no or infinitely many invariant manifolds tangent to eigenspaces may exist, most often without containing any periodic orbit. Having then identical terminology for two such vastly different concepts is clearly less than optimal. Furthermore, while both dissipative normal mode concepts are inspired by nonlinear dynamical systems theory, neither of the two has been placed on firm mathematical foundations comparable to other classic concepts in nonlinear dynamics, such as stable, unstable and center manifolds near equilibria (see, e.g., Guckenheimer and Holmes [16] for a survey).

Indeed, as Neild et al. [30] observe, the envisioned Shaw–Pierre-type invariant surfaces are already non-unique in the linearized system, and there is no known result guaranteeing their persistence as nonlinear normal modes in the full nonlinear system. These authors propose normal form theory as a more expedient computational tool to investigate near-equilibrium dynamics for model reduction purposes. Truncated normal forms, however, offer no a priori guarantee for the actual existence of the structures they predict either. Rather, the persistence of such structures needs to be investigated on a case-by-case basis either numerically or via mathematical analysis.

Cirillo et al. [10,11] also observe the non-uniqueness of invariant manifolds tangent to eigenspaces in a two-dimensional linear example. They point out that only one of these manifolds is infinitely many times differentiable and then state without further analysis that there is a unique, analytic Pierre–Shaw-type invariant surface tangent to any two-dimensional modal subspace of a nonlinear system. While a proof of this claim is yet to be provided, the authors also put forward a computational technique for the construction of invariant manifolds on larger domains of the phase space. Their proposed approach is actually a special case of the classic *parametrization method* (see, e.g., Cabré et al. [9] for a historical and technical survey), which forms the basis of some of the rigorous invariant manifold results we will use in the present paper.

The above concerns about an ambiguity in the definition of Shaw–Pierre-type normal modes have been sporadic in the literature. One reason might be the general expectation that if one manages to compute arbitrarily many terms in the Taylor-series approximation of an envisioned invariant surface, then that surface is bound to exist and be unique. While the success of a low-order numerical or Taylor approximation to an envisioned invariant manifold is certainly encouraging, by no means does it give any guarantee for the existence of a unique manifold. This classic issue is well documented for the divergence of Lindstedt series for invariant tori in conservative systems (Arnold [2]). For dissipative systems, an early example of a divergent expansion for an invariant manifold was already pointed out by Euler [13] (cf. Arnold [1]).

We recall Euler's example here briefly in a slightly altered form relevant to damped vibrations. Consider the planar dynamical system

$$\begin{aligned} \dot{x} &= -x^2, \\ \dot{y} &= -y + x, \end{aligned} \tag{1}$$

whose right-hand side is analytic on the whole $(x, y)$ plane. A formal Taylor series for a center manifold tangent to the $x$-axis at the origin is computable up to any order, but diverges for any $x \neq 0$. Therefore, the formal Taylor expansion of the center manifold does not converge to any analytic invariant manifold (cf. Appendix "Modified Euler example of a non-analytic but $C^\infty$ center manifold" for details). Accordingly, there is a continuous family of non-unique, non-analytic center manifolds with vastly different global shapes for $x > 0$ (cf. Fig. 1). None of these manifolds is distinguished in any way. Approximating any one of them analytically or numerically, then reducing the full system to this approximation leads to a highly arbitrary reduced model outside a neighborhood of the fixed point.

The global phase space dynamics of higher-dimensional systems cannot be visualized in such a simple way as in Fig. 1. Accordingly, the non-uniqueness of Shaw–Pierre-type invariant surfaces is often overlooked or ignored in computational studies for multi-degree-of-freedom problems (see Renson et al. [37] for a recent review). Some of these approaches solve a PDE for the invariant manifold with ill-posed boundary conditions; others use the modal subspaces of the linearization to set boundary conditions away from the fixed point; yet others envision a uniquely defined boundary condition that they determine by minimizing an ad hoc cost function (cf. Appendix "Uniqueness issues for invariant manifolds obtained from numerical

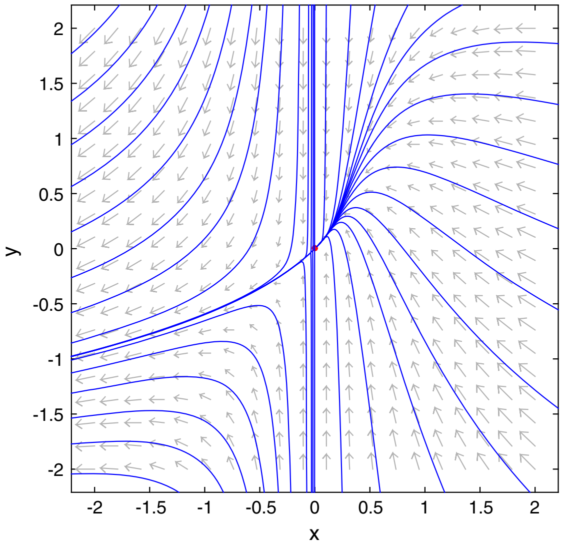



**Fig. 1** Phase portrait of the dynamical system (1) showing infinitely many $C^\infty$ invariant manifolds with vastly different global behaviors. The formal computability of the common Taylor expansion of these manifolds up to any order, therefore, does not imply their uniqueness

solutions of PDEs" for details). In all cases, the computed invariant manifold depends on the choice of basis functions, or domain boundaries or cost functions used in the process. The resulting ambiguities in the solutions are small close to the equilibrium, but are vastly amplified over larger domains where nonlinear normal mode analysis is meant to surpass the results from linearization (cf. Fig. 1).

Here we discuss a unified mathematical approach to nonlinear normal modes in dissipative systems to address these issues. First, we propose eliminating the ambiguity in the terminology itself. Borrowing the original concept of Rosenberg [38] from conservative systems, we call a near-equilibrium quasiperiodic motion in a dissipative, nonlinear system a *nonlinear normal mode* (NNM). Such NNMs are certainly special, but the invariant surfaces envisioned in the seminal work of Shaw and Pierre [39] are arguably more influential for the overall system dynamics, and can be viewed as invariant surfaces asymptoting to eigenspaces along a NNM. To emphasize this distinction, we will refer to the smoothest member of an invariant manifold family tangent to a modal subbundle along an NNM as a *spectral submanifold* (SSM). Our precise definitions of NNMs and SSMs (to be given in Definitions 1 and 2) are general enough to apply to both autonomous and externally forced systems with finitely many forcing frequencies.

With this terminology at hand, we employ classical invariant manifold results of Fenichel [15] and more recent invariant manifold results of Cabré et al. [8] and Haro and de la Llave [18] to deduce existence, uniqueness, regularity and robustness theorems for NNMs and SSMs, respectively. The conditions of these theorems are computable solely from the spectrum of the linearized system. Contrary to common expectation in vibration theory, however, the mathematical conditions for NNMs and SSMs are more affected by the real part of the spectrum, rather than the imaginary part (i.e., frequencies) of the oscillations. Therefore, even weak damping should be carefully considered and analyzed, rather than ignored, if one wishes to construct robust SSMs for model reduction purposes. We illustrate our results on simple, low-dimensional examples and discuss the relevance of our findings for model reduction. More detailed numerical examples of higher-dimensional mechanical systems will be treated elsewhere.

## 2 Setup

Our study is motivated by, but not restricted to, $n$-degree-of- freedom mechanical systems of the form

$$M\ddot{q} + (C+G)\,\dot{q} + (K+B)\,q = F_0(q,\dot{q}) + \epsilon F_1(q,\dot{q},\Omega_1 t,\ldots,\Omega_k t;\epsilon), \quad 0 \le \epsilon \ll 1, \tag{2}$$

$$F_0(q,\dot{q}) = \mathcal{O}\left(|q|^2, |q|\,|\dot{q}|, |\dot{q}|^2\right), \tag{3}$$

where $q = (q_1,\ldots,q_n) \in U \subset \mathbb{R}^n$ is the vector of generalized coordinates defined on an open set $U$; $M = M^T \in \mathbb{R}^{n\times n}$ is the positive definite mass matrix; $C = C^T \in \mathbb{R}^{n\times n}$ is a positive semi-definite damping matrix; $G = -G^T \in \mathbb{R}^{n\times n}$ is the gyroscopic matrix; $K = K^T \in \mathbb{R}^{n\times n}$ is a positive semidefinite stiffness matrix; $B = -B^T \in \mathbb{R}^{n\times n}$ is the coefficient matrix of follower forces; the vector $F_0 \in \mathbb{R}^n$ represents autonomous nonlinearities; and the vector $F_1 \in \mathbb{R}^n$ denotes external forcing with the frequency vector $\Omega = (\Omega_1,\ldots,\Omega_k) \in \mathbb{R}^k$ with $k \ge 0$. Note that $F_1(q,\dot{q},\Omega_1 t,\ldots,\Omega_k t)$ is not necessarily nonlinear and hence can in principle be large even when $|q|$ and $|\dot{q}|$ are small. In the special case of $k=0$, the external forcing is autonomous, while in the case of $k=1$, the external forcing is time-periodic. For $k > 1$, the external forcing is quasiperiodic if at least two of the frequencies $\Omega_j$ are rationally incommensurate. We assume both $F_0$ and $F_1$ to be of class $C^r$ in their arguments, where $r$ is either a nonnegative integer, $\infty$, or equal to $a$, with $C^a$ referring to analytic functions. In short, we assume

$$r \in \mathbb{N}^+ \cup \{\infty, a\}. \tag{4}$$

For $\epsilon = 0$, system (2) has an equilibrium point at $q = 0$. Linear oscillations around this equilibrium point are governed by the spectral properties of the linearized system on the left-hand side of (2). Our main interest here is the relevance of these linear oscillations for the dynamics of the full system (2). A strict mathematical relationship between linear and nonlinear oscillations can only be expected near the equilibrium (i..e, for small values of $|q|$ and $|\dot{q}|$) and for small values of the forcing parameter $\epsilon$. We seek to establish, however, the existence of nonlinear sets of solutions near the equilibrium that continue to extend to larger domains of the phase space and hence exert a more global influence on the system dynamics.

After the change of variables $x_1 = q, x_2 = \dot{q}$, the evolution of the vector $x = (x_1, x_2) \in \mathcal{U} = U \times \mathbb{R}^n$ is governed by the first-order differential equation

$$\dot{x} = Ax + f_0(x) + \epsilon f_1(x, \Omega t; \epsilon),$$
$$f_0(x) = \mathcal{O}(|x|^2), \quad 0 \leq \epsilon \ll 1, \tag{5}$$

with a constant matrix $A \in \mathbb{R}^{N \times N}$, and with the class $C^r$ functions $f_0 : \mathcal{U} \to \mathbb{R}^N$ and $f_1 : \mathcal{U} \times \mathbb{T}^k \to \mathbb{R}^N$, where $\mathbb{T}^k = S^1 \times \cdots \times S^1$ is the $k$-dimensional torus.

As long as $A$, $f_0$ and $f_1$ are of the general form stated above, their specific form will be unimportant for our forthcoming discussion, as we state all results in terms of the ODE (5). If, however, the ODE (5) arises from the mechanical system (2), then we specifically have $N = 2n$ and

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}(K+B) & -M^{-1}(C+G) \end{pmatrix},$$
$$f_0(x) = \begin{pmatrix} 0 \\ M^{-1}F_0(x_1, x_2) \end{pmatrix},$$
$$f_1(x, \Omega t) = \begin{pmatrix} 0 \\ M^{-1}F_1(x_1, x_2, \Omega_1 t, \ldots, \Omega_k t) \end{pmatrix}.$$

## 3 Linear spectral geometry: eigenspaces, normal modes, spectral subspaces and invariant manifolds

### 3.1 Eigenvalues

The linear, unperturbed part of system (5) is

$$\dot{x} = Ax. \tag{6}$$

The matrix $A$ has $N$ eigenvalues $\lambda_j \in \mathbb{C}, j = 1, \ldots, N$, with multiplicities counted. We order these eigenvalues so that their real parts form a decreasing sequence under increasing $j$ :

$$\text{Re}\lambda_N \leq \text{Re}\lambda_{N-1} \leq \cdots \leq \text{Re}\lambda_1. \tag{7}$$

We denote the algebraic multiplicity of $\lambda_j$ (i.e., its multiplicity as a root of the characteristic equation of $A$) by $\text{alg}(\lambda_j)$ and its geometric multiplicity (i.e., the number of independent eigenvectors corresponding to $\lambda_j$) by $\text{geo}(\lambda_j)$. We recall that $A$ is called semisimple if $\text{alg}(\lambda_j) = \text{geo}(\lambda_j)$ holds for all $\lambda_j$. This is always the case if all eigenvalues are distinct or $A$ is symmetric. When $A$ is not semisimple, then some of its eigenvalues satisfy $\text{alg}(\lambda_j) > \text{geo}(\lambda_j)$, leading to nontrivial blocks in the Jordan decomposition of $A$. A good reference for this and other forthcoming aspects of linear dynamical systems is Hirsch et al. [21].

### 3.2 Eigenspaces

For each distinct eigenvalue $\lambda_j$, there exists a real *eigenspace* $E_j \subset \mathbb{R}^N$ spanned by the imaginary and real parts of the corresponding eigenvectors and generalized eigenvectors of $A$. We have $\dim E_j = \text{alg}(\lambda_j)$ in case $\text{Im}\,\lambda_j = 0$, while we have $\dim E_j = 2 \times \text{alg}(\lambda_j)$ in case $\text{Im}\,\lambda_j \neq 0$. In the latter case, $E_j \equiv E_{j+1}$ because $\lambda_j = \bar{\lambda}_{j+1}$. That is, the real eigenspaces associated with each of two complex conjugate eigenvalues coincide with each other.

An eigenspace $E_j$ also represents an invariant subspace for the linearized system (6), filled with trajectories of this system corresponding to the eigenvalue $\lambda_j$. Specifically, we have

$$E_j = \operatorname*{span}_{t \in \mathbb{R}} \left\{ e^{\text{Re}\lambda_j t} \cos\left[\text{Im}\left(\lambda_j\right) t\right] \sum_{\alpha=1}^{\text{alg}(\lambda_j)} a_j^\alpha t^{\alpha-1}; \; e^{\text{Re}\lambda_j t} \sin\left[\text{Im}\left(\lambda_j\right) t\right] \sum_{\alpha=1}^{\text{alg}(\lambda_j)} b_j^\alpha t^{\alpha-1} \right\} \tag{8}$$

for appropriate real vectors $a_j^\alpha, b_j^\alpha \in \mathbb{R}^N$. In the generic case, $\lambda_j$ is a simple real or simple complex eigenvalue, in that case $E_j$ is one- or two-dimensional, respectively.

### 3.3 Linear normal modes

The classic definition of a *linear normal mode* refers to a periodic solution of the linear system (6), arising from an eigenvalue $\lambda_j$ with $\mathrm{Re}\lambda_j = 0$ and $\mathrm{alg}\,(\lambda_j) = \mathrm{geo}\,(\lambda_j)$. In this case, normal modes fill the full eigenspace of $\lambda_j$, i.e., we have

$$E_j = \underset{t\in\mathbb{R}}{\mathrm{span}} \left\{ a_j^1 \cos\left[\mathrm{Im}\,(\lambda_j)\,t\right], \ldots, a_j^{\mathrm{alg}\,(\lambda_j)} \cos\left[\mathrm{Im}\,(\lambda_j)\,t\right]; \right. \\ \left. b_j^1 \sin\left[\mathrm{Im}\,(\lambda_j)\,t\right], \ldots, b_j^{\mathrm{alg}\,(\lambda_j)} \sin\left[\mathrm{Im}\,(\lambda_j)\,t\right] \right\}, \qquad (9)$$

with the vectors $a_j^\alpha, b_j^\alpha$ appearing in (8) and with $\dim E_j = 2 \times \mathrm{alg}\,(\lambda_j)$. In case of a linear mechanical system without symmetries, the eigenvalues $\lambda_j = i\omega_j$ generating normal modes are typically simple. In that case, we have $\mathrm{alg}\,(\lambda_j) = \mathrm{geo}\,(\lambda_j) = 1$ and $\dim E_j = 2$. The normal mode family of period $T_j = 2\pi/\omega_j$ then spans the two-dimensional invariant plane $E_j$ in the phase space of the linear system (6)

The fixed point $x = 0$ of the linear system (6) can also be considered as a singular normal mode when viewed as a periodic motion of arbitrary period. This *trivial normal mode*, however, is isolated and does not form a family spanning a nontrivial subspace. Yet, this representation of the fixed point as a periodic orbit becomes useful when we seek its continuation under small forcing ($\epsilon > 0$) in the perturbed Eq. (5). The fixed point will generally not survive, but a unique periodic or quasiperiodic orbit mimicking the stability of the fixed point will often exist, as we discuss below.

### 3.4 Spectral subspaces

By linearity, a subspace spanned by any combination of eigenspaces is also invariant under the dynamics of the linear system (6). Specifically, a *spectral subspace*

$$E_{j_1,\ldots,j_q} = E_{j_1} \oplus E_{j_2} \oplus \ldots \oplus E_{j_q} \\ = \left\{ v \in \mathbb{R}^N \,:\, v = \sum_{i=1}^{q} v_i, \quad v_i \in E_{j_i}, \right. \\ \left. E_{j_l} \neq E_{j_k}, \quad k, l = 1, \ldots, q \right\}, \qquad (10)$$

with $\oplus$ denoting the direct sum of vector spaces, is an invariant subspace of system (6). The definition (10) avoids double-counting the same real eigenspace corresponding to complex conjugate eigenvalues. Also, by definition, any single eigenspace $E_j$ is also a spectral subspace.

Classic examples of spectral subspaces include the *stable subspace* $E^s$, the *unstable subspace* $E^u$ and the *center subspace* $E^c$. In the presence of $n_s, n_u$ and $n_c$ eigenvalues with negative, positive and zero real parts, respectively, these classic spectral subspaces are defined as

$$E^s = \left\{ v \in \mathbb{R}^N \,:\, v = \sum_{i=1}^{n_s} v_i, \quad v_i \in E_{j_i}, \right. \\ \left. \mathrm{Re}\lambda_{j_i} < 0, \quad i = 1, \ldots, n_s \right\}, \\ E^u = \left\{ v \in \mathbb{R}^N \,:\, v = \sum_{i=1}^{n_u} v_i, \quad v_i \in E_{j_i}, \right. \\ \left. \mathrm{Re}\lambda_{j_i} > 0, \quad i = 1, \ldots, n_u \right\}, \\ E^c = \left\{ v \in \mathbb{R}^N \,:\, v = \sum_{i=1}^{n_c} v_i, \quad v_i \in E_{j_i}, \right. \\ \left. \mathrm{Re}\lambda_{j_i} = 0, \quad i = 1, \ldots, n_c \right\}. \qquad (11)$$

Linearized oscillatory systems in mechanics often have only decaying solutions due to the presence of damping on an otherwise conservative system of oscillators. In that case, $E^s = \mathbb{R}^N$ and $E^u = E^c = \emptyset$. If, in addition, all eigenvalues $\lambda_j$ are distinct and complex, then the minimal spectral subspaces are formed by the two-dimensional eigenspaces $E_j$. Again, any direct sum of these two-dimensional eigenspaces is a spectral subspace by the above definition.

### 3.5 Invariant manifolds in the linearized system

For simplicity, we assume here that the matrix $A$ has only distinct eigenvalues. We make this assumption here only for ease of exposition and will drop it later in our results for the full nonlinear system.

In its eigenbasis, $A$ is then diagonal and the linearized system (6) can be written in the complexified form

$$\dot{y} = \Lambda y, \quad \Lambda = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_N\right), \tag{12}$$

where $y \in \mathbb{C}^N$ is a complex vector, with its $j$th coordinate $y_j$ denoting a coordinate along the (generally complex) eigenvector $e_j$ of $A$. Complexified equivalents of all real eigenspaces $E_j$ and spectral subspaces $E_{j_1,\ldots,j_q}$ are again invariant subspaces for the linearized dynamics (12). As invariant manifolds, not only are all these subspaces infinitely many times differentiable but also analytic. Indeed, their coordinate representations are given by the analytic graphs $y_l = f_l(y_{j_1}, \ldots, y_{j_q}) \equiv 0$, for all $l \notin \{j_1, \ldots, j_q\}$, over any spectral subspace $E_{j_1,\ldots,j_q}$.

There are, however, generally infinitely many other invariant manifolds in the linearized system (12) that are also graphs over $E_{j_1,\ldots,j_q}$ and are tangent $E_{j_1,\ldots,j_q}$ at the origin. Indeed, as we show in Appendix "Uniqueness and analyticity issues for invariant manifolds in linear systems," along *any* codimension-one surface $\Gamma \subset E_{j_1,\ldots,j_q}$, intersected transversely by the linear vector field (12) within $E_{j_1,\ldots,j_q}$, we can prescribe the $y_l$ coordinates of an invariant manifold via arbitrary smooth functions $y_l|_\Gamma = f_l^0(\Gamma)$ with $l \notin \{j_1, \ldots, j_q\}$ and obtain (under nonresonance conditions) a unique manifold satisfying this boundary condition. For two-dimensional systems, this arbitrariness in the boundary conditions leads to a one-parameter family of invariant surfaces (see. Fig. 2a). In the multi-dimensional case, illustrated in Fig. 2b, there is a substantially higher degree of non-uniqueness for invariant manifolds tangent to individual spectral subspaces. Indeed, both the choice of the codimension-one boundary surface $\Gamma$ and the choice of the boundary values $f_l^0(\Gamma)$ of the invariant manifold are arbitrary, as long as $\Gamma$ is transverse to the linear vector field.

A subset of these infinitely many solutions is simple to write down in the case of underdamped mechanical vibrations whereby we have $\mathrm{Im}\lambda_j \neq 0$ for all eigenvalues. Passing to amplitude-phase variables $(r_j, \varphi_j)$ by letting $(y_j, \bar{y}_j) \equiv (y_j, y_{j+1}) = r_j e^{i\varphi_j}$, we can rewrite system (12) in the simple amplitude-phase form

$$\dot{r}_j = -\mathrm{Re}\lambda_j\, r, \quad \dot{\varphi}_j = \mathrm{Im}\lambda_j, \quad j = 1, \ldots, n = N/2,$$

with $n$ denoting the number of degrees of freedom in the system (2). In this case, a family of invariant manifolds tangent to the spectral subspace $E_{j_1,\ldots,j_q}$ is given explicitly by the equations

$$r_l = f_{r_l}(r_{j_1}, \varphi_{j_1}, \ldots, r_{j_q}, \varphi_{j_q}) := \sum_{i=1}^{q} C_l^{j_i} r_{j_i}^{\frac{\mathrm{Re}\lambda_l}{\mathrm{Re}\lambda_{j_i}}},$$
$$\phi_l = f_{\varphi_l}(r_{j_1}, \varphi_{j_1}, \ldots, r_{j_q}, \varphi_{j_q}) := D_l^{j_i} + \frac{\mathrm{Im}\lambda_l}{\mathrm{Im}\lambda_{j_1}}\varphi_{j_1}, \tag{13}$$

for all $l \notin \{j_1, \ldots, j_q\}$, with $C_l^{j_i} \in \mathbb{R}$ and $D_l^{j_i} \in [0, 2\pi)$ denoting arbitrary constants. Under the nonresonance conditions $\lambda_l/\lambda_{j_i} \notin \mathbb{N}^+$, if

$$\mathrm{Re}\lambda_l < \mathrm{Re}\lambda_{j_i} < 0, \quad i = 1, \ldots, q, \quad l \notin \left\{j_1, \ldots, j_q\right\} \tag{14}$$

holds, then any nonzero solution (13) has only finitely many continuous derivatives at the origin. The only exceptions are the identically zero solutions for which $C_l^{j_i} = 0$ holds for all $j_i$ and $l$ values, giving $f_{r_l}(r_{j_1}, \varphi_{j_1}, \ldots, r_{j_q}, \varphi_{j_q}) \equiv 0$. These zero solutions are, in fact the unique smoothest ($C^\infty$ and even $C^a$) member of the solution family (13), representing the invariant spectral subspace $E_{j_1,\ldots,j_q}$ itself.

Condition (14), however, never holds in the case of $E_{j_1,\ldots,j_q} = E_{N-q+1,\ldots,N}$, i.e., when the invariant manifold is sought as a graph over the spectral subspace of the $q$ fastest decaying modes. In this case, $\mathrm{Re}\lambda_{j_i} < \mathrm{Re}\lambda_l < 0$ hold for all indices involved, and the only differentiable member of the solution family (13) at the origin is $f_l(y_{N-q+1}, \ldots, y_N) \equiv 0$. This is unique differentiable invariant manifold over $E_{N-q+1,\ldots,N}$, which also happens to be analytic. The uniqueness of $E_{N-q+1,\ldots,N}$ as a smooth invariant manifold with the prescribed tangency property does not just hold within the special solution family (13). Indeed, the classic strong stable manifold theorem (see, e.g., Hirsch et al. [20]) applied to the linear system (12) implies uniqueness for $E_{N-q+1,\ldots,N}$ among *all* invariant manifolds tangent to $E_{N-q+1,\ldots,N}$ at the origin. This uniqueness of fast invariant manifolds is also illustrated in Fig. 2a for the two-dimensional case and in Fig. 2b for the multi-dimensional case.

In summary, under appropriate nonresonance assumptions on the eigenvalues, there are infinitely many Shaw–Pierre-type invariant manifolds tangent to any non-fast spectral subspace $E_{j_1,\ldots,j_q}$ at the origin of the linearized system (6). Clearly, one cannot expect such manifolds to be unique in the nonlinear context studied by Shaw and Pierre [39] either. Thus, the common assumption in the nonlinear normal modes literature,

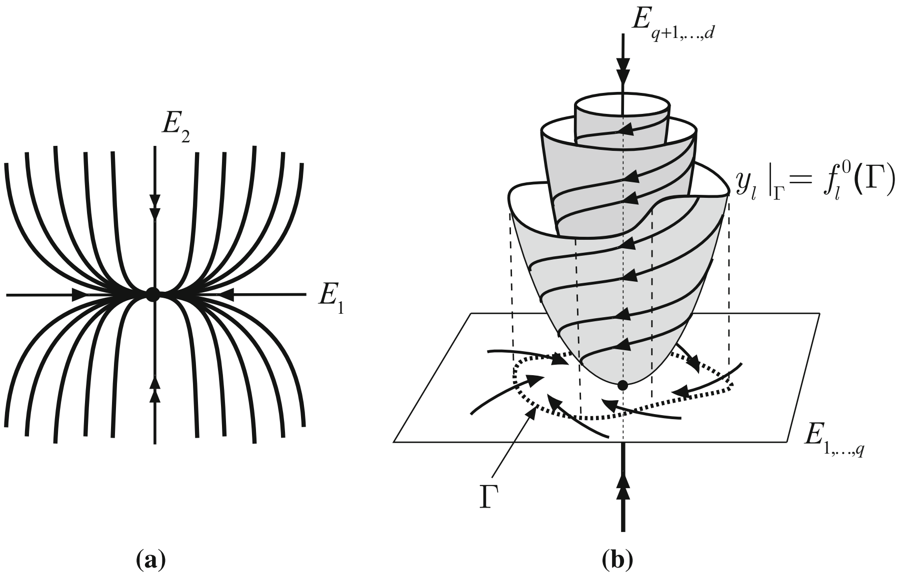



**Fig. 2** **a** Non-uniqueness of invariant manifolds tangent to the slower decaying spectral subspace of a planar, linear dynamical system. Note the uniqueness of the invariant manifold tangent to the faster-decaying spectral subspace. **b** Non-uniqueness of invariant manifolds tangent to the direct product $E_{1,\ldots,q}$ of $q$ slowest decaying spectral subspaces of a higher-dimensional, linear dynamical system. Under appropriate nonresonance conditions (cf. Appendix "Uniqueness and analyticity issues for invariant manifolds in linear systems"), any codimension-one boundary surface $\Gamma$ transverse to the flow within $E_{1,\ldots,q}$ yields an invariant manifold tangent to $E_{1,\ldots,q}$ at the fixed point, for *any* choice of the smooth functions $y_l = f_l^0(\Gamma)$, with $l \notin \{j_1, \ldots, j_q\}$. Again, note the uniqueness of the invariant manifold tangent to the spectral subspace of the remaining faster-decaying modes

that invariant manifolds tangent to eigenspaces will uniquely emerge from approximate operational procedures, is generally unjustified.

Observe, however, that despite the non-uniqueness of invariant manifolds tangent to a non-fast spectral subspace $E_{j_1,\ldots,j_q}$ at the origin of the linear system (12), the flat boundary condition $y_l = f_l(y_{j_1}, \ldots, y_{j_q}) \equiv 0$, with $l \notin \{j_1, \ldots, j_q\}$, yields the unique *analytic invariant manifold*, $E_{j_1,\ldots,j_q}$, provided that the non-resonance conditions $\lambda_p/\lambda_{j_i} \notin \mathbb{N}^+$ hold (see Appendix "Uniqueness and analyticity issues for invariant manifolds in linear systems" for details). This gives hope that perhaps there is a unique analytic (or at least a unique smoothest) continuation of spectral subspaces of the linearized system to locally smoothest manifolds in the nonlinear system (5) near the origin. As we show in later sections, this expectation turns out to be justified under certain conditions.

### 3.6 Spectral quotients

As we observed above, nontrivial solutions of the form (13) have only a finite number of continuous derivatives at the origin. Namely, if the graph is constructed over the spectral subspace $E_{j_1,\ldots,j_q}$, then only $\mathrm{Int}\left[\mathrm{Re}\lambda_l/\mathrm{Re}\lambda_{j_i}\right]$ continuous derivatives exist for the $r_l$ coordinate function, with $\mathrm{Int}\left[\,\cdot\,\right]$ denoting the integer part of a real number.

The smoothest non-flat invariant graphs in the family (13), therefore, satisfy

$$r_L = C_L^{j_I} r_{j_I}^{\frac{\mathrm{Re}\lambda_L}{\mathrm{Re}\lambda_{j_I}}}, \quad L = \arg\max_{l\notin\{j_1,\ldots,j_q\}} |\mathrm{Re}\lambda_L|,$$
$$I = \arg\min_{i\in\{1,\ldots,q\}} \left|\mathrm{Re}\lambda_{j_i}\right|,$$
$$r_l \equiv 0, \quad l \neq L,$$

with their degree of smoothness at the origin equal to $\mathrm{Int}\left[\mathrm{Re}_L/\mathrm{Re}\lambda_{j_I}\right]$. This is the maximal degree of smoothness that any non-flat member of the solution family (13) can attain. The only smoother invariant graph over $E_{j_1,\ldots,j_q}$ in the graph family (13) is the subspace $E_{j_1,\ldots,j_q}$ itself.

This maximal smoothness of the invariant graphs (13) is purely determined by the ratio of the fastest decay exponent outside $E_{j_1,\ldots,j_q}$ to the slowest decay exponent within $E_{j_1,\ldots,j_q}$. For later purposes, we now give a formal definition of the integer part of this ratio for any spectral subspace $E$ of the operator $A$. We also define another version of the same quotient, with the numerator replaced by the fastest decay exponent in the whole spectrum of $A$. Our notation for the full spectrum of $A$ is $\mathrm{Spect}(A)$, whereas we denote the spectrum of the restriction of $A$ to its spectral subspace $E$ by $\mathrm{Spect}(A|_E)$.

**Definition 1** For any spectral subspace of the linear operator $A$, we define the relative spectral quotient $\sigma(E)$ and the absolute spectral quotient $\Sigma(E)$ as

$$\sigma(E) = \mathrm{Int}\left[\frac{\min_{\lambda\in\mathrm{Spect}(A)-\mathrm{Spect}(A|_E)}\mathrm{Re}\lambda}{\max_{\lambda\in\mathrm{Spect}(A|_E)}\mathrm{Re}\lambda}\right], \qquad (15)$$

$$\Sigma(E) = \mathrm{Int}\left[\frac{\min_{\lambda\in\mathrm{Spect}(A)}\mathrm{Re}\lambda}{\max_{\lambda\in\mathrm{Spect}(A|_E)}\mathrm{Re}\lambda}\right]. \qquad (16)$$

These spectral quotients will play a major role in later sections when we discuss the existence and uniqueness of nonlinear continuations of invariant manifolds of the linearized system.

## 4 Nonlinear spectral geometry: nonlinear normal modes and spectral submanifolds

The fundamental assumption of nonlinear modal analysis is that appropriate generalizations of invariant manifolds of the linearized system persist under the full system (5) (see, e.g., Vakakis [44], Kerschen et al. [24], Peeters et al. [33], and Avramov and Mikhlin [3,4] for reviews).

The classic definition of Rosenberg [38] for autonomous, conservative systems states that nonlinear normal modes are synchronous periodic orbits, i.e., periodic motions that reach their extrema along all modal coordinate directions at the same time. A useful relaxation of this concept allows for general (not necessarily synchronous) periodic orbits in autonomous systems (see, e.g., Peeters et al. [33]).

Here we relax Rosenberg's definition even further for general dissipative systems, allowing a nonlinear normal mode to be a recurrent motion with a discrete Fourier spectrum of $f$ frequencies.[1] If $f > 1$ and the frequencies of the motion are rationally independent, then the motion is quasi-periodic and forms a noncompact set in the phase space. To this end, we use the closure of such a trajectory in our normal mode definition (with the closure including the trajectory as well as all its limit points). Specifically, the closure of a periodic orbit is just the periodic orbit itself, while the closure of a quasiperiodic orbit contains further points outside the trajectory, forming an invariant torus densely filled by the trajectory.

**Definition 2** *A nonlinear normal mode (NNM)* is the closure of a multi-frequency solution

$$x(t) = \sum_{|m|=1}^{\infty} x_m e^{i\langle m,\Omega\rangle t}, \quad m \in \mathbb{N}^f, \quad \Omega \in \mathbb{R}^f,$$

of the nonlinear system (5). Here $f \in \mathbb{N}$ is the number of frequencies; the vector $m$ is a multi-index of $f$ nonnegative integers; and $x_m \in \mathbb{C}^n$ are the complex Fourier amplitudes of the real solution $x(t)$ with respect to the frequencies in the frequency vector $\Omega = (\Omega_1, \ldots, \Omega_f)$. Special cases of NNMs include (see Fig. 3):

(1) *trivial NNM* ($f = 0$): a fixed point
(2) *periodic NNM* (either $f = 1$, or $f > 1$ and the elements of $\Omega$ are rationally commensurate): a periodic orbit
(3) *quasiperiodic NNM* ($f > 1$ and the elements of $\Omega$ are rationally incommensurate): an $f$-dimensional invariant torus

A further expectation in the nonlinear vibrations literature—put forward first by Shaw and Pierre [39] in its simplest form, then extended by Pescheck et al. [31], Shaw et al. [41], Jiang et al. [22]—is that an arbitrary spectral subspace $E_{j_1,\ldots,j_q}$ of the $x = 0$ fixed point will also persist under the addition of nonlinear and time-dependent terms in system (11). This would lead to a nonlinear continuation of the spectral subspace $E_{j_1,\ldots,j_q}$ into an invariant manifold $W_{j_1,\ldots,j_q}(\mathcal{N})$ along $\mathcal{N}$. While Shaw and Pierre [39] call such a $W_{j_1,\ldots,j_q}(\mathcal{N})$ a nonlinear normal mode, the dynamics in $W_{j_1,\ldots,j_q}(\mathcal{N})$ will not inherit the forward- and backward-bounded,

[1] Recurrent motions are typical in conservative systems with compact energy surfaces. Thus, recurrence by itself can only distinguish nonlinear normal modes in dissipative systems.

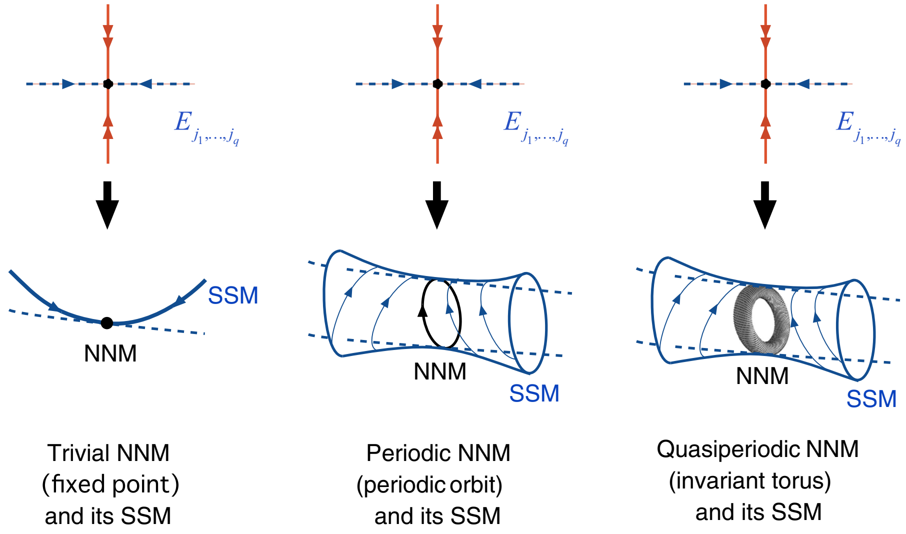



**Fig. 3** Schematics of the three main types of NNMs (trivial, periodic and quasiperiodic) and their corresponding SSMs (autonomous, periodic and quasiperiodic). In all cases, the NNM are, or are born out of, perturbations of a fixed point. The SSMs are always tangent to a subbundle along the NNM whose fibers are close to a specific spectral subspace $E_{j_1,\ldots,j_q}$ of the linearized system

recurrent nature of linear normal modes even in the simplest dissipative examples. To make this distinction from classic normal modes clear, we refer here to $W_{j_1,\ldots,j_q}(\mathcal{N})$ as a spectral submanifold.

**Definition 3** *A spectral submanifold (SSM)* of a NNM, $\mathcal{N}$, is an invariant manifold $W(\mathcal{N})$ of system (5) such that

(i) $W(\mathcal{N})$ is a subbundle of the normal bundle $N\mathcal{N}$ of $\mathcal{N}$, satisfying $\dim W(\mathcal{N}) = \dim E + \dim \mathcal{N}$ for some spectral subspace $E$ of the operator $A$.
(ii) The fibers of the bundle $W(\mathcal{N})$ perturb smoothly from the spectral subspace $E$ of the linearized system (6) under the addition of the nonlinear and $\mathcal{O}(\epsilon)$ terms in system (5).
(iii) $W(\mathcal{N})$ has strictly more continuous derivatives along $\mathcal{N}$ than any other invariant manifold satisfying (i) and (ii).

More specifically, in the case of zero external forcing ($\epsilon = 0$), an SSM is the smoothest invariant manifold $W(0)$ out of all invariant manifolds that are tangent to a spectral submanifold $E$ at $x = 0$ and have the same dimension as $E$. In the case of nonzero external forcing ($\epsilon \neq 0$), an SSM is the smoothest invariant manifold $W(\mathcal{N})$ out of all invariant manifolds that are $\mathcal{O}(\epsilon)$ $C^1$-close to the set $\mathcal{N} \times E$ along $\mathcal{N}$ and have the same dimension as $\mathcal{N} \times E$ does.

To be clear, there is no a priori guarantee that a unique smoothest member in a family of surfaces satisfying (i) and (ii) of Definition 3 actually exists. Indeed, no smooth surface might exist, or those that exist may be equally smooth. We will need to derive conditions under which SSMs are unique and hence well defined in the sense of Definition 3.

Special cases of SSMs include (see Fig. 3):

*(1) autonomous SSM* ($f = 0$): nonlinear continuations of spectral submanifolds discussed for linear systems in Sect. 3.5.
*(2) periodic SSM* (either $f = 1$, or $f > 1$ and the elements of $\Omega$ are rationally commensurate): a three-dimensional invariant manifold tangent to a spectral subbundle along a hyperbolic periodic orbit
(3) *quasiperiodic SSM* ($f > 1$ and the elements of $\Omega$ are rationally incommensurate): an invariant manifold tangent to a spectral subbundle of a hyperbolic invariant torus.

Classic examples of autonomous SSMs include the stable manifold $W^s(\mathcal{N})$ and the unstable manifold $W^u(\mathcal{N})$ of a fixed point $\mathcal{N}$ (i.e., of a trivial NNM). Classic examples of non-autonomous SSMs include the stable manifold $W^s(\mathcal{N})$ and the unstable manifold $W^u(\mathcal{N})$ of a periodic or quasiperiodic orbit $\mathcal{N}$. The SSMs of interest here are submanifolds of $W^s(\mathcal{N})$ that perturb smoothly from spectral subspaces within $\mathcal{N} \times E^s$. The construction of these surfaces has been the main question in the nonlinear modal analysis of autonomous and non-autonomous systems, to be discussed in detail in our Theorems 3 and 4 below.

There is a clear geometric distinction between our NNM definition (a generalization of the normal mode concept of Rosenberg) and our SSM definition (a generalization of the normal mode concept of Shaw and Pierre, with the highest smoothness requirement added). Both concepts are helpful, but refer to highly different dynamical structures in dissipative dynamical systems.

## 5 Existence and uniqueness of NNMs

As mentioned before, the survival of the trivial NNMs in the form of a nearby perturbed solution in system (5) is broadly expected in the nonlinear normal modes literature. These perturbed NNMs are routinely sought via formal asymptotic expansions with various a priori postulated time scales (see, e.g., Nayfeh [29] for a survey of such intuitive methods). There is generally limited concern for the validity of these formal approximations (see Verhulst [45] for a discussion). Formal computability of the first few terms of the assumed asymptotic expansion for NNMs, however, does not imply that the targeted structure actually exists, as we discussed in the Introduction.

Here, we would like to fill this conceptual gap by clarifying the existence and uniqueness of NNMs using classical invariant manifold theory. The same theory also allows us to conclude the existence of a special SSM, the stable manifold of the NNM. Here we only consider damped mechanical vibrations for which

$$\mathrm{Re}\lambda_j < 0, \quad j = 1, \ldots, N \tag{17}$$

holds in the linearized system(6). This assumption ensures that we are in the dissipative setting in which our NNM and SSM definitions are meaningful.

### 5.1 Trivial NNM under autonomous external forcing ($k = 0$)

For time-independent external forcing, (5) remains autonomous even under the inclusion of the remaining $\mathcal{O}(\epsilon)$ forcing terms. Because these autonomous forcing terms are not assumed to vanish at $x = 0$, the full system will generally no longer have a fixed point at $x = 0$. The following theorem nevertheless guarantees the existence of a nearby trivial NNM with spectral properties mimicking that of the origin.

**Theorem 1** (Existence, uniqueness and persistence of autonomous NNMs) *Assume that the external forcing is autonomous ($k = 0$) in (5). Assume further that (17) holds for the eigenvalues of the matrix $A$.*

*Then, for $\epsilon \neq 0$ small enough, there exists a unique, trivial NNM, $x_\epsilon = \tau(\epsilon)$, with $\tau(0) = 0$, in system (5). This NNM attracts all nearby trajectories and depends on $\epsilon$ in a $C^r$ fashion.*

*Proof* Since no zero eigenvalues are allowed for the linearized system, a unique, smoothly persisting fixed point (trivial NNM) will exist for small enough $\epsilon$ by the implicit function theorem. This persisting fixed point will be asymptotically stable by the classic stable manifold theorem applied to system (5), as described, e.g., in Guckenheimer and Holmes [16]. □

### 5.2 Periodic and quasiperiodic NNM under non-autonomous external forcing ($k \geq 1$)

The existence of a small-amplitude periodic solution under purely periodic forcing in system (5) is also routinely assumed in the nonlinear vibrations literature. These solutions are then sought via numerical continuation or finite Fourier expansions. Conditions guaranteeing the success of these formal procedures are generally omitted.

Next we deduce general mathematical conditions for system (5) under which the existence, uniqueness and even the stability type of a nontrivial NNM follow under general quasiperiodic forcing, including the case of periodic forcing ($k = 1$).

**Theorem 2** (Existence, uniqueness and persistence of non-autonomous NNMs) *Assume that the external forcing $f_1$ is quasi-periodic with $k \geq 1$ frequencies, and the eigenvalues of the matrix $A$ satisfy (17).*

*Then, for $\epsilon \neq 0$ small enough, there exists a unique NNM, $x_\epsilon(t) = \epsilon\tau(\Omega_1 t, \ldots, \Omega_k t; \epsilon)$ in the system (5), where the function $\tau$ is $2\pi$-periodic in each of its first $k$ arguments. This NNM attracts all nearby trajectories and depends on $\epsilon$ in a $C^r$ fashion.*

*Proof* For $r \in \mathbb{N}^+$, the theorem can be proven using classic invariant manifold results, as detailed in Appendix "Existence, uniqueness and persistence of non-autonomous NNMs." Proving case for $r \in \{0, \infty, a\}$ requires use of the existence results of Haro and de la Llave [18] for invariant tori which are directly applicable here. □

Theorem 2 gives a mathematical foundation to various formal expansion techniques (two-timing, harmonic balance, etc) and numerical continuation techniques used in the nonlinear vibrations literature. The existence of the NNMs and their domain of attraction are independent of any possible resonances between the forcing frequencies $\Omega_j$ and the imaginary parts of the eigenvalues of $A$. The nature of the NNM (periodic or quasiperiodic) will depend on the actual value of $\epsilon$ and will be captured by general multi-mode Fourier expansions, as we describe in Sect. 8.1.

Of relevance here is the recent work of Kuether et al. [26], who call a periodic NNM (as defined in Definition 2) nonlinear forced response, to distinguish it from nonlinear normal modes (defined as not necessarily synchronous periodic orbits of the unforced and undamped nonlinear system). Kuether et al. [26] investigate connections between NNM and forced responses via intuitive techniques. A firm connection between the quasiperiodic or periodic NNM and the $x = 0$ equilibrium is offered by Theorem 2 for small $|\epsilon|$ values. For large values of $|\epsilon|$, such a connection no longer exists, as the local phase space structure near the former equilibrium is drastically altered by large perturbations.

## 6 Spectral submanifolds in autonomous systems ($k = 0$)

In this section, we discuss spectral submanifolds in the sense of Definition 3, i.e., smoothest nonlinear continuations of spectral subspaces $E_{j_1,\ldots,j_q}$ in the nonlinear system (5). We assume here that $k = 0$ holds, in which case, after a possible shift of coordinates, all autonomous terms contained in the function $f_1$ on the right-hand side of system (5) can be subsumed either into the linear term $Ax$ or the autonomous nonlinear term $f_0(x)$. Thus, without any loss of generality, we can write the $k = 0$ case of system (5) in the form

$$\dot{x} = Ax + f_0(x), \quad f_0(x) = \mathcal{O}(|x|^2), \quad f_0 \in C^r, \tag{18}$$

where $r$ is selected as in (4).

### 6.1 Main result

The idea of seeking two-dimensional spectral submanifolds in system (18) is originally due to Shaw and Pierre [39]. They called such spectral submanifolds nonlinear normal modes, even though these surfaces generally do not contain periodic or even recurrent motions in the presence of damping. Shaw and Pierre [40] later extended their original idea to infinite-dimensional evolutionary equations arising in continuum oscillations. Furthermore, Pescheck et al. [31] extended the original Shaw–Pierre concept to the nonlinear continuation of an arbitrary, finite-dimensional spectral subspace. More recent reviews of the approach and its applications are given by Kerschen et al. [24] and Avramov and Mikhlin [3,4].

We restrict here the discussion to the case of a stable underlying NNM, the context in which the Shaw–Pierre invariant manifold concept was originally proposed. We thus assume throughout this section that

$$\mathrm{Re}\lambda_j < 0, \quad j = 1, \ldots, N, \tag{19}$$

implying that the origin is an asymptotically stable fixed point. By reversing the direction of time, we obtain similar results for unstable NNMs (repelling fixed points) with $\mathrm{Re}\lambda_j > 0, \quad j = 1, \ldots, N$.

To describe appropriate nonresonance conditions for a spectral subspace $E$, we will use linear combinations of eigenvalues associated with a spectral subspace $E$ with nonnegative integers $m_i$. Specifically, for a $q$-dimensional spectral subspace $E$, we denote such linear combinations as

$$\langle m, \lambda \rangle_E := m_1\lambda_{j_1} + \ldots + m_q\lambda_{j_q},$$
$$\lambda_{j_k} \in \mathrm{Spect}(A|_E), \quad m \in \mathbb{N}^q, \quad q = \dim E.$$

We define the order of the nonnegative integer vector $m$ as

$$|m| := m_1 + \ldots + m_q.$$

**Theorem 3** (Existence, uniqueness and persistence of autonomous SSM) Consider a spectral subspace $E$ and assume that the low-order nonresonance conditions

$$\langle m, \lambda \rangle_E \neq \lambda_l, \quad \lambda_l \notin \mathrm{Spect}(A|_E), \quad 2 \leq |m| \leq \sigma(E) \tag{20}$$

hold for all eigenvalues of $\lambda_l$ of $A$ that lie outside the spectrum of $A|_E$.

*Then the following statements hold:*

(i) *There exists a class* $C^r$ SSM, $W(0)$, *tangent to the spectral subspace* $E$ *at the trivial NNM*, $x = 0$. *Furthermore*, $\dim W(0) = \dim E$.
(ii) W(0) *is unique among all* $C^{\sigma(E)+1}$ *invariant manifolds with the properties listed in (i).*
(iii) *If* $f_0$ *is jointly* $C^r$ *in* $x$ *and an additional parameter vector* $\mu$, *then the SSM* $W(0)$ *is jointly* $C^r$ *in* $x$ *and* $\mu$. *In particular, if* $f_0(x, \mu)$ *is* $C^\infty$ *or analytic, then* $W(0)$ *persists under small perturbations in the parameter* $\mu$ *and will depend on these perturbations in a* $C^\infty$ *or analytic fashion, respectively.*

*Proof* We deduce the results from a more general theorem of Cabré et al. [8] in Appendix "Proof of Theorem 3." □

In short, Theorem 3 states that a unique smoothest Shaw–Pierre-type invariant surface, i.e., an SSM in the sense of Definition 3, exists and persists, as long as no low-order resonances arise between the master modes and the enslaved modes. The order of these nonresonance conditions varies from one type of SSM to the other, as we discuss next.

## 6.2 Application to specific spectral subspaces

We now spell out the meaning of Theorem 3 for different choices of the spectral subspace $E$. We specifically consider spectral subspaces $E_{j_1,\ldots,j_q}$, where the selected $q$ eigenvalues $\lambda_{j_1}, \ldots, \lambda_{j_q}$ are ordered so that their real parts form a nondecreasing sequence:

$$\mathrm{Re}\lambda_{j_1} \leq \cdots \leq \mathrm{Re}\lambda_{j_q} < 0. \tag{21}$$

We order the real parts of the remaining $N - q$ eigenvalues as

$$\mathrm{Re}\lambda_{j_{q+1}} \leq \cdots \leq \mathrm{Re}\lambda_{j_d} < 0. \tag{22}$$

Here $\mathrm{Re}\lambda_{j_{q+1}}$ may be larger or smaller than the real parts of any of the eigenvalues listed in (21).

We distinguish three types of SSMs in our discussion (cf. Fig. 4).

- *A fast spectral submanifold (fast SSM),* $W_{N-q+1,\ldots,N}(0)$, is an SSM in the sense of Definition 3, with $E_{N-q+1,\ldots,N}$ chosen as the subspace of the $q$ strongest decaying modes of the linearized system. Here $q \leq N$, with $q = N$ marking the special case of a fast spectral submanifold that coincides with the domain of attraction of the fixed point at $x = 0$.
- *An intermediate spectral submanifold (intermediate SSM),* $W_{j_1,\ldots,j_q}(0)$, is an SSM in the sense of Definition 3, serving as the nonlinear continuation of

$$E_{j_1,\ldots,j_q} = E_{j_1} \oplus E_{j_2} \oplus \cdots \oplus E_{j_q} \tag{23}$$

for a general choice of the $q < N$ eigenspaces $E_{j_1}, \ldots, E_{j_q}$.
- *A slow spectral submanifold (slow SSM),* $W_{1,\ldots,q}(0)$, is an SSM in the sense of Definition 3, with the underlying spectral subspace $E_{1,\ldots,q}$ chosen as the subspace of the $q < N$ slowest decaying modes of the linearized system.

In Fig. 4, we illustrate parts of the spectrum of $A$ that generate fast, intermediate and slow spectral subspaces, whose smoothest nonlinear continuations are the fast, intermediate and slow SSMs.

Table1 summarizes relevant relative spectral quotients and nonresonance conditions, as obtained from a direct application of Theorem 3 to fast, intermediate and slow spectral subspaces.

For fast SSMs, Table 1 requires no nonresonance condition, giving just a sharpened version of a classic result in dynamical systems, the strong stable manifold theorem (see, e.g., Hirsch et al. [20]). If the nonlinear function $f_0$ is analytic (class $C^a$) in a neighborhood of the origin, then so is the unique fast SSM, $W_{N-q+1,\ldots,N}(0)$. In that case, seeking the unique fast SSM as a Taylor-expanded graph over the fast stable subspace $E_{N-q+1,\ldots,N}$ leads to a convergent Taylor series for $W_{N-q+1,\ldots,N}(0)$. By statement (iii) of Theorem 3, the same holds for Taylor expansions with respect to any parameter $\mu$ on which the system may depend analytically.

That said, the relevance of fast SSMs for model reduction is generally limited. These manifolds contain atypical trajectories that reach the origin in the shortest possible time, practically unaffected by the remaining slower modes. Special cases of relevance may arise, for instance, if one wishes to control general motions that exhibit the fastest possible decay to the equilibrium.

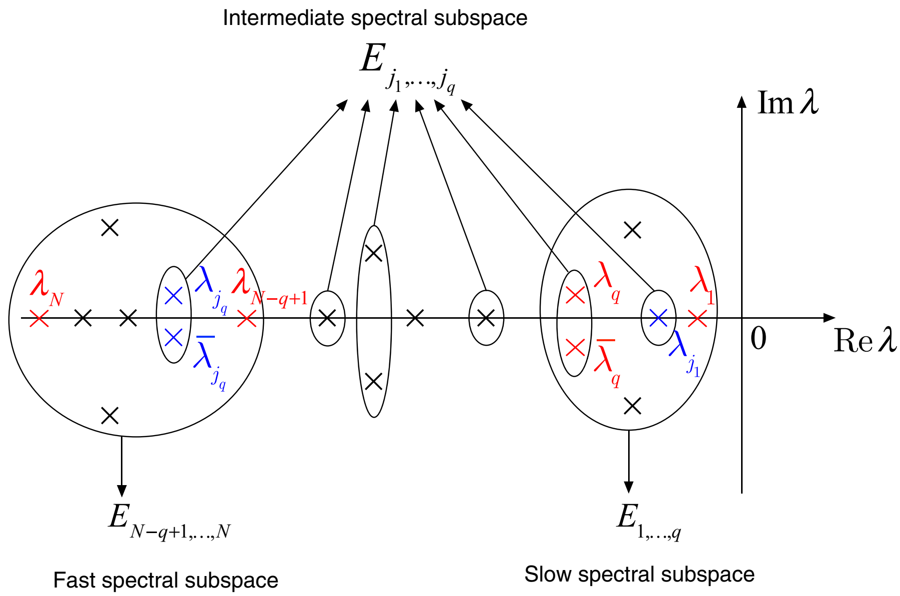



**Fig. 4** Fast, intermediate and slow spectral subspaces identified from the spectrum of $A$. The smoothest nonlinear continuations of these along an NNM are fast, intermediate and slow SSMs of the NNM

**Table 1** Conditions for different types of SSMs obtained from Theorem 3, with parameters $a, b \in \mathbb{N}^q$ and $l \in \mathbb{N}$

| | Fast SSM | Intermediate SSM | Slow SSM |
|---|---|---|---|
| $E$ | $E_{N-q+1,\ldots,N}$ | $E_{j_1,\ldots,j_q}$ | $E_{1,\ldots,q}$ |
| $\sigma(E)$ | 0 | $\text{Int}\left[\text{Re}\lambda_{j_{q+1}}/\text{Re}\lambda_{j_q}\right]$ | $\text{Int}\left[\text{Re}\lambda_N/\text{Re}\lambda_1\right]$ |
| Nonresonance | – | $\sum_{i=1}^q \left(a_i\lambda_{j_i} + b_i\bar{\lambda}_{j_i}\right) \neq \lambda_l$ <br> $\lvert a\rvert + \lvert b\rvert \in [2, \sigma(E)], \quad l \in \left[j_{q+1}, j_N\right]$ | $\sum_{i=1}^q \left(a_i\lambda_i + b_i\bar{\lambda}_i\right) \neq \lambda_l$ <br> $\lvert a\rvert + \lvert b\rvert \in [2, \sigma(E)], \quad l \in [q+1, N]$ |
| $W(0)$ | $W_{N-q+1,\ldots,N}(0)$ | $W_{j_1,\ldots,j_q}(0)$ | $W_{1,\ldots,q}(0)$ |

The two-dimensional invariant manifolds originally envisioned by Shaw and Pierre [39] generally fall in the category of intermediate SSMs, with $q = 1$, $\text{Im}\,\lambda_{j_1} \neq 0$, and $\dim E_{j_1} = 2$. In the later work by Peschek et al. [31], invariant surfaces defined over an arbitrary $q \geq 1$ number of internally resonant modes are envisioned, although the resonance among these modes is not exploited in the construction. By Table 1, all these intermediate SSMs exist in a rigorous mathematical sense, as long as the spectral subspaces over which they are constructed exhibit no low-order resonances with the remaining modes (resonances *within* those spectral subspaces are allowed). A low-order resonance is one whose order $|a| + |b|$ does not exceed $\sigma(E) = \text{Int}\left[\text{Re}\lambda_{j_{q+1}}/\text{Re}\lambda_{j_q}\right]$. Any such intermediate SSM is of class $C^r$, but is already unique in the class of $C^{\sigma(E)+1}$ invariant surfaces tangent to $E_{j_1,\ldots,j_q}$. This means that a Taylor expansion of order $\sigma(E) + 1$ or higher is only valid for a unique intermediate SSM.

Slow SSMs exist by Theorem 3 under the conditions detailed in the last column of Table 1. Again, no low-order resonances are allowed between the $q$ slowest decaying modes in $E_{1,\ldots,q}$ and the remaining faster modes outside $E_{1,\ldots,q}$. The order of the resonance is low if it does not exceed the relative spectral quotient $\sigma(E) = \text{Int}\,[\text{Re}\lambda_N/\text{Re}\lambda_1]$. Interestingly, this nonresonance order has no dependence on the number $q$ of slow modes considered. As intermediate SSMs, slow SSMs are unique among class $C^{\sigma(E)+1}$ invariant manifolds tangent to $E_{1,\ldots,q}$ at the trivial normal mode $x = 0$. For model reduction purposes, slow SSMs offer the most promising option, as we discuss in Sect. 8.

Shaw and Pierre [39,40], Elmegard [12], and Renson et al. [37] allude to the theory of normally hyperbolic invariant manifolds by Fenichel [15] as justification for the numerical computation of general SSMs. Another hint in the literature at a rigorous existence result for two-dimensional autonomous SSMs in analytic systems is given by Cirillo et al. [11], who invoke a classic analytic linearization theorem by Poincaré [34]. A closer inspection of these results reveals, however, that the applicability of the theorems of Fenichel and Poincaré is substantially limited in practical settings (see Appendices "Comparison with applicable results for normally hyperbolic invariant manifolds" and "Comparison with results deducible from analytic linearization theorems" for details).

*Example 1* (*Application of Theorem* 3) Consider the planar system

$$\dot{x} = -x,$$
$$\dot{y} = -\sqrt{24}y + x^2 + x^3 + x^4 + x^5, \tag{24}$$

which is analytic on the whole plane, i.e., we have $r = a$ in the notation of Theorem 1. The eigenvalues of the linearized system at the origin are $\lambda_2 = -\sqrt{24}$ and $\lambda_1 = -1$, giving $N = 2$ and $q = 1$ for the construction of a slow SSM $W_1(0)$ over the slow subspace $E_1 = \{(x, y) : y = 0\}$. The required order of nonresonance from Table 1 is, therefore,

$$\sigma(E) = \text{Int}\,[\text{Re}\lambda_N/\text{Re}\lambda_1] = \text{Int}\left[\sqrt{24}\right] = 4,$$

up to which the nonresonance condition

$$a_1 \cdot (-1) \neq -\sqrt{24}, \quad a_1 = 2, 3, 4$$

is satisfied. Then Theorem 3 guarantees the existence of an analytic (class $C^a$) slow SSM, $W_1(0)$, that is unique among all class $C^5$ invariant manifolds tangent to the $x$-axis at the origin. We seek this slow SSM in the form

$$y = h(x) = a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots, \tag{25}$$

the minimal Taylor expansion that only exists for the analytic SSM but not for the other invariant manifolds. Differentiation of (25) in time gives

$$\dot{y} = \left[2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \mathcal{O}(x^5)\right]\dot{x} = -2a_2x^2 - 3a_3x^3 - 4a_4x^4 - 5a_5x^5 + \mathcal{O}(x^6), \tag{26}$$

while substitution of (25) into the second equation in (24) gives

$$\dot{y} = \left(1 - \sqrt{24}a_2\right)x^2 + \left(1 - \sqrt{24}a_3\right)x^3 + \left(1 - \sqrt{24}a_4\right)x^4 + \left(1 - \sqrt{24}a_5\right)x^5 + \mathcal{O}(x^6). \tag{27}$$

Equating (26) and (27) gives

$$a_2 = \frac{1}{\sqrt{24}-2}, \quad a_3 = \frac{1}{\sqrt{24}-3}, \quad a_4 = \frac{1}{\sqrt{24}-4}, \quad a_5 = \frac{1}{\sqrt{24}-5}, \quad a_j = 0, \quad j \geq 6. \tag{28}$$

We also observe that the ODE (24) is explicitly solvable: A direct integration gives $x(t)$ which, upon substitution into the $y$ equation, yields an inhomogeneous linear ODE for $y(t)$. Combining the expressions for $x(t)$ and $y(t)$ enables us to eliminate the time variable $t$, giving the equation of trajectories in the form

$$y(x; x_0, y_0) = K(x_0, y_0)x^{\sqrt{24}} + \frac{x^2}{\sqrt{24}-2} + \frac{x^3}{\sqrt{24}-3} + \frac{x^4}{\sqrt{24}-4} + \frac{x^5}{\sqrt{24}-5},$$
$$K(x_0, y_0) = \frac{y_0}{x_0^{\sqrt{24}}} - \frac{x_0^{2-\sqrt{24}}}{\sqrt{24}-2} - \frac{x_0^{3-\sqrt{24}}}{\sqrt{24}-3} - \frac{x_0^{4-\sqrt{24}}}{\sqrt{24}-4} - \frac{x_0^{5-\sqrt{24}}}{\sqrt{24}-5},$$

with $(x_0, y_0)$ denoting an arbitrary initial condition on the trajectory. This shows that the graph $y(\cdot\,; x_0, y_0)$ of the slow SSM is generally only of class $C^4$, as the term $K(x_0, y_0)x^{\sqrt{24}}$ admits only four continuous derivatives at the origin. The only exception is the case $K(x_0, y_0) = 0$, for which $y(\cdot\,; x_0, y_0)$ becomes a quintic polynomial in $x$ and hence analytic over the whole plane. But $K(x_0, y_0) = 0$ holds only along the points

$$y_0 = \frac{x_0^2}{\sqrt{24}-2} + \frac{x_0^3}{\sqrt{24}-3} + \frac{x_0^4}{\sqrt{24}-4} + \frac{x_0^5}{\sqrt{24}-5}, \tag{29}$$

which lie precisely on the SSM, $W_1(0)$, whose Taylor expansion we computed in (28). This example, therefore, illustrates the sharpness of the results of Theorem 3: The analytic slow SSM, $W_1(0)$, is indeed unique among all five times continuously differentiable invariant manifolds tangent to the $x$-axis at the origin. We plot in red the unique analytic SSM for this example in Fig. 5a.

*Example 2* (*Optimality of Theorem* 3) Consider the planar dynamical system

$$\dot{x} = -x,$$

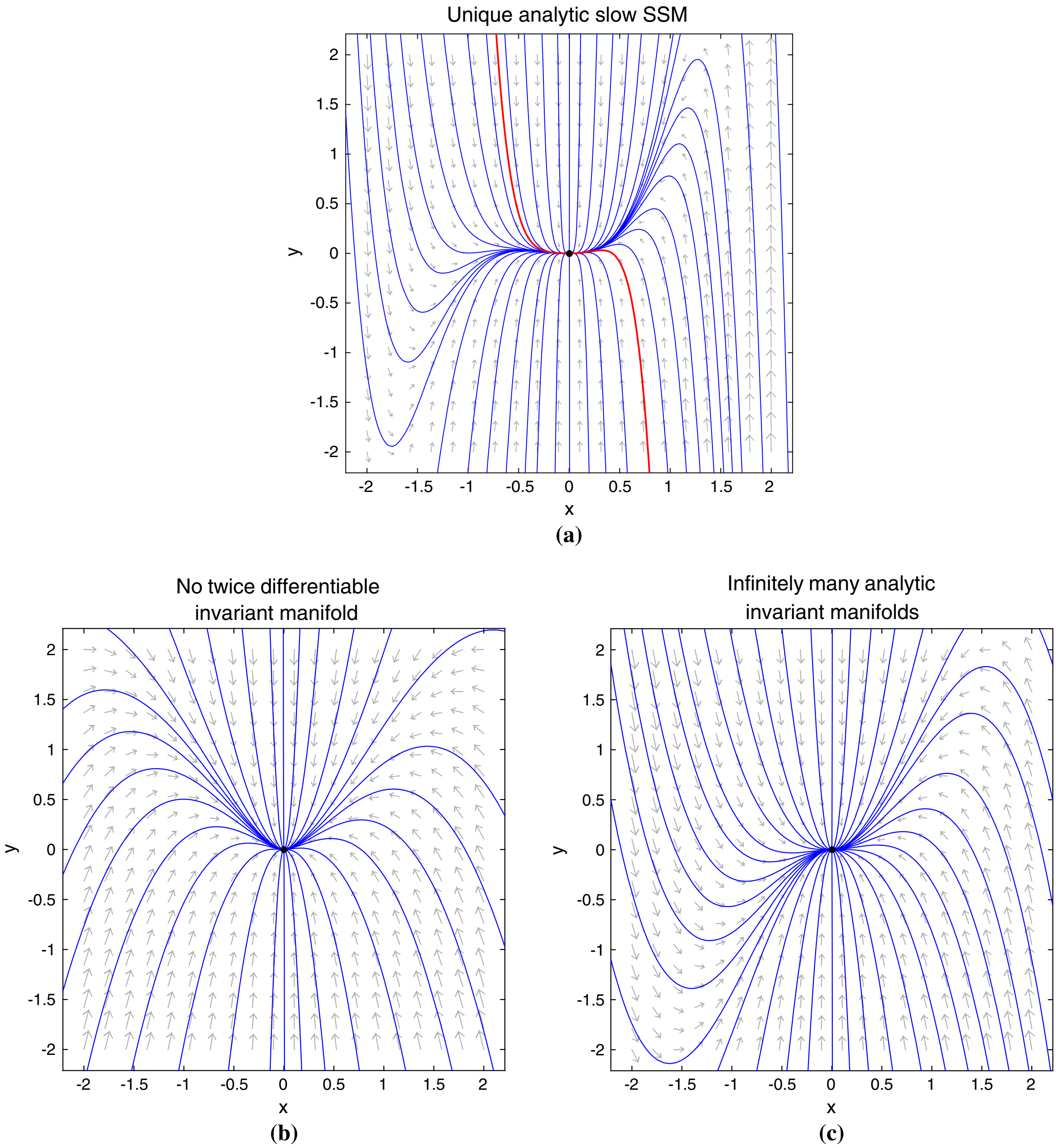



**Fig. 5** **a** Phase portrait of system (24), with the unique analytic SSM guaranteed by Theorem 3 computed explicitly (*red*). **b** Phase portrait of system (30). **c** Phase portrait of system (34). For all three plots: Trajectories are shown in *blue* and the vector field is indicated with *gray arrows*. (Color figure online)

$$\dot{y} = -2y + x^2, \tag{30}$$

with its phase portrait shown in Fig. 5b. The system is analytic over the whole plane and has a stable node-type fixed point at the origin with eigenvalues $\lambda_2 = -2$ and $\lambda_1 = -1$. This system, therefore, falls into the slow SSM case of Table 1 with $\sigma(E) = 2$. The corresponding nonresonance condition is, however, violated because

$a_1 \cdot (-1) = -2, \quad a_1 = 2.$

Theorem 3, therefore, fails to apply, and hence, we have no a priori mathematical guarantee for the existence or uniqueness of an at least $C^2$ slow SSM. To see if such a manifold nevertheless exists, we again seek a slow SSM in the form

$$y = h(x) = a_2 x^2 + a_3 x^3 + \cdots, \tag{31}$$

a graph with quadratic tangency to $E_1$ at the origin. Differentiation of this graph in time gives

$$\dot{y} = \left[2a_2 x + \mathcal{O}(x^2)\right]\dot{x} = -2a_2 x^2 + \mathcal{O}(x^3), \tag{32}$$

while substitution of the graph into the second equation in (30) gives

$$\dot{y} = (-2a_2 + 1)\, x^2 + \mathcal{O}(x^3). \tag{33}$$

Equating (32) and (33) gives no solution for $a_2$, and hence, no $C^2$ invariant manifold tangent to $E_1$ exists in this example. There are infinitely many invariant manifolds tangent to the spectral subspace $E_1$, but none of them is smoother than the other one: They all just have one continuous derivative at the origin. As a consequence, no SSM exists by Definition 3. Next, consider the slightly different dynamical system

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= -2y + x^3, \end{aligned} \tag{34}$$

with its phase portrait shown in Fig. 5c, which violates the same nonresonance condition as (30). This time, we find infinitely many analytic invariant manifolds tangent to the spectral subspace $E_1$. Indeed, any member of the analytic manifold family $y(x) = Cx^2 - x^3$, with the parameter $C \in \mathbb{R}$, is invariant and tangent to the spectral subspace $E_1$ of (34) at the origin. Thus, the violation of the nonresonance condition in the slow case of Table 1 may either lead to the nonexistence of a single $C^2$ invariant manifold, or to a high degree of non-uniqueness of smooth (even analytic) invariant manifolds.

*Example 3* (*Illustration of Theorem 3 on a mechanical example*) We reconsider here the damped nonlinear mechanical system studied by Shaw and Pierre [39]. As shown in Fig. 6, this two-degree-of-freedom mechanical system consists of two masses connected via springs to each other and to their environment. Two of the springs are linearly elastic and linearly damped, while the remaining spring is still elastic but has a cubic nonlinearity as well. The displacements $q_1$ and $q_2$, as well as the damping coefficient $c$, the spring constant $k$ and the coefficient $\gamma$ of the cubic nonlinearity, are all non-dimensionalized.

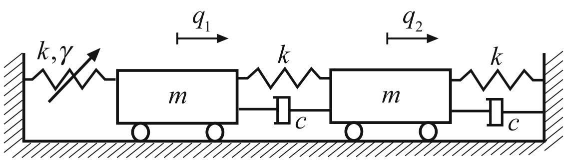



**Fig. 6** The two-degree-of-freedom mechanical model considered by Shaw and Pierre [39]

The equations of motion for this system are of the general form (2) with $n = N/2 = 2$ and $q = (q_1, q_2)$, and with the quantities

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad C = \begin{pmatrix} c & -c \\ -c & 2c \end{pmatrix},$$

$$K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}, \quad G = B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$F_0(q, \dot{q}) = \begin{pmatrix} -\gamma q_1^3 \\ 0 \end{pmatrix}, \quad \epsilon F_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In the variables $x_1 = q_1, x_2 = \dot{q}_1, x_3 = q_2, x_4 = \dot{q}_2$, the first-order form (5) of the system has

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{2k}{m} & -\frac{c}{m} & \frac{k}{m} & \frac{c}{m} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & \frac{c}{m} & -\frac{2k}{m} & -\frac{2c}{m} \end{pmatrix}, \quad f_0(x) = \begin{pmatrix} 0 \\ -\gamma x_1^3 \\ 0 \\ 0 \end{pmatrix},$$

$$\epsilon f_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Shaw and Pierre [39] fixed the parameter values

$$c = 0.3, \quad k = 1, \quad m = 1, \quad \gamma = 0.5, \tag{35}$$

and reported for this parameter setting the eigenvalues

$$\lambda_1 = -0.0741 \pm 1.0027i \quad \lambda_2 = -0.3759 \pm 1.6812i. \tag{36}$$

This implies the existence of two two-dimensional real invariant subspaces, $E_1$ and $E_2$, for the linearized system.

Shaw and Pierre calculated a formal cubic-order Taylor expansion for SSMs tangent to these subspaces at the origin. Since the function $f_0(x)$ is analytic on the whole phase space, Theorem 3 guarantees the existence of an analytic fast SSM, $W_2(0)$. Furthermore, since $\sigma(E_2) = 0$ holds by Table 1, $W_2(0)$ is unique among all $C^1$ invariant manifolds tangent to the fast spectral subspace $E_2$ at the fixed point $x = 0$. Theorem 3 also guarantees the existence of a unique analytic slow SSM, $W_1(0)$, as long as no resonance conditions (listed in the last column of Table 1) up to order

$$\sigma(E_1) = \mathrm{Int}\left[\frac{\mathrm{Re}\,\lambda_2}{\mathrm{Re}\,\lambda_1}\right] = 5$$

hold. These nonresonance conditions take the specific form

$$-0.0741\,(a_1 + b_1) + 1.0027\,(a_1 - b_1)\,i$$
$$\neq -0.3759 \pm 1.6812i, \quad |a_1| + |b_1| = 2, 3, 4, 5,$$

which are all satisfied, as seen by inspection.

We conclude that the analytic slow SSM $W_1(0)$ exists and is unique among all $C^6$ invariant manifolds tangent to the slow spectral subspace $E_1$ at the fixed point $x = 0$. Therefore, the cubic-order Taylor expansion of Shaw and Pierre [39] more than captures the fast SSM $W_2(0)$ uniquely, but fails to capture the slow SSM uniquely. Indeed, the latter cubic expansion holds for infinitely many $C^5$ invariant manifolds tangent to the origin along the slow spectral subspace. A 6th-order Taylor expansion would hold only for the unique analytic slow SSM, for which the expansion can be continued up to any order, giving a convergent power series in a neighborhood of the origin. The required order of expansion remains the 6th for general underdamped parameter values, but increases sharply with increasing overdamping (see Fig. 7).

We now carry out the computation of the slow SSM in detail for the parameter values (35). Applying a linear change of coordinates, we split the state vector $x \in \mathbb{R}^4$ as

$$x = (y, z) \in E_1 \times E_2, \tag{37}$$

which results in the transformed equations of motion

$$\dot{y} = A_y y + f_{0y}(y, z) = \begin{pmatrix} -0.0741 & 1.0027 \\ -1.0027 & -0.0741 \end{pmatrix} y + \begin{pmatrix} 1.0148 \\ -0.2162 \end{pmatrix} p(y, z), \tag{38}$$

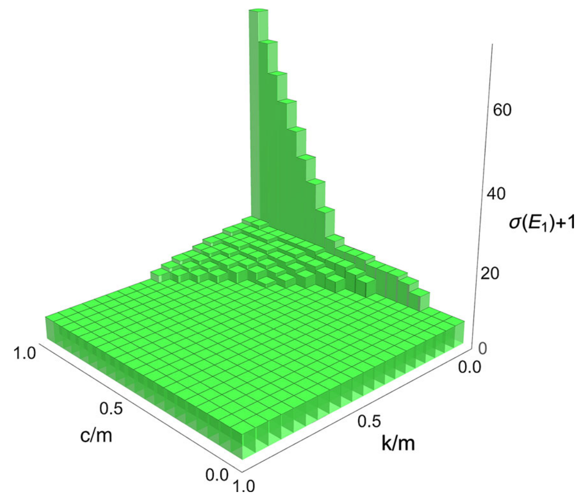



**Fig. 7** Dependence of the uniqueness class of the slow SSM in Example 3 on the parameters $k/m$ and $c/m$

$$\dot{z} = A_z z + f_{0z}(y, z) = \begin{pmatrix} -0.3759 & 1.6812 \\ -1.6812 & -0.3759 \end{pmatrix} z + \begin{pmatrix} 0.8046 \\ -0.1685 \end{pmatrix} p(y, z),$$
$$p(y, z) = -0.5\,(-0.0374\,y_1 - 0.5055\,y_2 - 0.1526\,z_1 - 0.3052\,z_2)^3\,. \tag{39}$$

We seek the slow SSM, $W_1(0)$, within the class of $C^6$ function in which the analytic SSM is already unique. This requires finding the coefficients in the 6th-order Taylor expansion

$$z = h(y) = \sum_{|p|=1}^{6} h_p y^p, \quad p = (p_1, p_2) \in \mathbb{N}^2,$$
$$y^p = y_1^{p_1} y_2^{p_2}, \quad h_p \in \mathbb{R}^2. \tag{40}$$

Differentiating this expression with respect to time and substituting $\dot{z}$ from (39) gives

$$\frac{\partial h(y)}{\partial y}\left[A_y y + f_{0y}\left(y, h(y)\right)\right] = A_z h(y) + f_{0z}\left(y, h(y)\right).$$

Equating powers of $y$ on both sides of this last expression, we obtain the unknown coefficients $h_p$ in (40) and hence the slow SSM in the form

$$z_1 = -0.0278y_1^3 + 0.0011y_1^2y_2 - 0.0026y_1y_2^2 + 0.0009y_2^3 + 0.0023y_1^5 - 0.0006y_1^4y_2 + 0.0026y_1^3y_2^2 - 0.0007y_1^2y_2^3 - 0.0010y_1y_2^4 + 0.0002y_2^5,$$

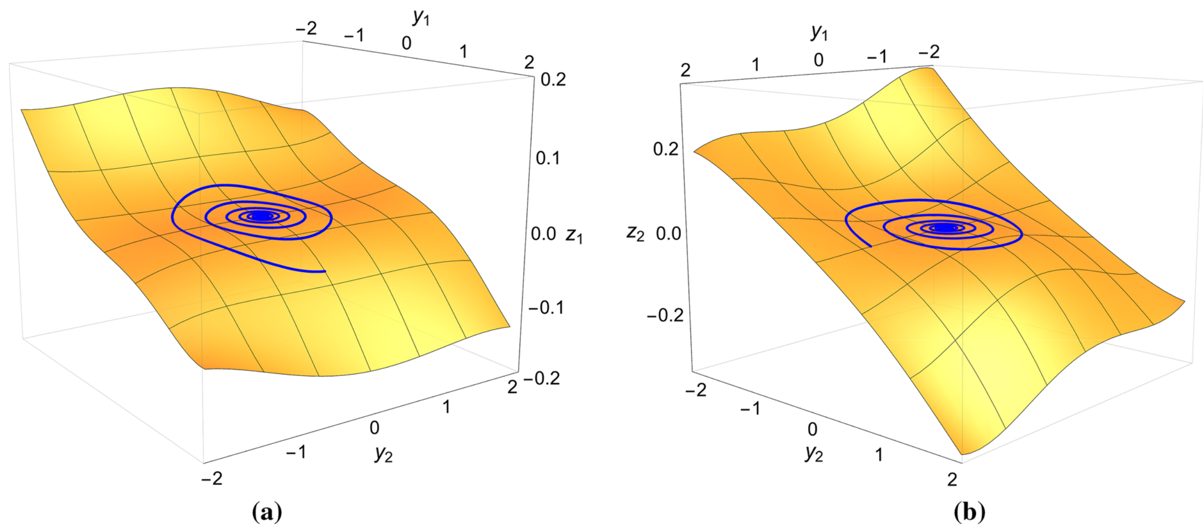



**Fig. 8** Two views of the slow SSM for the nonlinear oscillator system (38)–(39). In the plots (8a–8b), $z_1$ and $z_2$ are shown as a function of $y$, respectively. The *blue curve* is a trajectory starting on the SSM from the initial conditions $(y_1(0) = 1.2, y_2(0) = 0, z_1(y_1(0), y_2(0)) = -0.042, z_2(y_1(0), y_2(0)) = -0.0045)$. The trajectory remains close to the SSM and converges to a trivial NNM, the $(y, z) = (0, 0)$ fixed point. (Color figure online)

$$z_2 = -0.0032y_1^3 - 0.0470y_1^2y_2 - 0.0074y_1y_2^2 - 0.0323y_2^3 + 0.0004y_1^5 + 0.0039y_1^4y_2 + 0.0004y_1^3y_2^2 + 0.0065y_1^2y_2^3 - 0.0005y_1y_2^4 + 0.0011y_2^5.$$

Note that the 6th-order terms (as well as any other odd-order terms) vanish due to the particular form of the nonlinearity in this example. The slow SSM obtained in this fashion is shown in Fig. 8.

We now compare the accuracy of the third-order approximation employed by Shaw and Pierre [39] to the fifth-order approximation used here. By the nature of the nonlinearity, this is in fact just one step up in accuracy, as the fourth-order terms are absent in the Taylor expansion of the SSM. Figure 9 shows a Poincaré-map view of our comparison, with dots indicating the intersection of representative trajectories launched from the approximate slow SSMs with the $y_2 = 0$ Poincaré section. We conclude that the sixth-order (which is the same as the fifth-order) approximation to the slow SSM brings a major improvement in its accuracy. This is evidenced by significantly reduced trajectory oscillations arising from the lack of exact invariance of the approximate SSM.

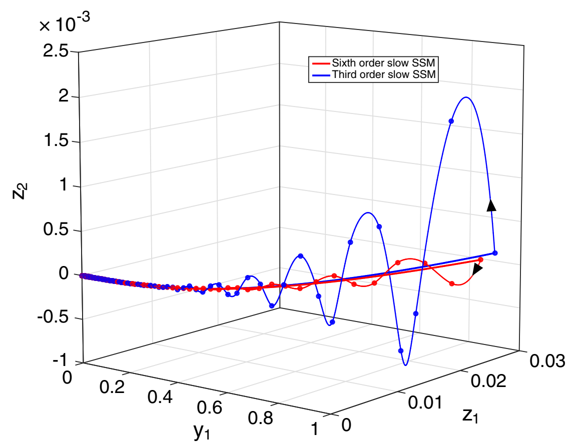



**Fig. 9** Poincaré map for trajectories launched on different approximation to the slow SSM. *Dots* indicate intersections of the trajectories with the $y_2 = 0$ hyperplane. *Arcs* connecting adjacent intersections are for illustration only, to give a sense of the trajectory evolution

## 7 Spectral submanifolds in non-autonomous systems ($k > 0$)

The idea of periodic SSMs ($k = 1$) was proposed first by Shaw, et al. [41] for undamped oscillatory systems and then later extended by Jiang et al. [22] for systems with damping. In these studies, the periodic time dependence appears as a perturbation, as in our Eq. (41). As a parallel development, Sinha et al. [42] considered systems with a time-periodic linear part and applied a Lyapunov–Floquet transformation to bring this linear part to an autonomous form before applying the SSM approach of Shaw and Pierre et al. This treatment appears to be the first one to give a general nonresonance condition for the Fourier expansion of the SSM to be at least formally computable (without consideration of convergence) up to a given order.

In later work, Redkar and Sinha [35] assume single-frequency external forcing and select the master modes (i.e., those constituting the spectral subspace of interest) as the ones in resonance or near-resonance with the external forcing. Gabale and Sinha [7] develop this approach further, selecting the master modes to be either in near-resonance with the forcing or to be those with eigenvalues that have dominant negative real parts (fast NNMs). The authors provide nonresonance conditions for formal computability up to any order, but the actual convergence of the approximation to a true invariant manifold is not discussed. As noted before, such a convergence is not guaranteed, as a PDE for an invariant surface can always be written down for any system, but it may not have a solution under the prescribed boundary conditions. Gabale and Sinha [7] also discuss the case of a time-periodic linear part, using a Lyapunov–Floquet transformation. This appears to be the first reference where the Shaw–Pierre invariant manifold approach is formally applied in the presence of two frequencies.

In summary, as in the autonomous case, only formal calculations of non-autonomous SSMs have appeared in the literature without mathematical arguments for existence and uniqueness. Unlike in the autonomous case, however, the connection of the assumed non-autonomous SSM to any surviving NNM (periodic orbit or invariant torus) has remained unexplored. It is therefore unclear in the literature what the orbits in the envisioned invariant manifolds should asymptote to. In the following, we address these conceptual gaps in the theory of non-autonomous SSMs.

## 7.1 Main result

We consider the full, perturbed non-autonomous dynamical system

$$\dot{x} = Ax + f_0(x) + \epsilon f_1(x, \Omega t; \epsilon),$$
$$f_0(x) = \mathcal{O}(|x|^2), \quad \Omega \in \mathbb{R}^k, \ k \geq 1; \quad 0 < \epsilon \ll 1. \tag{41}$$

Our smoothness assumptions on $f_0$ and $f_1$ will be spelled out in our main result below.

We continue to assume that the linear part of this system is asymptotically stable, i.e.,

$$\text{Re}\lambda_j < 0, \quad j = 1, \ldots, N. \tag{42}$$

As already noted in the autonomous case, this assumption on the dissipative nature of the system ensures that our NNM and SSM definitions indeed capture distinguished solution sets of the nonlinear oscillatory system.

**Theorem 4** (Existence, uniqueness and persistence of non-autonomous SSM) Consider a spectral subspace $E$ and assume that the low-order nonresonance conditions

$$\langle m, \text{Re}\lambda \rangle_E \neq \text{Re}\lambda_l, \quad \lambda_l \notin \text{Spect}(A|_E), \quad 2 \leq |m| \leq \Sigma(E) \tag{43}$$

hold for all eigenvalues $\lambda_l$ of $A$ that lie outside the spectrum of $A|_E$.

*Then the following holds:*

(i) *There exists an SSM,* $W(x_\epsilon(t))$ *that is of class* $C^{\Sigma(E)+1}$ *in the variable* $x$. For any fixed time $t_0$, the time slice $W(x_\epsilon(t_0))$ of the SSM is $\mathcal{O}(\epsilon)$ $C^r$-close to $E$ *along the quasiperiodic NNM,* $x_\epsilon(t) = \epsilon\tau(\Omega t; \epsilon)$. *Furthermore,* $\dim W(x_\epsilon(t)) = \dim E + k$.

(ii) $W(x_\epsilon(t))$ *is unique among all invariant manifolds that satisfy the properties listed in (i) and are at least of class* $C^{\Sigma(E)+1}$ *with respect to the* $x$ *variable along the NNM* $x_\epsilon(t)$,

(iii) If the functions $f_0$ and $f_1$ are $C^\infty$ or analytic, then $W(x_\epsilon(t))$ will depend on $\epsilon$ in a $C^\infty$ or analytic fashion, respectively.

*Proof* The results can be deduced from a more general result of Haro and de la Llave [18], as we show in Appendix "Proof of Theorem 4." □

According to Theorem 4, under the appropriate nonresonance conditions between the modes in the spectral subspace $E$ and those outside $E$, a well-defined periodic or quasiperiodic SSM attached to a periodic or quasiperiodic NNM exists. This gives precise mathematical conditions for the existence and uniqueness of the invariant surfaces envisioned by Jiang et al. [22] for the time-periodic case and extends their existence to the case of quasiperiodic forcing. The SSMs obtained in this fashion are unique among invariant surfaces that are at least $\Sigma(E)+1$-times continuously differentiable in the $x$ direction along the NNM.

## 7.2 Applications to specific spectral subspaces

We again consider a select group of $q$ master modes of the linearized system with

$$\mathrm{Re}\lambda_{j_1} \leq \cdots \leq \mathrm{Re}\lambda_{j_q} < 0, \tag{44}$$

and with the remaining modes ordered as

$$\mathrm{Re}\lambda_{j_{q+1}} \leq \cdots \leq \mathrm{Re}\lambda_{j_N} < 0.$$

In analogy with the autonomous case, we distinguish three types of non-autonomous SSMs in our discussion (cf. Fig. 4):

- *A fast spectral submanifold (fast SSM),* $W_{N-q+1,\ldots,N}(x_\epsilon(t))$, is an SSM in the sense of Definition 3, with the underlying spectral subspace chosen as $E_{N-q+1,\ldots,N}$, the subspace of the $q<N$ fastest decaying modes of the linearized system. The SSM $W_{N-q+1,\ldots,N}(x_\epsilon(t))$ is time-periodic if either $k=1$ or the elements of the frequency $\Omega$ are rationally commensurate for $k>1$. In all cases, $W_{N-q+1,\ldots,N}(x_\epsilon(t))$ is a surface in which trajectories are asymptotic to the nontrivial NNM, $x_\epsilon(\Omega t)$.
- *An intermediate spectral submanifold (intermediate SSM),* $W_{j_1,\ldots,j_q}(x_\epsilon(t))$, is an SSM in the sense of Definition 3, serving as the nonlinear continuation of

$$E_{j_1,\ldots,j_q} = E_{j_1} \oplus E_{j_2} \oplus \ldots \oplus E_{j_q} \tag{45}$$

  for a general choice of the $q<N$ eigenspaces $E_{j_1},\ldots,E_{j_q}$. Trajectories in $W_{j_1,\ldots,j_q}(x_\epsilon(t))$ are asymptotic to the nontrivial NNM, $x_\epsilon(t)$.
- *A slow spectral submanifold (slow SSM),* $W_{1,\ldots,q}(x_\epsilon(t))$, is an SSM in the sense of Definition 3, with the underlying spectral subspace chosen as $E_{1,\ldots,q}$, the subspace of the $q<N$ slowest decaying modes of the linearized system. Again, trajectories in $W_{1,\ldots,q}(x_\epsilon(t))$ are asymptotic to the nontrivial NNM, $x_\epsilon(t)$.

Table 2 summarizes the relevant absolute spectral quotients and nonresonance conditions, as deduced from Theorem 4, for specific choices of the spectral subspace $E$.

Much of our general discussion after Theorem 3 on the various choices of $E$ remains valid in the present non-autonomous context, with two main differences. First, even the existence of fast SSMs now requires a low-order nonresonance condition (cf. the first column of Table 2). Accordingly, a non-autonomous fast SSM is only guaranteed to be unique among at least $C^{\Sigma(E)+1}$ smooth invariant manifolds. Second, Table 2 only requires the real parts of the eigenvalues inside $E$ to be in nonresonance with the real parts of those outside $E$. Resonances, therefore, occur with a larger likelihood than those listed for the autonomous case in Table 1, since they now only involve a condition on the real parts of the eigenvalues.

Being as far as possible from resonances is also more important here than in the autonomous case, as the exact nonresonance condition ensuring the convergence of the Taylor approximation for non-autonomous SSMs is not explicitly known. Rather, this condition is only known to be $\mathcal{O}(\epsilon)$ close to that listed in the appropriate column of Table 2. This is because the spectrum of the infinite-dimensional transfer operator arising in the proof of the Theorem 4 is generally only computable for $\epsilon=0$, giving the nonresonance conditions listed in Table 2 (see Appendix "Proof of Theorem 4").

As in the autonomous case, one might ask if the existence of SSMs guaranteed by Theorem 4 could also be deduced directly from more classical dynamical systems results (cf. our related discussion in Appendices "Comparison with applicable results for normally hyperbolic invariant manifolds" and "Comparison with results deducible from analytic linearization theorems" for the autonomous case). It turns out that the shortcomings of Fenichel's invariant manifold theorem would be the same as in the autonomous case, while the non-autonomous extensions of Poincare's analytic linearization theorem would be even more restrictive than in the autonomous case (cf. Appendices "Comparison with applicable results for normally hyperbolic invariant manifolds" and "Comparison with results deducible from analytic linearization theorems" for details).

**Table 2** Conditions for different types of non-autonomous SSMs appearing in Theorem 4, with parameters $a \in \mathbb{N}^q$ and $l \in \mathbb{N}$

| | Fast SSM | Intermediate SSM | Slow SSM |
|---|---|---|---|
| $E$ | $E_{N-q+1,\dots,N}$ | $E_{j_1,\dots,j_q}$ | $E_{1,\dots,q}$ |
| $\Sigma(E)$ | $\mathrm{Int}\left[\mathrm{Re}\lambda_N/\mathrm{Re}\lambda_{N-q+1}\right]$ | $\mathrm{Int}\left[\mathrm{Re}\lambda_N/\mathrm{Re}\lambda_{j_q}\right]$ | $\mathrm{Int}\left[\mathrm{Re}\lambda_N/\mathrm{Re}\lambda_1\right]$ |
| Nonresonance | $\sum_{i=N-q+1}^{N} a_i\mathrm{Re}\lambda_i \neq \mathrm{Re}\lambda_l$ | $\sum_{i=1}^{q} a_i\mathrm{Re}\lambda_{j_i} \neq \mathrm{Re}\lambda_l$ | $\sum_{i=1}^{q} a_i\mathrm{Re}\lambda_i \neq \mathrm{Re}\lambda_l$ |
| | $\|a\| \in [2, \Sigma(E)],\quad l \in [1, N-q]$ | $\|a\| \in [2, \Sigma(E)],\quad l \in \left[j_{q+1}, j_N\right]$ | $\|a\| \in [2, \Sigma(E)],\quad l \in [q+1, N]$ |
| $W(x_\epsilon(t))$ | $W_{N-q+1,\dots,N}(x_\epsilon(t))$ | $W_{j_1,\dots,j_q}(x_\epsilon(t))$ | $W_{1,\dots,q}(x_\epsilon(t))$ |

*Example 4* (Periodic SSM from the application of Theorem 4) Consider a periodically forced version of Example 1, given by

$$\dot{x} = -x,$$
$$\dot{y} = -\sqrt{24}y + x^2 + x^3 + x^4 + x^5 + \epsilon \sin \Omega_1 t, \tag{46}$$

with $\Omega_1 = 1$. This system is analytic in all variables, and hence, we again have $r = a$ in the notation of Theorem 1. The same nonresonance conditions are satisfied as in Example 1. Therefore, Theorem 4 guarantees the existence of an analytic (i.e., class $C^a$) slow SSM, $W_1(x_\epsilon(t))$, that is unique among all class $C^5$ (in $x$) invariant manifolds tangent to the horizontal axis along the NNM. Near the origin, this slow SSM is guaranteed to be of the form

$$y = h(x,t) = a_0(t) + a_2(t)x^2 + a_3(t)x^3 + a_4(t)x^4 + a_5(t)x^5 + \cdots, a_j(t+2\pi) = a_j(t). \tag{47}$$

This is the minimal Taylor expansion that only exists for the analytic SSM but not for the other invariant manifolds tangent to the slow subbundle along the NNM. Differentiation of (47) in time gives

$$\dot{y} = \dot{a}_0 + \left[2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \mathcal{O}(x^5)\right] \dot{x} + \dot{a}_2x^2 + \dot{a}_3x^3 + \dot{a}_4x^4 + \dot{a}_5x^5 + \mathcal{O}(x^6)$$
$$= (\dot{a}_2 - 2a_2)\, x^2 + (\dot{a}_3 - 3a_3)\, x^3 + (\dot{a}_4 - 4a_4)\, x^4 + (\dot{a}_5 - 5a_5)\, x^5 + \mathcal{O}(x^6), \tag{48}$$

while substitution of (47) into the second equation in (46) gives

$$\dot{y} = -\sqrt{24}a_0 + \epsilon \sin t + \left(1 - \sqrt{24}a_2\right) x^2 + \left(1 - \sqrt{24}a_3\right) x^3 + \left(1 - \sqrt{24}a_4\right) x^4 + \left(1 - \sqrt{24}a_5\right) x^5 + \mathcal{O}(x^6). \tag{49}$$

Equating (48) and (49) gives

$$\dot{a}_0 = -\sqrt{24}a_0 + \epsilon \sin t, \quad \dot{a}_j = \left(j - \sqrt{24}\right) a_j + 1, \quad j = 2,3,4,5, \quad \dot{a}_k = \left(j - \sqrt{24}\right) a_k, \quad k \geq 6. \tag{50}$$

The requirement of $2\pi$-periodicity on $a_i(t)$ given in (47) defines a boundary-value problem for the ODEs in (50), whose unique solutions are

$$a_0(t) = \epsilon \frac{\sqrt{24}}{25}\left(\sin t - \frac{1}{\sqrt{24}} \cos t\right), \quad a_j(t) \equiv \frac{1}{\sqrt{24} - j}, \quad j = 2,3,4,5, \quad a_k(t) \equiv 0, \quad k \geq 6$$

Just as in Example 1, the ODE (46) is explicitly solvable: A direct integration gives $x(t)$ which, upon substitution into the $y$ equation, yields an inhomogeneous linear ODE for $y(t)$. Combining the expressions for $x(t)$ and $y(t)$ gives the solutions in the form

$$y(x; x_0, y_0, t) = K(x_0, y_0)x^{\sqrt{24}} + \frac{x^2}{\sqrt{24}-2} + \frac{x^3}{\sqrt{24}-3} + \frac{x^4}{\sqrt{24}-4} + \frac{x^5}{\sqrt{24}-5} + \epsilon\frac{\sqrt{24}}{25}\left[\sin t - \frac{1}{\sqrt{24}} \cos t\right],$$
$$K(x_0, y_0, t_0) = \frac{y_0 - \epsilon \frac{\sqrt{24}}{25}\left(\sin t_0 - \frac{1}{\sqrt{24}} \cos t_0\right)}{x_0^{\sqrt{24}}} - \frac{x_0^{2-\sqrt{24}}}{\sqrt{24}-2} - \frac{x_0^{3-\sqrt{24}}}{\sqrt{24}-3} - \frac{x_0^{4-\sqrt{24}}}{\sqrt{24}-4} - \frac{x_0^{5-\sqrt{24}}}{\sqrt{24}-5}, \tag{51}$$

with $(x_0, y_0)$ denoting an arbitrary initial condition for the solution at the initial time $t_0$.

This confirms the existence of a unique periodic NNM guaranteed by Theorem 2. Specifically,

$$\begin{pmatrix} x_\epsilon(t) \\ y_\epsilon(t) \end{pmatrix} = \epsilon\tau(\Omega_1 t;\epsilon) = \epsilon\left(\begin{array}{c} 0 \\ \frac{\sqrt{24}}{25}\left[\sin t - \frac{1}{\sqrt{24}}\cos t\right] \end{array}\right)$$

attracts all solutions from the SSM, which can be seen with $x = x_0 e^{-t}$ substituted into (51). The graph $y(\,\cdot\,;x_0,y_0;t)$ is a time-dependent representation of all invariant manifolds tangent to the slow subbundle of this NNM, which is parallel to the $x$-axis. As in the autonomous case, these invariant manifolds are generally only of class $C^4$, because the term $K(x_0,y_0,t_0)x^{\sqrt{24}}$ admits only four continuous derivatives along the NNM (which satisfies $x \equiv 0$). The only exception is the case $K(x_0,y_0) = 0$, for which $y(\,\cdot\,;x_0,y_0)$ becomes a quintic polynomial in $x$ plus sine and cosine functions of $t$, all of which are analytic. But $K(x_0,y_0) = 0$ holds only along the points

$$\begin{aligned} y_0(x_0,t_0) = {} & \frac{x_0^2}{\sqrt{24}-2} + \frac{x_0^3}{\sqrt{24}-3} + \frac{x_0^4}{\sqrt{24}-4} \\ & + \frac{x_0^5}{\sqrt{24}-5} + \epsilon\frac{\sqrt{24}}{25}\left(\sin t_0 - \frac{1}{\sqrt{24}}\cos t_0\right), \end{aligned} \tag{52}$$

which lie precisely on the $W_1(\tau_\epsilon(t_0))$ slice (or fiber) of the SSM, $W_1(\tau_\epsilon(t))$, whose Taylor expansion we computed in (50). We show the unique analytic SSM for this example in Fig. 10.

*Example 5* (*Quasiperiodic SSM from the application of Theorem 4*) Consider the system

$$\begin{aligned} \dot{x} = -x,\ \dot{y} = {} & -\sqrt{24}y + x^2 + x^3 + x^4 \\ & + x^5 + \epsilon(\sin\Omega_1 t + \sin\Omega_2 t), \end{aligned} \tag{53}$$

with $\Omega_1 = 1$ and $\Omega_2 = \sqrt{2}$. This is just the quasi-periodically forced version of Example 3. Based on the same reasoning as in that example, we conclude from Theorem 4 the existence of a unique quasiperiodic SSM in the form

$$\begin{aligned} y = h(x,\phi_1,\phi_2) = {} & a_0(\phi_1,\phi_2) \\ & + a_2(\phi_1,\phi_2)x^2 + a_3(\phi_1,\phi_2)x^3 \\ & + a_4(\phi_1,\phi_2)x^4 + a_5(\phi_1,\phi_2)x^5 + \cdots, \\ a_j(\phi_1,\phi_2) = {} & a_j(\phi_1 + 2\pi/\Omega_1,\phi_2), \\ a_j(\phi_1,\phi_2) = {} & a_j(\phi_1,\phi_2 + 2\pi/\sqrt{2}), \end{aligned} \tag{54}$$

with the phase variables satisfying $\dot{\phi}_1 = 1$, $\dot{\phi}_2 = \sqrt{2}$. Differentiation of (54) in time gives

$$\begin{aligned} \dot{y} = {} & \dot{a}_0 + \left[2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \mathcal{O}(x^5)\right] \\ & \times\dot{x} + \dot{a}_2x^2 + \dot{a}_3x^3 + \dot{a}_4x^4 + \dot{a}_5x^5 + \mathcal{O}(x^6) \\ = {} & (\dot{a}_2 - 2a_2)\,x^2 + (\dot{a}_3 - 3a_3)\,x^3 + (\dot{a}_4 - 4a_4)\,x^4 \\ & + (\dot{a}_5 - 5a_5)\,x^5 + \mathcal{O}(x^6) \end{aligned} \tag{55}$$

while substitution of (54) into the second equation in (53) gives

$$\begin{aligned} \dot{y} = {} & -\sqrt{24}a_0 + \epsilon\left(\sin t + \sin\sqrt{2}t\right) + \left(1 - \sqrt{24}a_2\right) \\ & \times x^2 + \left(1 - \sqrt{24}a_3\right)x^3 \qquad (56) \\ & + \left(1 - \sqrt{24}a_4\right)x^4 + \left(1 - \sqrt{24}a_5\right)x^5 + \mathcal{O}(x^6). \end{aligned} \tag{57}$$

Equating (55) and (56) gives

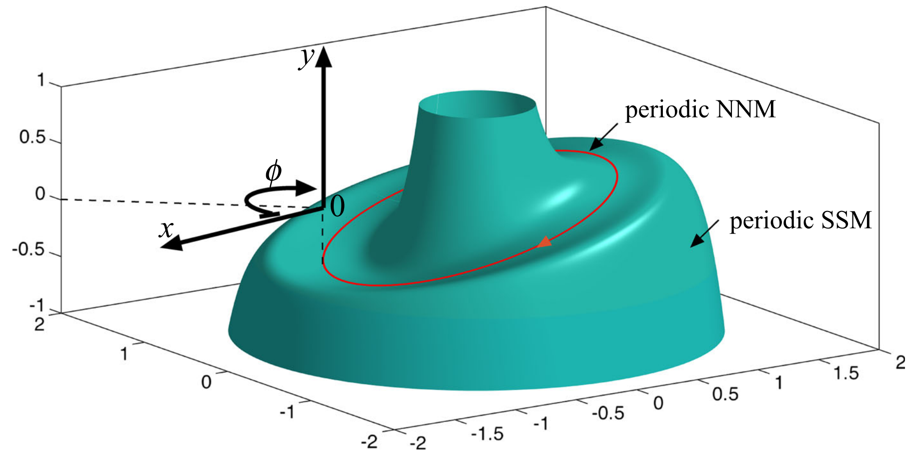
$$\begin{aligned} \dot{a}_0 &= -\sqrt{24}a_0 + \epsilon\left(\sin t + \sin\sqrt{2}t\right), \\ \dot{a}_j &= \left(j - \sqrt{24}\right)a_j + 1, \quad j \in [2,5], \\ \dot{a}_k &= \left(j - \sqrt{24}\right)a_k, \\ \dot{a}_k &= \left(j - \sqrt{24}\right)a_k, \quad k \geq 6. \end{aligned} \tag{58}$$

The quasi-periodicity requirements on $(\phi_1,\phi_2)$ given in (54) define a boundary-value problem for the PDEs in (58), whose unique solutions are

$$\begin{aligned} a_0(\phi_1,\phi_2) &= \epsilon\left(\frac{\sqrt{24}}{25}\left[\sin\phi_1 - \frac{1}{\sqrt{24}}\cos\phi_1\right]\right. \\ &\qquad \left. + \frac{\sqrt{24}}{26}\left[\sin\phi_2 - \frac{\sqrt{2}}{\sqrt{24}}\cos\phi_2\right]\right), \\ a_j(\phi_1,\phi_2) &\equiv \frac{1}{\sqrt{24}-j}, \quad j = 2,3,4,5, \\ a_k(\phi_1,\phi_2) &\equiv 0, \quad k \geq 6. \end{aligned}$$

At the same time, just as in Example 3, the ODE (53) is explicitly solvable: A direct integration gives $x(t)$ which, upon substitution into the $y$ equation, yields an inhomogeneous linear ODE for $y(t)$. Combining the expressions for $x(t)$ and $y(t)$ gives the solutions in the form

**Fig. 10** Phase portrait of system (46) in the extended phase space of $(x, y, \phi)$, where $\phi = t \mod 2\pi$. The *green* surface is the unique analytic SSM guaranteed by Theorem 4, emanating from the unique NNM (*red*) guaranteed by Theorem 2. The forcing parameter is selected as $\epsilon = 2$. (Color figure online)




$$y(x; x_0, y_0, t) = K(x_0, y_0)x^{\sqrt{24}} + \frac{x^2}{\sqrt{24}-2} + \frac{x^3}{\sqrt{24}-3} + \frac{x^4}{\sqrt{24}-4} + \frac{x^5}{\sqrt{24}-5}$$
$$+\epsilon\left(\frac{\sqrt{24}}{25}\left[\sin t - \frac{1}{\sqrt{24}}\cos t\right] + \frac{\sqrt{24}}{26}\left[\sin\sqrt{2}t - \frac{\sqrt{2}}{\sqrt{24}}\cos\sqrt{2}t\right]\right),$$
$$K(x_0, y_0, t_0) = \frac{y_0 - \epsilon\left(\frac{\sqrt{24}}{25}\left[\sin t_0 - \frac{1}{\sqrt{24}}\cos t_0\right] + \frac{\sqrt{24}}{26}\left[\sin\sqrt{2}t_0 - \frac{\sqrt{2}}{\sqrt{24}}\cos\sqrt{2}t_0\right]\right)}{x_0^{\sqrt{24}}}$$
$$-\frac{x_0^{2-\sqrt{24}}}{\sqrt{24}-2} - \frac{x_0^{3-\sqrt{24}}}{\sqrt{24}-3} - \frac{x_0^{4-\sqrt{24}}}{\sqrt{24}-4} - \frac{x_0^{5-\sqrt{24}}}{\sqrt{24}-5},$$

with $(x_0, y_0)$ denoting an arbitrary initial condition for the solution at the initial time $t_0$.

Again, all these solutions decay exponentially to a unique quasiperiodic NNM given by

$$\begin{pmatrix} x_\epsilon(t) \\ y_\epsilon(t) \end{pmatrix} = \epsilon\tau(\Omega_1, \Omega_2 t; \epsilon)$$
$$= \epsilon\begin{pmatrix} 0 \\ \frac{\sqrt{24}}{25}\left[\sin t - \frac{1}{\sqrt{24}}\cos t\right] \\ +\frac{\sqrt{24}}{26}\left[\sin\sqrt{2}t - \frac{\sqrt{2}}{\sqrt{24}}\cos\sqrt{2}t\right] \end{pmatrix}.$$

The graph $y(\cdot; x_0, y_0; t)$ is a time-dependent representation of all invariant manifolds tangent to the slow subbundle of this NNM. Any $t = t_0$ slice of this subbundle is parallel to the $x$-axis, i.e., to the spectral subspace $E_1$. As in the autonomous case, these invariant manifolds are generally only of class $C^4$, because the term $K(x_0, y_0, t_0)x^{\sqrt{24}}$ admits only four continuous derivatives along the NNM (which satisfies $x \equiv 0$). The only exception is the case $K(x_0, y_0) = 0$, for which $y(\cdot; x_0, y_0)$ becomes a quintic polynomial in $x$ plus sine and cosine functions of $t$, all of which are analytic. But $K(x_0, y_0) = 0$ holds only along the points

$$y_0(x_0, t_0) = \frac{x_0^2}{\sqrt{24}-2} + \frac{x_0^3}{\sqrt{24}-3} + \frac{x_0^4}{\sqrt{24}-4}$$
$$+\frac{x_0^5}{\sqrt{24}-5} + \epsilon\left(\frac{\sqrt{24}}{25}\left[\sin t_0 - \frac{1}{\sqrt{24}}\cos t_0\right]\right.$$
$$\left.+\frac{\sqrt{24}}{26}\left[\sin\sqrt{2}t_0 - \frac{\sqrt{2}}{\sqrt{24}}\cos\sqrt{2}t_0\right]\right), \qquad (59)$$

which lie precisely on the $t = t_0$ slice (fiber) of the SSM, $W_1(x_\epsilon(t))$ whose Taylor expansion we computed in (58). We show the unique analytic SSM for this example in Fig. 11.

*Example 6* (*Illustration of Theorem 4 on a mechanical example*) As a last example, we reconsider here Example 3 with time-dependent forcing. First, we illustrate

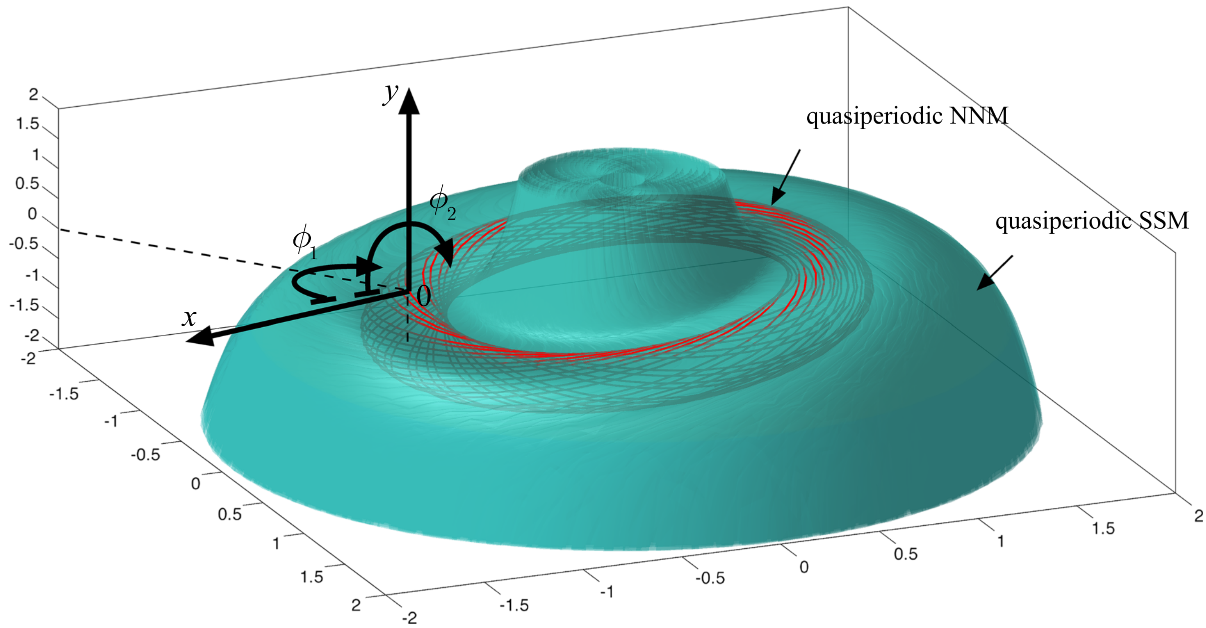



**Fig. 11** A projection of system (53) from the extended phase space of $(x, y, \phi_1, \phi_2)$, where $\phi_1 = (\phi_{10} + \Omega_1 t) \mod 2\pi$ and $\phi_2 = (\phi_{20} + \Omega_2 t) \mod 2\pi/\sqrt{2}$. The green surface is the unique analytic, quasiperiodic SSM guaranteed by Theorem 4, emanating from the unique quasiperiodic NNM (*red*) guaranteed by Theorem 2. The forcing parameter is selected as $\epsilon = 2$. The specific projection used in this visualization is $(x, \phi_1, \phi_2, y) \mapsto (x + 0.2\cos\phi_2 + 1)\cos\phi_1, (x + 0.2\cos\phi_2 + 1)\sin\phi_1, y(x, \phi_1, \phi_2))$. (Color figure online)

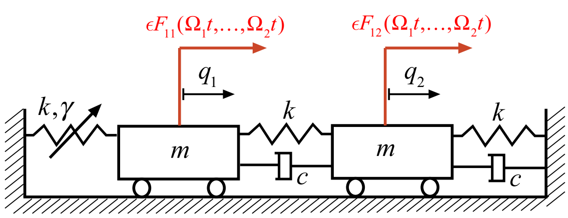



**Fig. 12** The quasiperiodically forced version of Example 3

the application of Theorem 4 to the general case of quasiperiodic forcing. Next, we restrict the forcing to be periodic and compute the periodic NNM and slow periodic SSM guaranteed by our results for this case.

Figure 12 shows the two-degree-of-freedom system already featured in Fig. 6, but now with multi-frequency parametric forcing

$$\epsilon F_1 = \epsilon \begin{pmatrix} F_{11}(\Omega_1 t, \ldots, \Omega_k t) \\ F_{12}(\Omega_1 t, \ldots, \Omega_k t) \end{pmatrix} \tag{60}$$

acting on both masses, with $k \geq 1$ arbitrary frequencies. All other details remain the same as in Example 3.

The eigenvalues of the linearized, unforced system are again those listed in (36), yielding the absolute spectral quotients

$$\Sigma(E_1) = \text{Int}\left[\frac{\text{Re}\,\lambda_2}{\text{Re}\,\lambda_1}\right] = 5, \quad \Sigma(E_2) = \text{Int}\left[\frac{\text{Re}\,\lambda_2}{\text{Re}\,\lambda_2}\right] = 1.$$

By Table 2, the relevant nonresonance condition for the slow non-autonomous SSM is

$$-0.0741 a_1 \neq -0.3759, \quad a_1 = 2, 3, 4, 5,$$

which is very close to being satisfied for $a_1 = 5$. This means that the existence of a non-autonomous SSM can only be concluded from Theorem 4 for very small values of $\epsilon$. Whenever it exists, the slow SSM is still analytic in the positions and velocities and unique among invariant manifolds that are at least of class $C^6$ in these variables. The dependence of this uniqueness class on the parameters is identical to that shown in Fig. 7. As for the fast SSM, Table 2 shows that no nonresonance conditions are required, because $\Sigma(E_2) < 2$. This time, however, the fast SSM can only be concluded to be unique in the function class $C^2$, given that $\Sigma(E_2) = 1$.

For simplicity, we now restrict our discussion to time-periodic forcing by selecting the forcing terms (60) as

$$\epsilon F_1 = \begin{pmatrix} 0 \\ \sin(\Omega_1 t) \end{pmatrix},$$

with $\Omega_1 = 1$. As concluded above already for more general forcing terms, Theorem 4 guarantees the existence of a unique analytic slow SSM, $W_1(x_\epsilon(t))$, for $\epsilon > 0$ small enough. This SSM is already unique among class $C^6$ invariant manifolds tangent to the slow spectral subbundle of a small-amplitude, periodic NNM, which is guaranteed to exist by Theorem 2. To compute this NNM and its slow SSM, we again use a linear change of coordinates (37) to obtain the equations of motion in the form

$$\begin{aligned}
\dot{y} &= A_y y + f_{0y}(y,z) + \epsilon f_{1y}(\Omega_1 t;\epsilon) \\
&= \begin{pmatrix} -0.0741 & 1.0027 \\ -1.0027 & -0.0741 \end{pmatrix} y + \begin{pmatrix} 1.0148 \\ -0.2162 \end{pmatrix} p(y,z) \\
&\quad + \epsilon \begin{pmatrix} 1.0016 \\ -0.0660 \end{pmatrix} \sin t, \qquad (61) \\
\dot{z} &= A_z z + f_{0z}(y,z) + \epsilon f_{1z}(\Omega_1 t;\epsilon) \\
&= \begin{pmatrix} -0.3759 & 1.6812 \\ -1.6812 & -0.3759 \end{pmatrix} z + \begin{pmatrix} 0.8046 \\ -0.1685 \end{pmatrix} p(y,z) \\
&\quad + \epsilon \begin{pmatrix} -0.7987 \\ 0.3861 \end{pmatrix} \sin t, \\
p(y,z) &= -0.5\,(-0.0374\,y_1 - 0.5055\,y_2 \\
&\quad - 0.1526\,z_1 - 0.3052\,z_2)^3. \qquad (62)
\end{aligned}$$

First, we seek the unique periodic NNM of this system in the form of a Taylor expansion in the perturbation parameter $\epsilon$. By Theorem 2, this NNM can be written in the form

$$x_\epsilon(t) = \epsilon \tau_1(t) + \mathcal{O}(\epsilon^2). \qquad (63)$$

Substitution of this expression into (61)–(62) and collection of the $\mathcal{O}(\epsilon)$ terms gives

$$\dot{\tau}_1(t) = A_{yz}\tau_1(t) + c \cdot \sin t, \quad A_{yz} = \begin{pmatrix} A_y & 0 \\ 0 & A_z \end{pmatrix},$$
$$c = \begin{pmatrix} 1.0016 \\ -0.0660 \\ -0.7987 \\ 0.3861 \end{pmatrix}. \qquad (64)$$

The unique, periodic particular solution of this inhomogeneous system of linear differential equations can be sought in the form

$$\tau_1(t) = a \cdot \sin t + b \cdot \cos t, \quad a, b \in \mathbb{R}^4. \qquad (65)$$

Substituting (65) into (64) gives algebraic equations for the vectors $a$ and $b$, whose solutions are explicitly computable as

$$a = -A_{yz}\left(A_{yz}^2 + I\right)^{-1} c, \quad b = -\left(A_{yz}^2 + I\right)^{-1} c.$$

With the relevant parameter values substituted into (63), we obtain the leading-order approximation of the attracting periodic NNM in the form

$$x_\epsilon(t) = \epsilon \begin{pmatrix} 6.7213 \\ -0.9408 \\ 0.0194 \\ 0.7253 \end{pmatrix} \sin t + \epsilon \begin{pmatrix} 0.4402 \\ 6.7357 \\ -0.4134 \\ -0.0809 \end{pmatrix} \cos t + \mathcal{O}(\epsilon^2). \qquad (66)$$

To obtain the unique slow SSM, $W_1(x_\epsilon(t))$, guaranteed by Theorem 4, we use the time-periodic Taylor expansion

$$z = h(y,t) = \sum_{|p|=0}^{6} h_p(t) y^p, \quad p = (p_1, p_2) \in \mathbb{N}^2,$$
$$y^p = y_1^{p_1} y_2^{p_2}, \quad h_p(t) = h_p(t + 2\pi) \in \mathbb{R}^2. \qquad (67)$$

Differentiating (67) with respect to time and substituting $\dot{y}$ and $\dot{z}$ from (61)–(62) gives

$$\frac{\partial h(y,t)}{\partial y}\left[A_y y + f_{0y}(y, h(y,t)) + \epsilon f_{1y}(\Omega_1 t;\epsilon)\right]$$
$$+ \sum_{|p|=0}^{6} \dot{h}_p y^p = A_z h(y,t) + f_{0z}(y, h(y,t))$$
$$+ \epsilon f_{1z}(\Omega_1 t;\epsilon).$$

Comparing equal powers of $y$ in this last expression leads to a set of coupled ODEs for $h_p(t)$. The $2\pi$-periodicity requirement on $h_p(t)$ given in (67) defines a boundary-value problem for these ODEs, which we solve numerically. The slow SSM surface obtained in this fashion is shown in the extended phase space in Fig. 13, along with the periodic NNM (red) obtained in (66).

As an alternative view, an instantaneous projection of the dynamics on the slow SSM from the four-dimensional $(y_1, y_2, z_1, z_2)$ phase space is shown in Fig. 14.

## 8 Relevance for model reduction

### 8.1 Expansions for NNMs

Theorems 1 and 2 provide existence, uniqueness and robustness results for NNMs in both the autonomous

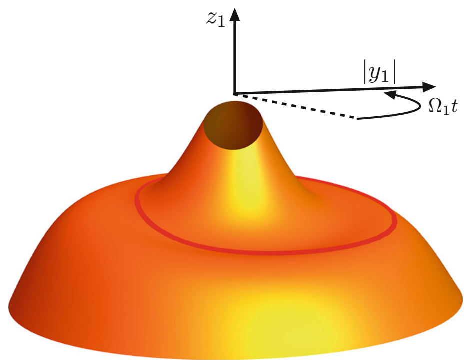



**Fig. 13** Projection of the SSM and NNM for the periodically forced Shaw–Pierre example from the extended phase space of $(y, z, \Omega_1 t \bmod 2\pi)$. The forcing parameter is $\epsilon = 0.1$. (see the related online supplemental movie for animation)

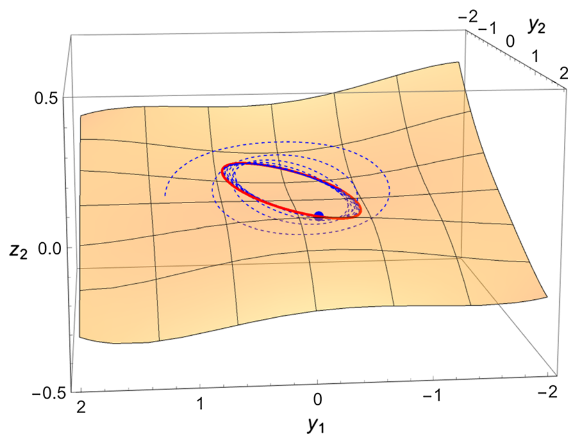



**Fig. 14** Instantaneous projection of the analytically computed periodic NNM and slow SSM for the periodically forced Shaw–Pierre example from the $(y_1, y_2, z_1, z_2)$ phase space for $\epsilon = 0.1$. Shown is an instantaneous position of the SSM surface along with the history of a trajectory (*blue*) launched from the SSM at an earlier time. Note that the trajectory has converged to the analytically computed approximation to the NNM (*red*). We find the mean squared error between the independently computed NNM and its projection onto the SSM to be $\mathcal{O}(\epsilon^3)$ over one time period (see the related online supplemental movie for animation). (Color figure online)

and the non-autonomous settings. Specifically, by Theorem 1, the unique NNM $x_\epsilon$ in the autonomous case $(k = 0)$ depends on $\epsilon$ in a $C^r$ fashion and hence can be approximated in the form of a Taylor series

$$x_\epsilon = \epsilon\tau(\epsilon) = \sum_{l=1}^{r} \xi_l \epsilon^l + o\left(\epsilon^r\right),$$

with the vector $\xi_l \in \mathbb{R}^N$ denoting the $l$th order Taylor coefficient of the function $x_\epsilon$.

In the non-autonomous case $(k > 0)$, Theorem 2 guarantees a unique NNM, $x_\epsilon(t) = \epsilon\tau(\Omega_1 t, \ldots, \Omega_k t; \epsilon)$ in system (5) that depends on $\epsilon$ in a $C^r$ fashion. Thus, $x_\epsilon(t)$ can be approximated in the form of a Taylor–Fourier series

$$\begin{aligned} x_\epsilon(t) &= \epsilon\tau(\Omega_1 t, \ldots, \Omega_k t; \epsilon) \\ &= \sum_{l=1}^{r} \epsilon^l \xi_l(\Omega_1 t, \ldots, \Omega_k t) + o\left(\epsilon^r\right) = \sum_{l=1}^{r} \sum_{|m|=1}^{\infty} \epsilon^l \\ &\quad \times \left[ A_m^l \sin(\langle m, \Omega\rangle\, t) + B_m^l \cos(\langle m, \Omega\rangle\, t) \right] + o\left(\epsilon^r\right), \end{aligned}$$

with the vectors $A_m^l, B_m^l \in \mathbb{R}^N$ denoting the multi-frequency Fourier coefficients of the function $x_\epsilon(t)$ corresponding to the multi-index $m = (m_1, \ldots, m_k) \in \mathbb{N}^k$.

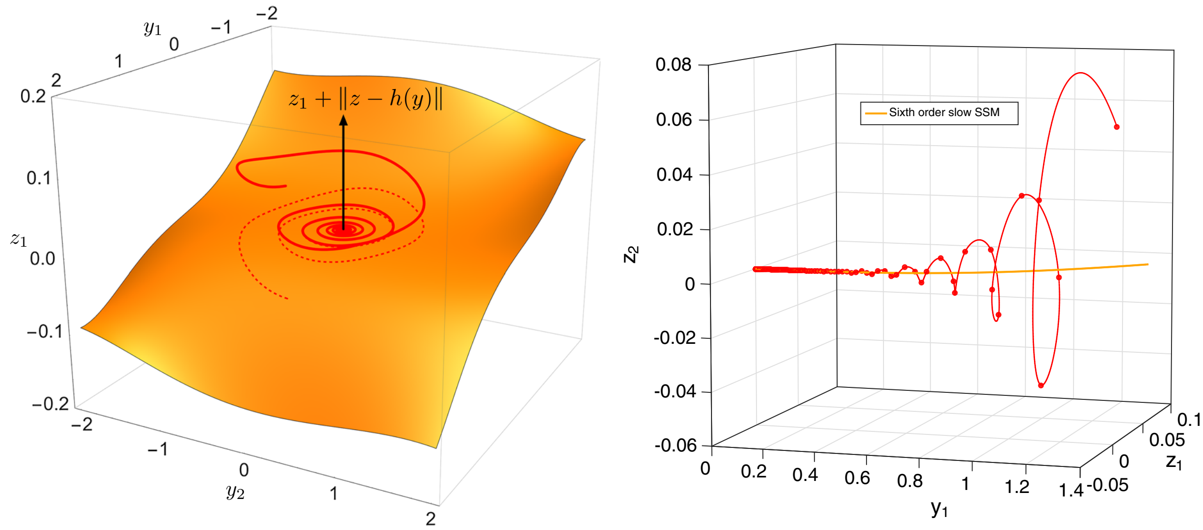
## 8.2 Expansions for slow SSMs

Theorems 3 and 4 provide a theoretical underpinning for the construction of reduced-order models over appropriately chosen spectral subspaces of the linearized system. Specifically, approximations to the flow on an SSM may simplify the study of long-term system dynamics.

Of highest relevance to such model reduction are slow SSMs. Since all linearized solutions decay to an NNM in our setting, slow SSMs contain the trajectories that resist this trend as much as possible and remain active for the longest time. These SSMs can be constructed under the conditions spelled out in the last columns of Tables 1 and 2.

To approximate uniquely a slow SSM, we need to use a Taylor expansion of at least order $\sigma(E) + 1$ or $\Sigma(E) + 1$, respectively. This order depends solely on the damping rates associated with the fastest and slowest decaying modes. Even if the real part of the whole spectrum of $A$ is close to zero, $\sigma(E)$ and $\Sigma(E)$ may well be large, as seen in the mechanical systems considered in Examples 3 and 6.

*Example 7* (Illustration of model reduction on a mechanical example) Here we illustrate the relevance of slow SSMs in model reduction for the unforced oscillator system in Example 3. Figure 15a, b shows different visualization of the fast convergence of a generic trajectory first to the slow SSM and then to the stable equilibrium along the SSM.




**(a)** **(b)**

**Fig. 15** **a** Fast convergence of a generic trajectory to the slow SSM and then subsequently to the equilibrium point along the SSM. Initial conditions for the trajectory were chosen off the SSM with the coordinates $y_1(0) = 1.2, y_2(0) = 0, z_1(y_1(0), y_2(0)) + \Delta z_1 = -0.042 + 0.1, z_2(y_1(0), y_2(0)) + \Delta z_2 = -0.0045 + 0.1$). The *vertical axis* in the figure represents the difference between $(z_1, z_2)$ and $h(y_1, y_2)$, which decays in time due to attraction to the SSM. **b** A different view on the same convergences shown by the Poincaré map already used in Fig. 9, with the damping now decreased to $c = 0.03$ to increase the number of intersection with the Poincaré section for clarity

## 8.3 The optimal dimension of the slow SSM

The integer $q$ in the choice of the slow spectral subspace $E_{1,\ldots,q}$ is a free parameter. This integer is best selected in a way so that the resulting slow SSM, $W_{1,\ldots,q}(x_\epsilon)$, is the most prevalent low-dimensional attractor containing the underlying NNM $x_\epsilon(t)$ described in Theorems 1 and 2.

Generally, can can construct a nested hierarchy of such prevalent slow manifolds. At any step in this hierarchy, the remaining slow spectrum can further be divided along the next largest gap in the real part of the eigenvalues $\lambda_j$ of the linearized system (6). Dividing the spectrum along this spectral gap provides the most readily observable decay rate separation for the trajectories inside of, and toward, the slow SSM. Defining the index sequence $q_j$ as

$$q_1 = \arg \max_{j\in[1,N-1]} \left|\mathrm{Re}\lambda_{j+1} - \mathrm{Re}\lambda_j\right|,$$
$$q_2 = \arg \max_{j\in[1,q_1-1]} \left|\mathrm{Re}\lambda_{j+1} - \mathrm{Re}\lambda_j\right|,$$
$$\vdots \quad \vdots \quad \vdots$$
$$q_l = \arg \max_{j\in[1,q_k-1]} \left|\mathrm{Re}\lambda_{j+1} - \mathrm{Re}\lambda_j\right|,$$
$$\vdots \quad \vdots \quad \vdots$$
$$q_w = 1,$$

gives the nested sequence

$$E_{1,\ldots,q_1} \supset E_{1,\ldots,q_2} \supset \ldots \supset E_{1,\ldots,q_l} \supset \ldots \supset E_1$$

of $w$ spectral subspaces. If the appropriate nonresonance conditions of Table 1 or 2 are satisfied for each element of this nested sequence, then a nested sequence of $w$ slow SSMs exists, asymptotic to an NNM of the full nonlinear system. In the autonomous case, this nested sequence of slow SSMs is

$$W_{1,\ldots,q_1}(0) \supset W_{1,\ldots,q_2}(0) \supset \ldots \supset W_{1,\ldots,q_l}(0) \supset \ldots \supset W_1(0), \quad (68)$$

while in the non-autonomous case, we have

$$W_{1,\ldots,q_1}(x_\epsilon(t)) \supset W_{1,\ldots,q_2}(x_\epsilon(t)) \supset \ldots \supset W_{1,\ldots,q_l}(x_\epsilon(t)) \supset \ldots \supset W_1(x_\epsilon(t)). \quad (69)$$

In the autonomous case, therefore, the minimal slow SSM is $W_1(0)$, tangent to the slowest eigenspace $E_1$

at $x = 0$ with $\dim W_1(0) = \dim E_1$. In the non-autonomous case, the minimal slow SSM is $W_1(x_\epsilon(t))$ which is $\mathcal{O}(\epsilon)$ $C^r$-close to $\{x_\epsilon\} \times E_1$ in the $x$ variable.

Reducing the full dynamical system (5) to the minimal slow SSM brings the largest reduction in the number of dimensions: The dimension of the reduced model obtained in this fashion is equal to the algebraic multiplicity of the eigenvalue $\lambda_1$ that lies closest to zero. If this eigenvalue is simple and complex, then the dimension of the reduced system on the slowest SSM is two. If the eigenvalue is simple and real, then this reduced dimension is one.

Reducing the dynamic to the minimal (slowest) SSM, however, only captures the correct system dynamics over very long time scales in case the spectral gap between $\text{Re}\lambda_1$ and $\text{Re}\lambda_2$ is small. This is because in that case, solution components decaying transverse the slowest SSM may take a long time to die out. More generally, the optimal choice of the SSM in the nested sequences (68)–(69) depends on the time scale over which the approximation of the reduced flow on the SSM is to be used as a model for the behavior of the full system. In the absence of a definitive target time scale, a reasonable choice is $W_{1,\ldots,q_1}(0)$ or $W_{1,\ldots,q_1}(x_\epsilon(t))$, i.e., the slow SSM corresponding to the largest gap in the real part of the spectrum of $A$.

## 8.4 Implications for the computation of NNMs and slow SSMs

Theorems 3 and 4 provide a mathematical foundation for a systematic computation of slow SSMs. Without going into technical details, we briefly mention the main computational implications that follow from the application of these theorems.

### *8.4.1 Local Taylor–Fourier expansion for slow SSMs*

In our terminology, all slow SSMs are unique and anchored to a unique NNM, which may be trivial (a fixed point), periodic (a closed orbit) or quasiperiodic (an invariant torus). The most common nonlinearities used in mechanical modeling are analytic functions, i.e., have everywhere convergent Taylor-series expansion in terms of the $x$ and $\epsilon$ variables. To this end, we will assume here that the right-hand side of the dynamical system (5) is analytic near the origin in all its arguments, i.e.,

$$f_0, f_1 \in C^a.$$

Theorems 3 and 4 then guarantee that under appropriate low-order nonresonance conditions, the slow SSMs of the system also admit convergent Taylor expansions about the NNMs they are anchored to.

Consider a spectral subspace $E_{1,\ldots q}$ with $u := \dim E_{1,\ldots,q}$, satisfying the nonresonance conditions of Table 1. After a linear change of coordinates, the variable $x$ can be split as

$$x = (y, z) \in E_{1,\ldots q} \times E_{q+1,\ldots,N}.$$

In these coordinates, system (5) takes the form

$$\begin{aligned} \dot{y} &= A_y y + f_{0y}(y, z) + \epsilon f_{1y}(y, z, \Omega t; \epsilon), \\ \dot{z} &= A_z z + f_{0z}(y, z) + \epsilon f_{1z}(y, z, \Omega t; \epsilon), \end{aligned} \tag{70}$$

with the constant matrices

$$A_y \in \mathbb{R}^{u\times u}, \quad A_z \in \mathbb{R}^{(N-u)\times(N-u)},$$

and with appropriate $C^r$ functions $f_{0y}$, $f_{0z}$, $f_{1y}$ and $f_{1z}$.

In the autonomous case, the unique slow SSM $W_{1,\ldots,q}(0)$ can then locally be written in the form of a convergent Taylor series

$$z = h^0(y) = \sum_{|p|=1}^{\infty} h_p^0 y^p, \quad p = (p_1, \ldots, p_u),$$
$$y^p := \left(y_1^{p_1}, \ldots, y_u^{p_u}\right), \quad h_p^0 \in \mathbb{R}^{N-u}.$$

By Theorem 3, this expansion can be truncated at an order

$$\sigma(E_{1,\ldots,q}) + 1 = \text{Int}\left[\text{Re}\lambda_N/\text{Re}\lambda_1\right] + 1,$$

as an approximation to the unique slow SSM $W_{1,\ldots,q}(0)$. Lower-order truncations of $h^0(y)$ also approximate a multitude of other invariant manifolds tangent to $E_{1,\ldots q}$.

In the non-autonomous case, the slow SSM, $W_{1,\ldots,q}(x_\epsilon, \Omega t)$, can locally be written in the form of a convergent Fourier–Taylor series

$$\begin{aligned} z = h^\epsilon(y,t) &= h^0(y) + \epsilon h^1(y, \Omega_1 t, \ldots, \Omega_k t; \epsilon) \\ &= \sum_{|p|=1}^{\infty} h_p(t) y^p \\ &= \sum_{|p|=1}^{\infty} h_p^0 y^p + \sum_{l=1}^{\infty}\sum_{|p|=1}^{\infty}\sum_{|m|=1}^{\infty} \epsilon^l y^p \\ &\quad \left[C_{lmp}\sin(\langle m, \Omega\rangle\, t) + D_{lmp}\cos(\langle m, \Omega\rangle\, t)\right], \end{aligned} \tag{71}$$

with $C_{lmp}, D_{lmp} \in \mathbb{R}^{N-u}$. Again, by Theorem 4, the convergent power series $h^0$ and $h^1$ of $y$ can be truncated at an order

$$\Sigma(E_{1,\ldots,q}) + 1 = \text{Int}\,[\text{Re}\lambda_N/\text{Re}\lambda_1] + 1,$$

serving as an approximation to the unique slow SSM, $W_{1,\ldots,q}(x_\epsilon(t))$. Lower-order truncations of the series will also approximate an infinity of other invariant manifolds with similar properties.

For an illustration of these computations in a simple setting, we refer the reader to Example 6. In that example, the Taylor expansion was carried out up to sixth order, and the Fourier expansion in formula (71) was replaced by the direct numerical solution of the boundary-value problems defining the time-periodic Taylor coefficients $h_p(t)$.

### *8.4.2 Local PDEs for slow SSMs*

Once the existence and uniqueness of the slow SSMs in the appropriate function class is clarified from Theorems 3 and 4, we may also write down a PDE for these manifolds using their invariance properties. As mentioned in the Introduction (see also Appendix "Uniqueness issues for invariant manifolds obtained from numerical solutions of PDEs"), such PDEs are solved in the literature without specific concern for the uniqueness of their solution under ill-posed or undetermined boundary conditions.

The relevant lesson from Theorems 3 and 4 is that approximate numerical solutions of these PDE in any set of basis functions should be constructed in a way that the infinitely many less smooth invariant manifolds are excluded from consideration. For instance, in the autonomous case covered by Theorems 3, cost functions penalizing the magnitude of numerically computed derivatives of order $\sigma(E_{1,\ldots,q}) + 1$, or $\Sigma(E_{1,\ldots,q}) + 1$, respectively, could be employed for a defendable approximation to the SSM.

### *8.4.3 Global parametrization of slow SSMs*

Classic invariant manifold techniques (see, e.g., Fenichel [15]) construct the invariant surfaces in question as graphs over an appropriate set of variables. In our present context, this translates to seeking an SSM as a graph of the form $z = h^0(y)$ or $z = h^\epsilon(y, t)$, as assumed in the Taylor–Fourier- and PDE-based approaches discussed above. Both of these approaches are local in nature, capturing only a subset of the SSM that can be viewed as a graph over the underlying $E_{1,\ldots,q}$ spectral subspace. The construction of the SSM, therefore, breaks down once the SSM develops a fold over $E_{1,\ldots,q}$, i.e., becomes a multi-valued graph over $E_{1,\ldots,q}$ (cf. Fig. 16)

The proofs of the results underlying Theorems 3–4, however, do not assume such a graph property. Rather, they construct the SSM by the parametrization method pioneered by Cabré et al. [8]. This method renders the SSMs as an embedding of $E_{1,\ldots,q}$ into the phase space $\mathbb{R}^N$, rather than a graph over the subspace $E_{1,\ldots,q}$ of $\mathbb{R}^N$. Moreover, the flow on the SSM is exactly conjugated to a polynomial function of a parametrization of $E_{1,\ldots,q}$. The order of this polynomial is no larger than $K = \Sigma(E_{1,\ldots,q})$.

More specifically, with the notation $X(x, \phi) = f_0(x) + \epsilon f_1(x, \phi, \epsilon)$, our dynamical system (5) and its associated flow map $F^t(x, \phi)\colon \mathbb{R}^N \times \mathbb{T}^k \to \mathbb{R}^N$ can be written as

$$\dot{x} = X(x, \phi), \quad \dot{\phi} = \Omega,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} F^t(x, \phi) = X\left(F^t(x, \phi), \phi + \Omega t\right), \quad F^0(x, \phi) = x.$$

An SSM can then be sought as the image of $E_{1,\ldots,q}$ under an embedding

$$W : E_{1,\ldots,q} \times \mathbb{T}^k \to \mathbb{R}^N,$$
$$(\eta, \phi) \mapsto x,$$

such that the reduced model flow on $E_{1,\ldots,q}$ has the associated differential equation and flow map

$$\dot{\eta} = \Lambda(\eta, \phi), \quad \dot{\phi} = \Omega,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} G^t(\eta, \phi) = \Lambda(G^t(\eta, \phi), \phi + \Omega t), \quad G^0(\eta, \phi) = \eta. \tag{72}$$

Our model flow is defined over all of the spectral subspace $E_{1,\ldots,q}$. We may seek this model flow map in the form of a Fourier–Taylor expansion

$$G^t(\eta, \phi) = \sum_{|j|=1}^{K} g_j(\phi, t)\eta^j,$$

which, substituted into (72), gives

$$\Lambda\left(\sum_{|j|=1}^{K} g_j(\phi, t)\eta^j, \phi + \Omega t\right)$$
$$= \sum_{|j|=1}^{K} \left[D_\phi g_j(\phi, t)\Omega + D_t g_j(\phi, t)\right]\eta^j$$
$$+ g_j(\phi, t)\Lambda(\eta, \phi).$$

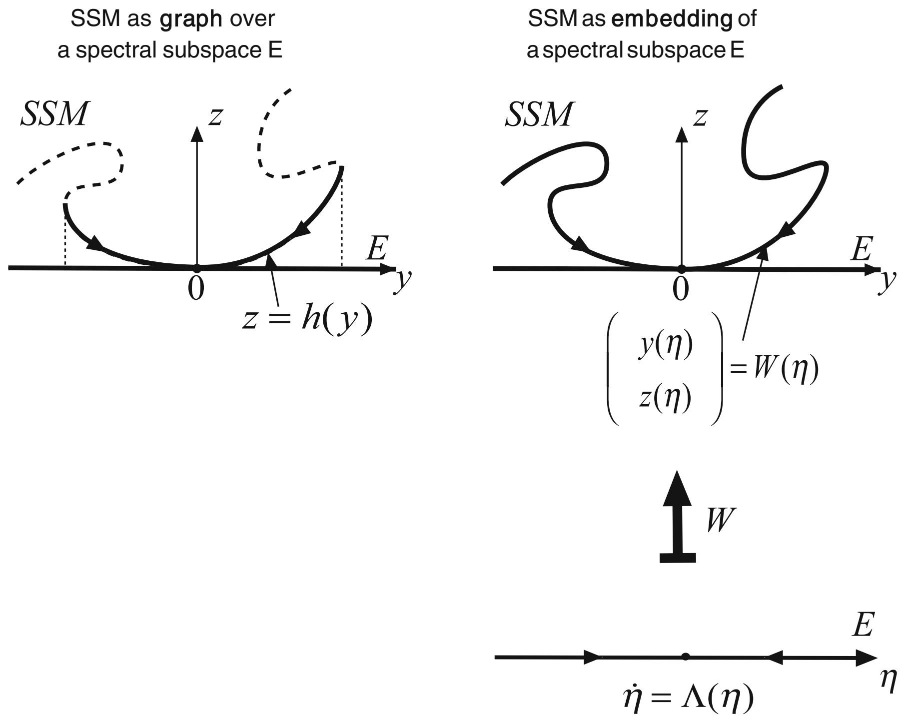



**Fig. 16** An illustration of the idea of the parametrization method for autonomous systems (no dependence on $\phi$): constructing an SSM as a graph over a spectral subspace $E$ versus as an embedding of the spectral subspace $E$

The invariance of the SSM can then be expressed by the equation

$$F^t\left(W(\eta,\phi),\phi\right) = W\left(G^t(\eta,\phi),\phi+\Omega t\right).$$

Differentiating this equation in time and setting $t = 0$ yields the infinitesimal invariance condition

$$X\left(W(\eta,\phi),\phi\right) = D_\eta W\left(\eta,\phi\right)\Lambda(\eta,\phi) + D_\phi W(\eta,\phi)\Omega. \tag{73}$$

Substituting the analytic Taylor–Fourier expansions

$$W\left(\eta,\phi\right) = \sum_{l=1}^{\infty}\sum_{|p|=1}^{\infty}\sum_{|m|=1}^{\infty}\epsilon^l\eta^p\left[E_{lmp}\sin(\langle m,\Omega\rangle t) + F_{lmp}\cos(\langle m,\Omega\rangle t)\right],$$

$$\Lambda(\eta,\phi) = \sum_{l=1}^{\infty}\sum_{|p|=1}^{\infty}\sum_{|m|=1}^{\infty}\epsilon^l\eta^p\left[G_{lmp}\sin(\langle m,\Omega\rangle t) + H_{lmp}\cos(\langle m,\Omega\rangle t)\right],$$

into the invariance condition (73), one can recursively solve for the coefficients of the embedding $W(\eta,\phi)$ of the SSMs together with the coefficients of the right-hand side $\Lambda(\eta,\phi)$ of the differential equation (72), describing the reduced-order dynamics on the slow SSM.

Practical hints on the numerical implementation of the above parametrization method are described by Haro et al. [19] and Mireles–James [28]. As mentioned in the Introduction, Cirillo et al. [10] have recently suggested a computational technique for a two-dimensional autonomous SSM that is identical to the parametrization method in their setting.

## 9 Conclusions

We have proposed a unified terminology in the nonlinear modal analysis of dissipative systems, deriving rigorous existence, uniqueness, smoothness and robustness results for the nonlinear normal modes (NNMs) and their spectral submanifolds (SSMs) covered by this terminology.

The NNMs defined here generalize the original nonlinear normal mode concept of Rosenberg to dissipative yet eternally recurrent motions with finitely many frequencies, including fixed points, periodic motions and quasiperiodic motions. In contrast, the SSMs introduced here are the smoothest invariant manifolds asymptotic to such generalized NNMs along their spectral subbundles. As such, SSMs build on the Shaw–Pierre normal mode concept and clarify its relationship with Rosenberg's concept in a general dissipative, multi-degree-of-freedom system, possibly subject to time-periodic or quasiperiodic forcing.

In our setting, NNMs are locally unique in the phase space, admitting a unique SSM over any of their spectral subspaces (or subbundles) that have no low-order resonances with the remaining part of the linearized spectrum. In the autonomous case, the order of these nonresonance conditions is fully governed by the relative spectral quotient $\sigma(E)$ of the spectral subspace of interest. In the non-autonomous case, the role of $\sigma(E)$ is taken over by the absolute spectral quotient $\Sigma(E)$. Both of these spectral quotients can be a priori determined from the spectrum of the linearized system (see Tables 1, 2).

Our results cover three classes of SSMs: fast, intermediate and slow. Out of these classes, fast SSMs have unrestricted uniqueness among all differentiable invariant surfaces in the autonomous case, but are generally the least relevant for model reduction. In contrast, slow SSMs are the most relevant to model reduction, but have the most restricted uniqueness properties. Namely, the minimal order of a Taylor expansion distinguishing any slow SSM from other invariant manifolds is the smallest integer that is larger than the ratio of the strongest and the weakest decay rate of the linearized system. This spectral ratio may well be large even for weakly damped systems; thus, a careful consideration of damping is essential for rigorous SSM-based model reduction approaches.

Our results are meant to aid the construction of formal expansions and intuitive computations of NNMs and SSMs. As we discussed, most of these operational approaches tend to hide the fundamental non-uniqueness of invariant manifolds tangent to modal subspaces. The ambiguity in the results is inherently small close to the underlying fixed point but is magnified significantly away from fixed points (see, e.g., Fig. 5a) and becomes an obstacle to extending invariant manifolds in a defendable fashion to larger domains of the phase space. The use of SSMs eliminates this ambiguity and should therefore be useful in expanding the range of nonlinear modal analysis in a well-understood fashion.

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## Appendix: Existence, uniqueness and analyticity issues for invariant manifolds tangent to eigenspaces

### Modified Euler example of a non-analytic but $C^\infty$ center manifold

For the system (1), the origin is a fixed point with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -1$ and corresponding eigenvectors $e_1 = (1, 1)$ and $e_2 = (0, 1)$. Therefore, the classic center manifold theorem (see, e.g., Guckenheimer and Holmes [16]) guarantees the existence of a center manifold $W^c(0)$, tangent to the $x$-axis at the origin. We seek $W^c(0)$ in the form of a Taylor expansion

$$y = h(x) = x + \sum_{j=2}^{\infty} a_j x^j,$$

which we differentiate in time to obtain

$$\dot{y} = \left(1 + \sum_{j=2}^{\infty} j a_j x^{j-1}\right) \dot{x} = -\left(1 + \sum_{j=2}^{\infty} j a_j x^{j-1}\right) x^2 = -x^2 - \sum_{j=2}^{\infty} j a_j x^{j+1} = -\sum_{j=2}^{\infty} (j-1) a_{j-1} x^j, \tag{74}$$

where we have let $a_1 = 1$. At the same, we evaluate the second equation in (1) on the manifold $W^c(0)$ to obtain

$$\dot{y} = -h(x) + x = -\sum_{j=2}^{\infty} a_j x^j. \tag{75}$$

Equating (74) and (75) gives the recursion $a_j = (j-1)a_{j-1}$ with $a_1 = 1$, which implies $a_j = (j-1)!$. We therefore obtain the explicit form

$$h(x) = \sum_{j=1}^{\infty} (j-1)! x^j \tag{76}$$

tas a formal expansion of the center manifold, as stated in the Introduction. The formal series $h(x) = \sum_{j=1}^{\infty}(j-1)!x^j$, however, diverges for any $x \neq 0$; thus, the center manifold is $C^\infty$ but not analytic in any open neighborhood of the origin.

Uniqueness and analyticity issues for invariant manifolds in linear systems

Any invariant manifold through the origin of the linearized system (12) is locally a graph over $q$ of the elements of the vector $y$. Such a graph is of the general form

$$y_l = f_l(y_{j_1}, \ldots, y_{j_q}), \quad l \notin \{j_1, \ldots, j_q\}. \tag{77}$$

By the invariance of these surfaces, one can substitute full trajectories into (77) and differentiate in time to obtain the PDE

$$\lambda_l f_l = \sum_{i=1}^{q} \lambda_{j_i} y_{j_i} \partial_{y_{j_i}} f_l, \quad l \notin \{j_1, \ldots, j_q\}. \tag{78}$$

This linear PDE can be solved locally by the method of characteristics (see, e.g., Evans [14]), once we prescribe the value of $f_l$ along an appropriate codimension-one set $\Gamma(s_1, \ldots, s_{q-1})$ of the spectral subspace $E_{j_1,\ldots,j_q}$. Here the real variables $s = (s_1, \ldots, s_{q-1})$ parametrize the surface $\Gamma$. For instance, $\Gamma$ can be selected as a $q-1$ dimensional sphere in $E_{j_1,\ldots,j_q}$ that surrounds the origin.

Fixing a boundary condition

$$f_l(\Gamma(s_1, \ldots, s_{q-1})) = f_l^0(s_1, \ldots, s_{q-1}) \tag{79}$$

gives the equation for characteristics:

$$y_{j_i}(t) = \Gamma_i(s_1, \ldots, s_{q-1})e^{\lambda_{j_i} t}, \quad i = 1, \ldots, q. \tag{80}$$

$$f_l(y_{j_1}(t), \ldots, y_{j_q}(t)) = f_l^0(s_1, \ldots, s_{q-1})e^{\lambda_p t}. \tag{81}$$

Then, the strategy to obtain a solution for the PDE (78) is the following: express the variables $(s_1, \ldots, s_{q-1}, t)$ as a function of $(y_{j_1}, \ldots, y_{j_q}) = (y_{j_1}(t), \ldots, y_{j_q}(t))$ from the $q$ algebraic equations (80) in the vicinity of $\Gamma$ and substitute the result into (81) to obtain a solution $f_l(y_{j_1}, \ldots, y_{j_q})$ to (78) that satisfies the boundary condition (79).

To this end, we rewrite (80) as

$$\Gamma_i(s_1, \ldots, s_{q-1})e^{\lambda_{j_i} t} - y_{j_i} = 0, \quad i = 1, \ldots, q, \tag{82}$$

and observe that this system of $q$ algebraic equations is solved by $t = 0$ and $y_{j_i}^0 = y_{j_i}(0) = \Gamma_i(s_1, \ldots, s_{q-1})$. By the implicit function theorem, the variables $(s_1, \ldots, s_{q-1}, t)$ can be expressed from (82) near $\Gamma$ as a function of $y_{j_i}$ if the Jacobian

$$D_{s_1,\ldots,s_{q-1},t} \begin{bmatrix} \Gamma_1(s_1, \ldots, s_{q-1})e^{\lambda_{j_1} t} - y_{j_1} \\ \vdots \\ \Gamma_q(s_1, \ldots, s_{q-1})e^{\lambda_{j_q} t} - y_{j_q} \end{bmatrix}_{(y_{j_i}=y_{j_i}^0, t=0)} = \left[ D_s\Gamma, \ -\Lambda y|_{E_{j_1,\ldots,j_q}} \right], \tag{83}$$

is non-degenerate. In other words, along the surface $\Gamma$, all tangent vectors of $\Gamma$ should be linearly independent of the vector field $\Lambda y$ restricted to its invariant subspace $E_{j_1,\ldots,j_q}$. In the language of linear PDEs, the boundary surface $\Gamma$ should be a *non-characteristic surface* for a unique, local solution to exist near $\Gamma$ for any boundary condition posed over $\Gamma$. This argument just reproduces the classic local existence and uniqueness result for linear first-order PDEs (see, e.g., Evans [14]).

Under these conditions, therefore, we have a unique, local solution for any initial function $f_l^0(s_1, \ldots, s_{q-1})$ defined on $\Gamma$. There are infinitely many different choices both for the surface $\Gamma$ and the boundary values $f_l^0$. Since the Jacobian (83) is non-degenerate for any $y \neq 0$, each of these infinitely many choices leads to a local invariant surface satisfying (78) in the vicinity of $\Gamma$, which in turn can be propagated all the day to the $y = 0$ fixed point along characteristics of the PDE. Accordingly, we obtain *infinitely many* invariant surfaces tangent to the spectral subspace $E_{j_1,\ldots,j_q}$ in the linearized system (12). Applying the more general Theorem 3 in the current linear setting, however, we obtain that only one analytic solution exists to the PDE (6) for any fixed subspace $E_{j_1,\ldots,j_q}$ under the nonresonance conditions detailed in Theorem 3. Since $f_l(y_{j_1}, \ldots, y_{j_q}) \equiv 0$ is analytic, this flat solution must be the unique analytic solution of (78). All other solutions are only finitely many times differentiable and hence are not even $C^\infty$.

Uniqueness issues for invariant manifolds obtained from numerical solutions of PDEs

The PDE approach we described in Sect. 1 is broadly used in the literature to compute Shaw–Pierre-type invariant surfaces for nonlinear systems. This approach was originally suggested by Shaw and Pierre [39], explored first in detail first by Peschek et al. [32], then developed and applied further by various authors (see Renson et al. [37] for a recent review). Interestingly, none of these studies report or discuss non-uniqueness of solutions, which appears to be in contradiction with our conclusions in Sect. 1. Here we take a closer look to understand the reason behind this paradox.

In the simplified setting of Sect. 1, one may seek invariant manifolds of the form $y_l = f_l(y_{j_1}, \dots, y_{j_q})$, $l \notin \{j_1, \dots, j_q\}$ in a nonlinear system

$$\dot{y} = \Lambda y + g(y), \quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_N), \\ g(y) = \mathcal{O}\left(|y|^2\right), \tag{84}$$

over a spectral subspace $E_{j_1,\dots,j_q}$ of the operator $A$. The same argument we used in the linear case now leads to a quasilinear version of the linear system of PDEs (78). This quasilinear system of PDEs is of the form

$$\lambda_l f_l + g_l(y_j, f) = \sum_{i=1}^{q} \left[\lambda_{j_i} y_{j_i} + g_{j_i}(y_j, f)\right] \partial_{y_{j_i}} f_l, \\ l \notin \{j_1, \dots, j_q\}, \tag{85}$$

with $y_j = (y_{j_1}, \dots, y_{j_q})$ and $f$ denoting the vector of the $f_l$ functions.

The local existence and uniqueness theory relevant to this PDE is identical to that for its linear counterpart (cf. Evans [14]). Specifically, as in Sect. 1, boundary conditions

$$f_l(\Gamma(s_1, \dots, s_{q-1})) = f_l^0(s_1, \dots, s_{q-1}), \tag{86}$$

must be posed on a non-characteristic, codimension-one boundary surface $\Gamma$ inside the subspace $E_{j_1,\dots,j_q}$ for the PDE (85) to have a unique local solution near $\Gamma$. Here the required non-characteristic property of $\Gamma$ is that the projected vector field $\dot{y}_j = [\Lambda y + g(y)]_j$ over $E_{j_1,\dots,j_q}$ should be transverse to $\Gamma$ at all points. Since this boundary condition is arbitrary, one again obtains infinitely many local Shaw–Pierre-type invariant manifolds near the boundary surface $\Gamma$ for the nonlinear problem (84): one for any boundary condition posed over any non-characteristic surface $\Gamma$. In the general case, all of these are also global solutions that extend smoothly to the origin and give a smooth solution to the PDE (85) in a whole neighborhood of the fixed point. The only exception is when the invariant manifold is sought as a graph over the $q$ fastest modes. In this case, the strong stable manifold theorem (Hirsch et al. [20]) guarantees the existence of a unique invariant manifold. In this case, while infinitely many local solutions still exist near a non-characteristic boundary surface $\Gamma$, these local solutions do not extend smoothly to the origin.

Surprisingly, all available numerical algorithms aiming to solve (85) in the nonlinear normal modes literature ignore this non-uniqueness issue. They are typically validated or illustrated on the computation of two-dimensional invariant manifolds tangent to the single, slowest decaying spectral subspace ($q = 1$, $\dim E_1 = 2$). Already in this simplest case, the high degree of non-uniqueness illustrated in Fig. 2 definitely applies. This raises the question: How do these studies obtain a unique invariant manifold? There are different reasons for each numerical algorithm, as we review next.

Peschek et al. [32] consider a spectral subspace $E_1$ corresponding to a simple, complex conjugate pair of eigenvalues. They pass to amplitude-phase variables $(a, \varphi)$ by letting $y_{j_1} = ae^{i\varphi}$ and reconsider the quasilinear PDE (85) posed for the unknown functions $f_l(a, \varphi)$. As domain boundary $\Gamma$, they then consider the $a = 0$ axis, over which they prescribe $f_l(0, \varphi) = 0$ and $\partial_a f_l(0, \varphi) = 0$. This is consistent with the fact that the origin $y_{j_1} = 0$ is mapped, due to the singularity of the polar coordinate change, to the $a = 0$ of the $(a, \phi)$ coordinate space, and hence, the surface should have a quadratic tangency with this line. However, the $a = 0$ line is invariant under the transformed nonlinear vector field $(\dot{a}, \dot{\varphi})$, given that it is the image of the fixed point of the original nonlinear system, which satisfies $\dot{a} = 0$. As a consequence, $\Gamma$ is a characteristic surface, and hence, local existence and uniqueness are not guaranteed for the quasilinear PDE (85) with this boundary condition. As we discussed above, the PDE is in fact known to have infinitely many solutions, all of which have a quadratic tangency with the origin and hence satisfy the singular boundary conditions $f_l(0, \phi) = 0$ and $\partial_a f_l(0, \phi) = 0$ in polar coordinates. Therefore, the problem considered by Peschek et al. [32] only has a unique solution for invariant manifolds over the fast modes, but not over the slow or intermediate modes. The same holds true for all other studies utilizing the approach developed by Peschek et al. [32].

Renson et al. [36] solve the same quasilinear PDE (85) in the setting of Peschek et al. [32] (autonomous system with $q = 1$ and with $\dim E_1 = 2$). In the conservative case, they seek to construct solutions using a closed boundary curve $\Gamma$ to which the nonlinear vector field $\dot{y}_j$ is tangent at each point. For damped systems, they solve the PDE outward from the equilibrium, first over an elliptic domain and then gradually outward over a nested sequence of annuli. The boundaries of all these domains are selected as non-characteristic curves; thus, a unique solution can be constructed over each domain in the nested sequence. Over the initial (elliptic) domain boundary, however, the spectral subspace itself is chosen as initial condition ($f_l^0(\Gamma) = 0$ for all $l > 2$), which singles out one special solution out of the arbitrarily many. The perceived uniqueness is, therefore, the artifact of the numerical procedure.

Finally, Blanc et al. [6] start out by correctly selecting a non-characteristic boundary curve $\Gamma$ in the amplitude–phase–coordinate setting of Peschek et al. [32] discussed above. This curve is just the $\varphi = 0$ line of the $(a, \varphi)$ coordinate plane, to which the characteristics of the PDE are transverse in a neighborhood of the origin, as required for the local existence and uniqueness of solutions near $\Gamma$. In this case, any initial profile $f_l(a, 0) = f_l^0(a)$ with $f_l^0(0) = 0$ and $f_l^{0\prime}(0) = 0$ would lead to a Shaw–Pierre-type invariant manifold, thereby revealing the inherent non-uniqueness of this numerical approach. Instead of realizing this, Blanc et al. [6] assert that there is a single correct boundary condition that they need to find by an optimization process.

In this optimization process, Blanc et al. [6] modify the initial boundary condition iteratively so that the computed PDE solution along the line $\varphi = 2\pi$, given by $f_l(a, 2\pi)$, is as close to $f_l(a, 0) = f_l^0(a)$ as possible in the $L^2$ norm. Should they enforce the exact periodicity of the solution of the PDE on the periodic domain $(a, \varphi) \in [0, a_{max}] \times [0, 2\pi]$ (say, by a spectral method), they would always have $f_l(a, \varphi) \equiv f_l(a, 0)$ on any solution, so minimizing the error in this identity would lead to a vacuous process. In other words, the seemingly unique solution in this approach is the surface along which the error arising from an inaccurate handling of the periodic boundary conditions is minimal in a particular norm.

### Existence, uniqueness and persistence of non-autonomous NNMs

We rewrite system (5) in the form of a $(N + k)$-dimensional autonomous system

$$\dot{x} = Ax + f_0(x) + \epsilon f_1(x, \phi; \epsilon),$$
$$\dot{\phi} = \Omega, \qquad (87)$$

defined on the phase space $\mathcal{P} = \mathcal{U} \times \mathbb{T}^k$. For $\epsilon = 0$, the trivial normal mode $x = 0$ now appears as an invariant, $k$-dimensional torus

$$T_0 = \left\{ (x, \phi) \in \mathcal{P} \; : \; x = 0, \;\; \phi \in \mathbb{T}^k \right\}$$

for system (87).

Assume that all eigenvalues of $A$ satisfy the condition $\mathrm{Re}\lambda_i \neq 0$. This means that all possible exponential contraction and expansion rates transverse to $T_0$ dominate (the zero) expansion and contraction rates in directions tangent to $T_0$, along the $\phi$ coordinates. In the language of the theory of normally hyperbolic invariant manifolds, the torus $T_0$ is a compact, $r$-normally hyperbolic invariant manifold for any integer $r \geq 1$ (Fenichel [15]).

Fenichel's general result on invariant manifolds does not allow, however, to conclude the persistence of $C^0$, $C^\infty$ or $C^a$ normally hyperbolic invariant manifolds. Instead, such persistence is established by Haro and de la Llave [18], who specifically study persistence of invariant tori in systems of the form of (87).

### Existence, uniqueness and persistence for autonomous SSMs ($k = 0$)

First, we recall a more abstract result of Cabré et al. [8] on mappings in Banach spaces, which we subsequently apply to our setting.

#### Spectral submanifolds for mappings on complex Banach spaces

We denote by $\mathcal{P}$ a real or complex Banach space and by $\mathcal{U} \subset \mathcal{P}$ an open set. We let $C^r(\mathcal{U}, Y)$ denote the set of functions $f : U \to Y$ that have continuous and bounded derivatives up to order $r$ in $\mathcal{U}$. Let the space $C^\infty(\mathcal{U}, Y)$ denote the set of those functions $f$ that are in the class $C^r(\mathcal{U}, Y)$ for every $r \in \mathbb{N}$, and let $C^a(\mathcal{U}, Y)$ denote the set of functions $f$ that are bounded and analytic in $U$.

Let $0 \in \mathcal{U}$ be a fixed point for a $C^r$ map $\mathcal{F} \colon \mathcal{U} \to \mathcal{P}$, where $r \in \mathbb{N} \cup \{\infty, a\}$. We denote the linearized map at the fixed point by $\mathcal{A} = D\mathcal{F}(0)$ and its spectrum by $\mathrm{spec}(\mathcal{A})$.

We also assume a direct sum decomposition $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$, with the subspaces $\mathcal{P}_1$ and $\mathcal{P}_2$ to be described shortly in terms of the spectral properties of $\mathcal{A}$. We denote the projections from the full space $\mathcal{P}$ onto these two subspaces by $\pi_1 : \mathcal{P} \to \mathcal{P}_1$ and $\pi_2 : \mathcal{P} \to \mathcal{P}_2$, and assume that both projections are bounded. Finally, for any set $S$ and positive integer $k$, we will use the notation

$$S^k = \underbrace{S \times \ldots \times S}_{k}$$

for the $k$-fold direct product of $S$ with itself.

Assume now that

(0) $\mathcal{A}$ is invertible
(1) The subspace $\mathcal{P}_1$ is invariant under the map $\mathcal{A}$, i.e.,

$$\mathcal{A}\mathcal{P}_1 \subset \mathcal{P}_1.$$

As a result, we have a representation of $\mathcal{A}$ with respect to above decomposition as

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{B} \\ 0 & \mathcal{A}_2 \end{pmatrix}, \tag{88}$$

with the operators $\mathcal{A}_1 = \pi_1 \mathcal{A}|_{\mathcal{P}_1}$, $\mathcal{A}_2 = \pi_2 \mathcal{A}|_{\mathcal{P}_2}$, and $\mathcal{B} = \pi_1 \mathcal{A}|_{\mathcal{P}_2}$. If $\mathcal{P}_2$ is also an invariant subspace for $\mathcal{A}$, then we have $\mathcal{B} = 0$.
(2) The spectrum of $\mathcal{A}_1$ lies strictly inside the complex unit circle, i.e., $\mathrm{Spect}(\mathcal{A}_1) \subset \{z \in \mathbb{C} : |z| < 1\}$.
(3) The spectrum of $\mathcal{A}_2$ does not contain zero, i.e., $0 \notin \mathrm{Spect}(\mathcal{A}_2)$.
(4) For the smallest integer $L \geq 1$ satisfying

$$\left[\mathrm{Spect}(\mathcal{A}_1)\right]^{L+1} \mathrm{Spect}(\mathcal{A}_2^{-1}) \subset \{z \in \mathbb{C} : |z| < 1\}, \tag{89}$$

we have

$$\left[\mathrm{Spect}(\mathcal{A}_1)\right]^{i} \cap \mathrm{Spect}(\mathcal{A}_2) = \emptyset \tag{90}$$

for every integer $i \in [2, L]$ (in case $L \geq 2$).
(5) $L + 1 \leq r$.

We then have the following result:

**Theorem 5** (Theorems 1.1 and 1.2, Cabré, Fontich and de la Llave [8]) *Under assumptions (0–5):*

(i) There exists a $C^r$ manifold $\mathcal{M}_1$ that is invariant under $\mathcal{F}$ and tangent to the subspace $\mathcal{P}_1$ at 0.
(ii) The invariant manifold $\mathcal{M}_1$ is unique among all $C^{L+1}$ invariant manifolds of $\mathcal{F}$ that are tangent to the subspace $\mathcal{P}_1$ at 0. That is, every two $C^{L+1}$ invariant manifolds with this tangency property will coincide in a neighborhood of 0.
(iii) There exists a polynomial map $R : \mathcal{P}_1 \to \mathcal{P}_1$ of degree not larger than $L$ and a $C^r$ map $K : \mathcal{U}_1 \subset \mathcal{P}_1 \to \mathcal{P}$, defined over an open neighborhood $\mathcal{U}_1$ of 0, satisfying

$$R(0) = 0, \quad DR(0) = \mathcal{A}_1, \quad K(0) = 0,$$
$$\pi_1 DK(0) = I, \quad \pi_2 DK(0) = 0,$$

such that $K$ serves as an embedding of $\mathcal{M}_1$ from $\mathcal{P}_1$ to $\mathcal{P}$, and $R$ represents the pull-back of the dynamics on $\mathcal{M}_1$ to $\mathcal{U}_1$ under this embedding. Specifically, we have

$$\mathcal{F} \circ K = K \circ R.$$

(iv) If, furthermore, $\left[\mathrm{Spec}(\mathcal{A}_1)\right]^i \cap \mathrm{Spec}(\mathcal{A}_1) = \emptyset$ holds for every integer $i \in [L_-, L]$, then $R$ can be chosen to be a polynomial of degree not larger than $L_- - 1$.
(v) Dependence on parameters: If $\mathcal{F}$ is jointly $C^r$ in $x$ and a parameter $\mu$, the invariant manifold $\mathcal{M}_1$ is jointly $C^{r-L-1}$ in space and the parameter $\mu$. In particular, $C^\infty$ and analytic maps will have invariant manifolds that are $C^\infty$ and analytic, respectively, with respect to any parameters in the system.

Proof of Theorem 3

We now apply Theorem 5 to system (18). In this context, the space $\mathcal{P}$ is the finite-dimensional, real vector space $\mathcal{P} = \mathbb{R}^N$, and the mapping is the time-one map $\mathcal{F} = F^1 : \mathcal{U} \subset \mathcal{P} \to \mathcal{P}$ of system (18). We further have

$$\mathcal{F}(0) = 0, \quad \mathcal{A} = D\mathcal{F}(0) = DF^1(0) = e^A, \tag{91}$$

and hence $\mathcal{A}$ is invertible. We have the spectra

$$\mathrm{spec}(\mathcal{A}) = \left\{e^{\lambda_1}, e^{\bar{\lambda}_1}, \ldots, e^{\lambda_N}, e^{\bar{\lambda}_N}\right\},$$
$$\mathrm{spec}(\mathcal{A}^{-1}) = \left\{e^{-\lambda_1}, e^{-\bar{\lambda}_1}, \ldots, e^{-\lambda_N}, e^{-\bar{\lambda}_N}\right\}, \tag{92}$$

where we have ordered the eigenvalues in an increasing order based on their real parts, i.e.,

$$\mathrm{Re}\lambda_N \leq \ldots \leq \mathrm{Re}\lambda_1 < 0,$$

and listed purely real elements of the spectrum of $\mathcal{A}$ and $\mathcal{A}^{-1}$ twice to simplify our notation. Equation (92) implies that condition (0) of Theorem 5 is always satisfied.

For a given spectral subspace $E$, we let $\mathcal{P}_1 = E$, so that assumption (1) of Theorem 5 is satisfied. Because the real part of the spectrum of $A$ is assumed to be strictly negative, the operator $\mathcal{A}$ defined in (91) satisfies assumptions (2–3) of Theorem 5.

Next we note that the smallest integer $L$ satisfying

$$\left[\mathrm{Spect}(\mathcal{A}_1)\right]^{L+1}\mathrm{Spect}(\mathcal{A}_2^{-1})\subset\{z\in\mathbb{C}\,:\,|z|<1\},$$

is just the smallest integer that satisfies

$$\left[e^{\max_{\lambda\in\mathrm{Spect}(A|_E)}\mathrm{Re}\lambda}\right]^{L+1}e^{\min_{\lambda\in\mathrm{Spect}(A)-\mathrm{Spect}(A|_E)}\mathrm{Re}\lambda}<1.$$

The solution of this inequality for a general real number $L$ is

$$L>\frac{\min_{\lambda\in\mathrm{Spect}(A)-\mathrm{Spect}(A|_E)}\mathrm{Re}\lambda}{\max_{\lambda\in\mathrm{Spect}(A|_E)}\mathrm{Re}\lambda}-1,$$

which, restricted to integer solutions, becomes

$$L\geq\sigma(E),$$

with the relative spectral quotient $\sigma(E)$ defined in (15). The nonresonance condition (90) can then be written in our setting precisely in the form (20). Thus, under the assumptions of Theorem 3, the conditions of Theorem 5 are satisfied, and the statements of Theorem 3 are restatements of Theorem 5 in our present context.

Comparison with applicable results for normally hyperbolic invariant manifolds

Out of the three types of SSMs covered by Theorem 3, the existence of the slow SSMs (last column in Table 1) can also be deduced in a substantially weaker form from the classical theory of inflowing invariant normally hyperbolic invariant manifolds (Fenichel [15]). To show this, we first rescale variables via $x\to\delta x$ in system (18) to obtain the rescaled autonomous problem

$$\dot{x}=Ax+\delta\tilde{f}_0(x;\delta),\quad\tilde{f}_0(x;\delta):=\frac{1}{\delta^2}f_0(\delta x).\qquad(93)$$

For $\delta=0$, this system coincides with the linearized system (6), while for $\delta>0$, it is equivalent to the full autonomous nonlinear system (18).

Assume now that the slow spectral subspace $E_{1,\ldots,q}$ featured in Table 1 satisfies the strict inequality

$$\mathrm{Re}\lambda_{q+1}<\mathrm{Re}\lambda_q.$$

This implies that $E_{1,\ldots,q}$ is normally hyperbolic, i.e., all decay rates of the linearized system within $E_{1,\ldots,q}$ are weaker than any decay rate transverse to $E_{1,\ldots,q}$. Furthermore, a small compact manifold $\tilde{E}_{1,\ldots,q}\subset E_{1,\ldots,q}$ with boundary can be selected for the unperturbed limit ($\delta=0$) of system (93) such that $\dim\tilde{E}_{1,\ldots,q}=\dim E_{1,\ldots,q}$ and $\tilde{E}_{1,\ldots,q}$ is inflowing invariant under the unperturbed limit of (93). This means that $Ax$ points strictly outward on the boundary $\partial\tilde{E}_{1,\ldots,q}$. Then, for $\delta>0$ small enough, the classic results of Fenichel [15] imply the existence of an invariant manifold $\tilde{W}(0)$ with boundary in system (93) that is $C^1$-close to $\tilde{E}_{1,\ldots,q}$. Furthermore, $\dim\tilde{W}(0)=\dim E_{1,\ldots,q}$ and the manifold $\tilde{W}(0)$ is of class $C^\gamma$, with

$$\gamma=\min\left(r,\,\mathrm{Int}\left[\frac{\mathrm{Re}\lambda_{q+1}}{\mathrm{Re}\lambda_q}\right]\right),\qquad(94)$$

which is the minimum of the degree of smoothness of (93) and the integer part of the ratio of the weakest decay rate normal to $\tilde{E}_{1,\ldots,q}$ to the strongest decay rate inside $\tilde{E}_{1,\ldots,q}$. Since $\delta>0$ has to be selected small in this result to keep the norm $\delta|\tilde{f}(0)|$ small enough, the above conclusion on the existence of $\tilde{W}(0)$ holds in a small enough neighborhood of $x=0$ in system (18).

This result might seem attractive at the first sight, as it requires no nonresonance conditions among the eigenvalues of the operator $A$. At the same time, the properties of $\tilde{W}(0)$ are substantially weaker than those obtained for $W_{1,\ldots,q}(0)$ in Theorem 3. First, the degree $\gamma$ of differentiability for $\tilde{W}(0)$ (cf. formula (94)) is generally much lower than $r$, the degree of smoothness of system (18). In particular, even if (18) is analytic, the manifold $\tilde{W}(0)$ may well just be once continuously differentiable and hence cannot be sought in the form of a convergent Taylor expansion. Second, no uniqueness is guaranteed by the normal hyperbolicity results of Fenichel [15] for $\tilde{W}(0)$ within any class of invariant manifolds. Third, the whole argument is only applicable to slow SSMs, but not to intermediate and fast SSMs.

Comparison with results deducible from analytic linearization theorems

The analytic linearization theorem of Poincaré [34] concerns complex systems of differential equations of the form

$$\dot{y}=\Lambda y+g(y),\quad g(y)=\mathcal{O}\left(|y|^2\right),\qquad(95)$$

where $\Lambda \in \mathbb{C}^{N\times N}$ is diagonalizable and $g(y)$ is analytic. If

1. all eigenvalues of $\Lambda$ lie in the same open half plane in the complex plane (e.g, $\mathrm{Re}\lambda_j < 0$ for all $j$, as in our case), and
2. the nonresonance conditions $\langle m, \lambda\rangle \neq \lambda_j$ hold for all $l = 1, \dots, N$ for all integer vectors $m = (m_1, \dots, m_N)$ with $m_i \geq 0$, and $\sum_i m_i \geq 2$,

then there exists an analytic, invertible change of coordinates $z = h(y)$ in a neighborhood of the origin under which system (95) transforms to the linear system

$$\dot{z} = \Lambda z. \tag{96}$$

The spectral subspaces of this linear system are all defined by analytic functions (trivially, flat graphs over themselves). As we discussed in Sect. 1, the spectral subspaces of nonresonant linear systems are in fact the only analytic invariant manifolds that are graphs over spectral subspaces.

Recall that the composition of two analytic functions is analytic and the inverse of an invertible analytic function is also analytic. We can, therefore, transform back the spectral subspaces of (96) under the analytic inverse mapping $h^{-1}(z)$ to conclude that (95) also has unique analytic SSMs tangent at the origin to any selected spectral subspace of the operator $\Lambda$. (Indeed, if (95) had more than one such analytic SSMs, then those would have to transform to nontrivial analytic SSMs of (96) under $h(y)$, but no such nontrivial analytic SSMs exist in (96).) The unique analytic SSMs over spectral subspaces of (95) can in turn be extended to smooth global invariant manifolds under the reverse flow map of (95) up to the maximum time of definition of backward solutions.

Cirillo et al. [11] touch on parts of this argument for the existence of two-dimensional SSMs in autonomous nonlinear systems, without establishing uniqueness and analyticity in detail. These authors involve the Koopman operator (cf. Mezić [27]) in their arguments, but all spectral subspaces of a linear mapping are well defined without the need to view them as zero sets of Koopman eigenfunctions. (These subspaces are in fact the only invariant manifolds of the linearized system (96) out of the infinitely many that are expressible as zero sets of Koopman eigenfunctions under the nonresonance conditions given above.) Furthermore, as shown by the argument above, the restriction to two-dimensional SSMs is not necessary either.

The line of reasoning we gave above for the existence of autonomous SSMs is complete but applicable only under assumptions that limit its applicability in practice. Specifically, SSMs obtained from the analytic linearization are applicable only when the linear operator $A$ in system (6) has no resonances, not even inside any of the spectral subspaces. This latter assumption is a limitation, as the main motivation in the nonlinear normal mode literature for multi-mode Pierre–Shaw-type invariant surfaces is precisely to deal with internal resonances inside a spectral subspace $E_{j_1,\dots,j_q}$. Furthermore, unlike Theorem 3, Poincaré's result guarantees uniqueness only for analytic dynamical systems and only within the class of analytic SSMs. This is again a limitation in practice, as no finite order can be deduced over which a Taylor expansion will only approximate the unique SSM. A relaxation of Poincaré's analytic setting to the case of finite differentiability is available (Sternberg [43]). In that setting, however, the uniqueness of SSMs can no longer be concluded within any function class, given that the local linearizing transformation $h(y)$ is no longer unique.

**Existence, uniqueness and persistence for non-autonomous SSMs ($k > 0$)**

First, we recall a more abstract result of Haro and de la Llave [18] on quasiperiodic mappings and their subwhiskers, which we subsequently apply to our setting.

Invariant tori and their spectral subwhiskers in quasi-periodic maps

We fix the finite-dimensional phase space $\mathcal{P} = \mathbb{R}^N \times \mathbb{T}^k$. On an open subset $\mathcal{U} = U \times \mathbb{T}^k \subset \mathcal{P}$ of this phase space, we consider a map $\mathcal{F}_1 : \mathcal{U} \to \mathbb{R}^N$. For some $r \in \mathbb{N} \cup \{\infty, a\}$ and $s \geq 2$, we will say that the map $\mathcal{F}_1$ is of class $C^{r,s}$, if $\mathcal{F}_1(x, \phi)$ is $C^r$ in its second argument $\phi \in \mathbb{T}^k$, and jointly $C^{r+s}$ in both of its arguments $(x, \phi) \in U \times \mathbb{T}^k$. In other words, if $\mathcal{F}_1 \in C^{r,s}$ then $\partial_\phi^i \partial_x^j \mathcal{F}_1$ exists and is continuous for all indices $(i, j) \in \mathbb{N}^2$ satisfying $i \leq r$ and $i + j \leq r + s$.

Next we assume that for any $\phi \in \mathbb{T}^k$, the map $\mathcal{F}_1(\cdot, \phi)$ is a local diffeomorphism. For a constant phase shift vector $\Delta \in \mathbb{R}^k$, we define the quasiperiodic mapping $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : \mathcal{U} \times \mathbb{T}^k \to \mathcal{P}$ as

$$\mathcal{F}(x,\phi) = (\mathcal{F}_1(x,\phi), \mathcal{F}_2(\phi)) := (\mathcal{F}_1(x,\phi), \phi + \Delta).$$

Assume that $\mathcal{F}_1(0,\phi) = 0$, i.e., $\mathcal{K} = \{0\} \times \mathbb{T}^k$ is an invariant torus for the map $\mathcal{F}$. Let $K : \mathbb{T}^k \to \mathbb{R}^n$ be a parametrization of the torus $\mathcal{K}$.

Next, we define the torus-transverse Jacobian

$$M(\phi) = D_x\mathcal{F}_1(0,\phi) \tag{97}$$

of the mapping component $\mathcal{F}_1$, and let $\nu\colon \mathbb{T}^k \to \mathbb{R}^N$ be any bounded mapping from the $k$-dimensional torus into $\mathbb{R}^N$. We then define the *transfer operator* $\mathcal{T}_\Delta\colon \nu \mapsto \mathcal{T}_\Delta \nu$ as a functional that maps the function $\nu$ into the function

$$[\mathcal{T}_\Delta \nu](\phi) = D_x\mathcal{F}_1(0,\phi-\Delta)\nu(\phi-\Delta). \tag{98}$$

Note that $\mathcal{T}_\Delta$ is just the torus-transverse component of the mapping $(\phi-\Delta, \nu(\phi-\Delta)) \mapsto (\phi, [\mathcal{T}_\Delta\nu](\phi))$ which maps the vector $\nu(\phi-\Delta)$, an element of the normal space of the torus $\mathcal{K}$ at the base point $(0,\phi-\Delta)$, under the linearized map $D\mathcal{F}$ into a vector in the normal space of $\mathcal{K}$ at the base point $(0,\phi)$.

As long as $\nu$ is taken from the class of bounded functions, the spectrum of the operator $\mathcal{T}_\Delta$ does not depend on the smoothness properties of $\nu$ (see Theorem 2.12, Haro and de la Llave [18]). We will need the *annular hull of the spectrum* of $\mathcal{T}_\Delta$, defined as

$$\mathcal{A} = \left\{ze^{i\alpha} \;:\; z \in \mathrm{Spect}\mathcal{T}_\Delta,\; \alpha \in \mathbb{R}\right\}. \tag{99}$$

This set is a union of circles in the complex plane, with each circle obtained by rotating an element of the spectrum of $\mathcal{T}_\Delta$.

We make the following assumptions:

(0) The spectrum of the operator $\mathcal{T}_\Delta$ does not intersect the complex unit circle, i.e.,

$$\mathrm{Spect}\mathcal{T}_\Delta \cap \{z \in \mathbb{C} \;:\; |z| = 1\} = \emptyset.$$

(1) There exists a decomposition of $N\mathcal{K}$, the normal bundle of $\mathcal{K}$, into a direct sum

$$N\mathcal{K} = P_1 \oplus P_2$$

of two $C^r$ subbundles, $P_1, P_2 \subset N\mathcal{K}$, such that $P_1$ is invariant under $M(\phi)$. As a consequence, a representation of $M(\phi)$ with respect to this decomposition is given by

$$M = \begin{pmatrix} M_1(\phi) & B(\phi) \\ 0 & M_2(\phi) \end{pmatrix}.$$

The corresponding restrictions of the transfer operator $\mathcal{T}_\Delta$ onto functions mapping into $P_1$ and $P_2$ will be denoted as $\mathcal{T}_{1,\Delta}$ and $\mathcal{T}_{2,\Delta}$. The annular hulls $\mathcal{A}_j$ of the spectra of these restricted operators can be defined similarly to $\mathcal{A}$:

$$\mathcal{A}_j = \left\{ze^{i\alpha} \;:\; z \in \mathrm{Spect}\mathcal{T}_{j,\Delta},\; \alpha \in \mathbb{R}\right\}, \quad j = 1,2,$$
$$\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}. \tag{100}$$

(2) The annular hull of $\mathrm{Spect}(\mathcal{T}_{1,\Delta})$ lies strictly inside the complex unit circle, i.e., $\mathcal{A}_1 \subset \{z \in \mathbb{C} \;:\; |z| < 1\}$

(3) For the smallest integer $L \geq 1$ satisfying

$$\mathcal{A}_1^{L+1}\mathcal{A}^{-1} \subset \{z \in \mathbb{C} \;:\; |z| < 1\}, \tag{101}$$

we have

$$\mathcal{A}_1^i \cap \mathcal{A}_2 = \emptyset \tag{102}$$

for every integer $i \in [2, L]$ (in case $L \geq 2$)

(5) $L + 1 \leq s$

We then have the following result:

**Theorem 6** (Haro and de la Llave [18]) *Under assumptions (0–5):*

(i) There exists an invariant manifold $\mathcal{M}_1 \subset \mathcal{P}$ that is a $C^{r,s}$ embedding of the subbundle $P_1$ into $\mathcal{P}$, and is tangent to $P_1$ along the torus $\mathcal{K}$.

(ii) The invariant manifold $\mathcal{M}_1$ is unique among all $C^{r,L+1}$ invariant manifolds of $\mathcal{F}$ that are tangent to the subbundle $P_1$ along the torus $\mathcal{K}$. That is, every two $C^{r,L+1}$ invariant manifolds with this tangency property will coincide in a neighborhood of $\mathcal{K}$.

(iii) There exists a map $R : P_1 \to P_1$ that is a polynomial of degree not larger than $L$ in the variable $\Delta$, of class $C^r$ in $x$ and $C^\infty$ in $\phi$, and there exists a $C^{r,s}$ map $W\colon U_1 \subset P_1 \to \mathcal{P}$, defined over an open tubular neighborhood $U_1$ of the zero section of $P_1$, satisfying

$$R(0,\phi) = 0, \quad D_1R(0,\phi) = M_1,$$
$$W(0,\phi) = K(\phi), \quad \pi_{P_1}D_1W(0,\phi) = I_{P_1},$$
$$\pi_{E_2}D_2W(0,\phi) = 0$$

for all $\phi \in \mathbb{T}^k$, such that $W$ serves as an embedding of $\mathcal{M}_1$ from $P_1$ to $\mathcal{P}$, and $R$ represents the pull-back of the dynamics on $\mathcal{M}_1$ to $U_1$ under this embedding. Specifically, we have

$$\mathcal{F}_1(W(\eta,\phi),\phi) = W(R(\eta,\phi),\phi+\Delta)$$

in the tubular neighborhood $U_1$.

(iv) If we further assume that for some integer $L_- \geq 2$, we have $\mathcal{A}_1^i \cap \mathcal{A}_1 = \emptyset$ for every integer $i \in [L_-, L]$, then $R$ can be chosen to be a polynomial of degree not larger than $L_- - 1$.

(v) If $\mathcal{A}_2 \cap \{z \in \mathbb{C} : |z| = 1 = \emptyset\}$ (i.e., the torus $\mathcal{K}$ is normally hyperbolic), then statements (i)–(iv) remain valid under small enough $C^{r,s}$ perturbations of the map $\mathcal{F}_1$. In particular, the invariant manifold $\mathcal{M}_1$ and its parametrization persist smoothly under small enough changes in parameters $\mu \in \mathbb{R}^p$ as long as for the new variable $\tilde{\phi} = (\phi, \mu)$, the function $\mathcal{F}_1(x, \tilde{\phi})$ is of class $C^{r,s}$.

These results have been collected, with minor notational changes, from Theorem 4.1 and Remark 4.7 of Haro and de la Llave [18].

Proof of Theorem 4

We consider eq. (41) but will work with its equivalent autonomous form

$$\dot{x} = Ax + f_0(x) + \epsilon f_1(x, \phi, \epsilon),$$
$$\dot{\phi} = \Omega. \quad (103)$$

We will state the smoothness assumptions on $f_0$ and $f_1$ in more detail later. By (v) of Theorem 6, we can first establish the existence of various spectral submanifolds attached to the invariant torus $\mathcal{K}_0 = \{0\} \times \Pi^k$ of the $\epsilon = 0$ limit of (103). We then conclude the existence of similar submanifolds attached to the quasiperiodic normal mode $x_\epsilon(t)$, represented by a perturbed invariant torus $\mathcal{K}_\epsilon$ for $\epsilon > 0$ in the full perturbed system (103).

In the context of the above theorem, we are working on the phase space $\mathcal{P} = \mathbb{R}^N \times \mathbb{T}^k$ and an open neighborhood $\mathcal{U} = U \times \mathbb{T}^k$, where $U \subset \mathbb{R}^N$ is an open neighborhood of the fixed point $x = 0$ of (41). We define the mapping $\mathcal{F}$ as the time-one map of the autonomous system (103) for $\epsilon = 0.$ , i.e.,

$$\mathcal{F}(x, \phi) = \left(F_0^1(x), \phi + \Omega\right) : \mathcal{U} \to \mathcal{P},$$
$$\mathcal{F}_1(x) = F_0^1(x),$$
$$\mathcal{F}_2(\phi) = \phi + \Omega, \quad (104)$$

with the map $F_0^1$ denoting the time-one map of $\dot{x} = Ax + f_0(x)$. By our assumptions, we have $\mathcal{F}_1(0) = 0$, and hence, the torus $\mathcal{K}_0 =$ is an invariant torus for the map $\mathcal{F}$ for $\epsilon = 0$.

The Jacobian of the $x$-dynamics at $x = 0$, as defined in (97), is

$$M(\phi) = D_x F_0^1(0) = e^A,$$

and the transfer operator defined in (98) takes the form

$$[\mathcal{T}_\Omega v](\phi) = e^A v(\phi - \Omega).$$

We now Fourier expand the general function $v \colon \mathbb{T}^k \to \mathbb{R}^N$ as

$$v(\phi) = \sum_{|m|=1}^{\infty} v_m e^{i\langle m, \phi\rangle}, \quad m \in \mathbb{Z}^n.$$

Be definition, $\lambda \in \mathbb{C}$ is in the spectrum of the operator $\mathcal{T}_\Omega$ if $[\lambda I - \mathcal{T}_\Omega]^{-1}$ does not exist. After Fourier-expanding $\mathcal{T}_\Omega v$, we see that the non-invertibility of $\lambda I - \mathcal{T}_\Omega$ is equivalent to the non-solvability of

$$\sum_{|m|=1}^{\infty} \left(\lambda I - e^{-i\langle m, \Omega\rangle} e^A\right) v_m e^{i\langle m, \phi\rangle} = \sum_{|m|=1}^{\infty} \tilde{v}_m e^{i\langle m, \phi\rangle}$$

for the coefficients $v_m$, where $\tilde{v}_m$ is arbitrary but fixed. This non-solvability arises precisely when

$$\det\left[e^A - \lambda e^{i\langle m, \Omega\rangle} I\right] = 0,$$

i.e., when $\lambda e^{i\langle m, \Omega\rangle}$ is contained in the spectrum $e^A$. We conclude that the spectrum of $\mathcal{T}_\Omega$ is given by

$$\text{Spect}\,(\mathcal{T}_\Omega) = \left\{e^{\lambda_j - i\langle m, \Omega\rangle} : \ j = 1, \ldots, d; \ \ m \in \mathbb{N}^k\right\}, \quad (105)$$

where $\lambda_j$ are the eigenvalues of $A$, listed in (7). By the definition (99), the annular hull of Spect$\mathcal{T}_\Omega$ is therefore

$$\mathcal{A} = \left\{z \in \mathbb{C} : |z| = e^{\text{Re}\lambda_j} : \ j = 1, \ldots, d\right\}. \quad (106)$$

For later reference, the analogous annular hull defined for the inverse of $A$ is then

$$\mathcal{A}^{-1} = \left\{z \in \mathbb{C} : |z| = e^{-\text{Re}\lambda_j} : \ j = 1, \ldots, d\right\}.$$

By assumption (42), Eq. (105) implies that hypotheses (0–2) of Theorem 6 are satisfied. To verify the remaining assumptions of the theorem, we note that the smallest integer $L$ satisfying

$$\mathcal{A}_1^{L+1} \mathcal{A}^{-1} \subset \{z \in \mathbb{C} : |z| < 1\}$$

is just the smallest integer that satisfies

$$\left[e^{\max_{\lambda \in \text{Spect}(A|_E)} \text{Re}\lambda}\right]^{L+1} e^{\min_{\lambda \in \text{Spect}(A)} \text{Re}\lambda} < 1.$$

The solution of this inequality for a general real $L$ is given by

$$L > \frac{\min_{\lambda \in \text{Spect}(A)} \text{Re}\lambda}{\max_{\lambda \in \text{Spect}(A|_E)} \text{Re}\lambda} - 1.$$

The integer solutions of this inequality therefore satisfy

$$L \geq \Sigma(E),$$

with the absolute spectral quotient $\sigma(E)$ defined in (16). The nonresonance condition (102) can be written in our setting precisely in the form (43). Thus, under the assumptions of Theorem 4, the conditions of Theorem 6 are satisfied. The statements of Theorem 4 are then just restatements of Theorem 6 in our present context.

## Comparison with applicable results for normally hyperbolic invariant manifolds

As in the autonomous case, the existence of slow non-autonomous SSMs (last column of Table 2) could also be deduced in a substantially weaker form from the classic theory of inflowing invariant normally hyperbolic invariant manifolds (Fenichel [15]).

Following the approach taken in Appendix "Comparison with applicable results for normally hyperbolic invariant manifolds" for the autonomous case, we let $\delta = \sqrt{\epsilon}$ and use the rescaling $x \to \delta x$ in system (103) to obtain the equivalent dynamical system

$$\dot{x} = Ax + \delta \left[ \tilde{f}_0(x; \delta) + f_1(\delta x, \phi) \right],$$
$$\dot{\phi} = \Omega. \tag{107}$$

Assume that the slow spectral subspace $E_{1,\ldots,q}$ featured in row (1) Table 2 satisfies the strict inequality

$$\text{Re}\lambda_{q+1} < \text{Re}\lambda_q.$$

This implies that in the $\delta = 0$ limit of system (107), the torus bundle $\mathcal{K}_0 \times E_{1,\ldots,q}$ is a normally hyperbolic invariant manifold, i.e., all decay rates of the linearized system within $\mathcal{K}_0 \times E_{1,\ldots,q}$ are weaker than any decay rate transverse to $E_{1,\ldots,q}$. Furthermore, a small compact manifold $\mathcal{K}_0 \times \tilde{E}_{1,\ldots,q} \subset \mathcal{K}_0 \times E_{1,\ldots,q}$ with boundary can be selected such that $\dim \tilde{E}_{1,\ldots,q} = \dim E_{1,\ldots,q}$ and $\mathcal{K}_0 \times \tilde{E}_{1,\ldots,q}$ is inflowing invariant under the flow of (107) for $\delta = 0$. This specifically means that the vector field $(Ax, \Omega)$ points strictly outward on the boundary $\partial(\mathcal{K}_0 \times \tilde{E}_{1,\ldots,q}) = \mathcal{K}_0 \times \partial\tilde{E}_{1,\ldots,q}$ of $\mathcal{K}_0 \times \tilde{E}_{1,\ldots,q}$. Then, for $\delta > 0$ small enough, the results of Fenichel [15] imply the existence of an invariant manifold $\tilde{W}$ with boundary in system (107) that is $\mathcal{O}(\delta)$ $C^1$-close to $\mathcal{K}_0 \times \tilde{E}_{1,\ldots,q}$ within a small neighborhood of $\mathcal{K}_0$. Furthermore, $\dim \tilde{W} = \dim \tilde{E}_{1,\ldots,q} + k$ and the manifold $\tilde{W}$ is of class $C^\gamma$ with the integer $\gamma$ defined in (94).

The limitations of this approach are identical to those discussed in Appendix "Comparison with applicable results for normally hyperbolic invariant manifolds."

## Comparison with results deducible from analytic linearization theorems

A time-quasiperiodic extension of the linearization theorem of Poincaré [34] (cf. Appendix "Comparison with results deducible from analytic linearization theorems") is given by Belaga [5] (cf. Arnold [1]), covering differential equations of the form

$$\dot{y} = \Lambda y + g(y, \phi), \quad g(y, \phi) = \mathcal{O}\left(|y|^2\right), \tag{108}$$
$$\dot{\phi} = \Omega, \tag{109}$$

where $\Lambda \in \mathbb{C}^{N \times N}$ is diagonalizable, $\phi \in \mathbb{T}^k$ and $g(y, \phi)$ is analytic. If

1. all eigenvalues of $\Lambda$ lie in the same open half plane in the complex plane (e.g, $\text{Re}\lambda_j < 0$ for all $j$ in our setting), and
2. the nonresonance conditions $\lambda_l \neq \langle m, \lambda \rangle + i \langle p, \Omega \rangle$ hold for all integer vectors $m \in (m_1, \ldots, m_N)$, with $m_i \geq 0$, and $\sum_i m_i \geq 2$, and for all $p \in \mathbb{Z}^k$ ,

then there exists an analytic, invertible change of coordinates $z = h(y)$ in a neighborhood of the origin under which system (108) transforms to

$$\dot{z} = \Lambda z,$$
$$\dot{\phi} = \Omega. \tag{110}$$

The spectral subbundles of the trivial normal mode $\{z = 0\} \times \mathbb{T}^k$ in this system are all defined by analytic functions, given as direct products of flat graphs over any spectral subspace of $\Lambda$ with the torus $\mathbb{T}^k$. It follows from our discussion in Sect. 1 that these flat subbundles are the only analytic spectral subbundles of (110). Then, following the argument in Appendix section "Comparison with results deducible from analytic linearization theorems," we conclude that (108) also has unique analytic, quasiperiodic SSMs, tangent at the origin to any selected spectral subspace of the operator $\Lambda$. These unique analytic SSMs over spectral subspaces of (108) can in turn be extended to smooth global invariant manifolds under the reverse flow map of (108) up to the maximum time of definition of backward solutions.

This construct has all the practical limitations already discussed Appendix "Comparison with results deducible from analytic linearization theorems," plus two more. First, resonances with the external forcing are also excluded by the above nonresonance assumptions. Second, the term representing external, time-dependent forcing must be fully nonlinear in the phase space variables. The latter is rarely the case in mechanical models.

We close by noting that in the case of $k = 1$ (single-frequency forcing), the above results of Belaga can be extended to cover time-periodic dependence in the linear operator $\Lambda$ as well (see Arnold [1]). This is the mechanical setting for the formal manifold calculations of Sinha et al. [42] and Redkar et al. [35]. The limitations of the linearization approach discussed above remain valid for this extension as well. In contrast, a direct application of Theorem 5 to the Poincaré map of (108) with $k = 1$ gives sharp existence, persistence and uniqueness results for SSMs, assuming that the Floquet multipliers associated with the time-dependent linearization are known.

Similarly, if $\Lambda$ has quasiperiodic ($k > 1$) dependence on $\phi$, Theorem 6 formally applies to the quasiperiodic map associated with the linearized system, giving sharp existence, persistence and uniqueness results for SSMs in the nonlinear system. In this general case, however, the spectrum of the transfer operator $[\mathcal{T}_{\Delta}\Omega](\phi)$ defined in (98) is not known and requires a case-by-case analysis. For this reason, we have assumed throughout this paper the common mechanical setting in which the operator $A$ of the linearized system is time-independent.

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