Homoclinic Jumping in the Perturbed Nonlinear Schrödinger Equation

G. HALLER
Brown University

Abstract

We study the damped, periodically forced, focusing NLS equation with even, periodic boundary conditions. We prove the existence of complicated solutions that repeatedly leave and come back to the vicinity of a quasi-periodic plane wave with two time scales. For pure forcing, we prove the existence of a complicated, self-similar family of homoclinic bifurcations. For mode-independent damping, we construct "jumping" transients. For mode-dependent damping, we find generalized Šilnikov-type solutions that connect a periodic plane wave to itself through repeated jumps. We also study the breakdown of the unstable manifold of plane waves through repeated jumping. Our results give a direct explanation for the numerical observations of Bishop et al. © 1999 John Wiley & Sons, Inc.

1 Introduction

The perturbed nonlinear Schrödinger (NLS) equation we study in this paper can be written in the form

$$iu_t = u_{xx} + 2\left|u\right|^2 u + i\varepsilon\left[\hat{D}u + \Gamma e^{i2\Omega^2 t}\right], \tag{1.1}$$

for which the forcing amplitude Γ and frequency $2\Omega^2$ are real numbers, $\hat{D}$ is a bounded, negative operator, and $\varepsilon \geq 0$ is a small parameter. The function $u(x,t)$ is even and periodic with period $L = 2\pi$ in the spatial variable x. Equation (1.1) is a well-known example of a partial differential equation that exhibits easily observable chaotic behavior in the time domain. The peculiar jumping of solutions around plane waves of (1.1) was first observed numerically by Bishop et al. [2], who used the simple linear damping term $\hat{D}u \equiv -\alpha u$. From a series of studies of finite-dimensional models (see [1, 3, 6, 7, 11, 12, 13, 16, 17], etc.) it has become clear that the jumping behavior of the perturbed NLS equation should be related to the presence of homoclinic solutions in the unperturbed limit.

Dynamically most interesting is the orbit family that is homoclinic to periodic plane waves in a one-to-one resonance with the periodic forcing term. Li et al. [14] showed that if the linear damping term $-\alpha u$ is amended with a "smoothed" Laplacian operator, then the perturbed partial differential equation (1.1) will also admit a pair of homoclinic orbits for certain parameter values. These orbits connect a quasi-periodic plane wave to itself and can be considered as infinite-dimensional

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analogs of Šilnikov's homoclinic orbit as they "spiral" back to the plane wave. Recently, Li [15] showed that the presence of such a symmetric pair of orbits implies the existence of Smale horseshoes. He then used symbolic dynamics to show the existence of chaotic jumping solutions for open sets in the parameter space.

In this paper we investigate a different mechanism for irregular jumping. This mechanism is provided by families of orbits that leave the set of plane waves of equation (1.1), exhibit irregular jumps near the "wings" of the destroyed homoclinic structure, and then finally return to a small vicinity of the plane waves, where they settle for long times. To construct such orbits, we do not need to introduce the smoothed Laplacian operator or other extra terms. As a consequence, our results apply directly to the perturbed NLS studied numerically by Bishop et al. In fact, the orbits we find are so robust that they even continue to exist in the limit of zero damping. In this Hamiltonian limit they exhibit a complicated sequence of homoclinic bifurcations as the forcing frequency is varied.

The methods we use are based on an infinite-dimensional extension of Fenichel's geometric singular perturbation theory, the study of infinite-dimensional Poincaré maps, and detailed, long-term energy estimates. The analysis is technical for two main reasons. First, the flow associated with the perturbed NLS equation is not smooth in the time variable t, which results in nonsmooth Poincaré maps. Second, these maps become singular in the limit of $\varepsilon = 0$. The first problem is present throughout our analysis but is finally resolved by restricting to H^{∞} initial conditions. The second problem is more serious and requires a detailed study of long-term passages of solutions near the set of plane waves. The multipulse orbits are constructed by matching the energy of a point on the returning solution with the energy of its projection onto a "large" stable manifold that guides solutions back to a vicinity of plane waves. If the two energies match, the two points coincide. Such a coincidence is inferred from the transverse zeros of an appropriately defined *energy function*. The calculation of zeros is very simple because, as opposed to earlier studies, we do not rely on invariants of the unperturbed NLS that are not known explicitly.

The paper is organized as follows: In Section 2 we describe the main properties of the unperturbed NLS equation. In Section 3 we invoke some invariant-manifold results from Li et al. [14] that are crucial in our construction and enable us to derive a convenient local normal form near the set of plane waves. The normal form is then used in Section 4 in our estimates for solutions that perform a long-time passage near the set of plane waves. These estimates are heavily used in Section 4, where we set up a local and a global Poincaré map to track solutions that exhibit large excursions from, and local passages near, the resonant plane waves. Section 5 is devoted to estimates on the change of energy on such solutions, and all these ingredients are put together in Section 6, where we prove our main theorem on the existence of multipulse orbits homoclinic to a small vicinity of resonant plane waves. This result is then applied in Section 7 to study multipulse jumping in the purely forced NLS equation and then in Section 8 to the forced and linearly damped

NLS equation. In Section 9 we show how our methods yield multipulse analogs of the Šilnikov-type orbits of Li et al. [14]. Finally, in Section 9 we describe how the unstable manifold of the plane waves breaks down into pieces with different jumping behavior. *We believe that this last result provides the most direct explanation available for the irregular jumping in the NLS observed by Bishop et al.*

2 Setup

2.1 The NLS as an Evolution Equation

We remove the explicit time dependence from equation (1.1) by applying the transformation $u \to ue^{-i2\Omega^2 t}$, which yields the new equation

$$u_t = -iu_{xx} - 2i\left[|u|^2 - \Omega^2\right]u + \varepsilon\left[\hat{D}u - \Gamma\right]. \tag{2.1}$$

We consider this equation as an evolution equation on the phase space

$$\mathcal{P} = \left\{u \in H^1_{\mathbb{C}} \mid u(x) = u(x+2\pi),\ u(x) = u(-x)\right\}.$$

Here $H^k_{\mathbb{C}}$ denotes the Sobolev space of complex-valued functions defined on the line that are square-integrable on $[0, 2\pi)$ together with their first k distributional derivatives. We will use the notation H^k for the subspace of real-valued elements of $H^k_{\mathbb{C}}$. On the phase space $\mathcal{P}$, equation (2.1) can be viewed as a perturbed Hamiltonian system

$$u_t = i\nabla_{\bar{u}}\left[H_0(u,\bar{u}) + \varepsilon H_1(u,\bar{u})\right] + \varepsilon g(u,\bar{u}) \tag{2.2}$$

with

$$\begin{aligned} H_0(u,\bar{u}) &= \frac{1}{2\pi}\int_0^{2\pi} |u_x|^2 + 2\Omega^2|u|^2 - |u|^4\,dx\,, \\ H_1(u,\bar{u}) &= \frac{i\Gamma}{2\pi}\int_0^{2\pi} \bar{u} - u\,dx\,, \\ g(u,\bar{u}) &= \hat{D}u\,. \end{aligned} \tag{2.3}$$

(Throughout this paper, $\nabla_a f$ refers to the gradient of the function f defined as $Df(a)\cdot v = \frac{1}{2\pi}\int_0^{2\pi} \nabla_a f(a(x))v(x)\,dx$.) The symplectic form on $\mathcal{P}$ for the Hamiltonian part of equation (2.2) is given by

$$\omega(c,d) = i\left(\langle d,c\rangle_{L^2} - \langle c,d\rangle_{L^2}\right) \tag{2.4}$$

with $c, d \in T_p\mathcal{P}$. (We use the L^2 inner product $\langle a,b\rangle_{L^2} = \frac{1}{2\pi}\int_0^{2\pi} a(x)\bar{b}(x)\,dx$.) Note that the unperturbed Hamiltonian H_0 naturally splits into an unbounded term

H_{00} and a bounded term H_{01}:

$$H_0 = H_{00} + H_{01}\,,$$
$$H_{00} = \frac{1}{2\pi}\int_0^{2\pi} |u_x|^2 \; dx\,,$$
$$H_{01} = \frac{1}{2\pi}\int_0^{2\pi} 2\Omega^2 |u|^2 - |u|^4 \; dx\,.$$

The contribution from the unbounded part to equation (2.2) is given by the linear term

$$i\nabla_{\bar{u}} H_{00}(u,\bar{u}) = M_0 u$$

with the linear operator $M_0 = -i\partial_{xx}$. Note that M_0 maps any function $u \in H^3_{\mathbb{C}}$ into an $H^1_{\mathbb{C}}$-function when defined in the sense of distributions. Since the space $H^3_{\mathbb{C}}$ is dense in $H^1_{\mathbb{C}}$, the domain of the operator M_0 is dense in the phase space $\mathcal{P}$. Furthermore, for any $u \in H^k_{\mathbb{C}}$ we have

$$\begin{aligned} \|M_0 u\|_{H^{k-2}} &= \|u_{xx}\|_{H^{k-2}} = \|\partial_x^2 u\|_{L^2} + \|\partial_x^3 u\|_{L^2} + \cdots + \|\partial^k u\|_{L^2} \\ &\le \|u\|_{L^2} + \|\partial_x u\|_{L^2} + \|\partial_x^2 u\|_{L^2} + \|\partial_x^3 u\|_{L^2} + \cdots + \|\partial_x^k u\|_{L^2} \\ &= \|u\|_{H^k}\,; \end{aligned} \tag{2.5}$$

thus M_0 is bounded in the H^{k-2} norm when it acts on $H^k_{\mathbb{C}}$ functions.

An important fact about (2.1) is that it admits a flow $F^t\colon \mathcal{P} \to \mathcal{P}$ (see Li et al. [14]). The flow operator F^t is continuous in t and is C^r in u and ε for fixed t. Furthermore, $H^k_{\mathbb{C}}$ initial conditions remain in $H^k_{\mathbb{C}}$ for all times; i.e., all $H^k_{\mathbb{C}}$ spaces are invariant with respect to the flow.

2.2 Resonance in the NLS Equation

We can write any solution $u(x,t)$ as

$$u(x,t) = c(t) + b(x,t)\,, \quad \langle b\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} b(x,t)dx = 0\,. \tag{2.6}$$

Here c is the spatial mean of u, b is the deviation from that mean, and $\langle b\rangle$ is the spatial average of b. As is well-known, one of the invariants of the integrable NLS equation is given by

$$I(u,\bar{u}) = \|u\|_{L^2} = \langle |u|^2\rangle = |c|^2 + \langle |b|^2\rangle\,.$$

As in Li et al. [14], we rewrite c in the form

$$c(t) = |c(t)|e^{i\phi(t)} = \sqrt{I(t) - \langle |b(x,t)|^2\rangle}\, e^{i\phi(t)}\,. \tag{2.7}$$

We denote the set of spatially independent solutions (i.e., solutions with $\partial_x u \equiv 0$) by Π. This two-dimensional subspace of $\mathcal{P}$ contains plane waves that belong to the space $H^k_{\mathbb{C}}$ for any integer k; i.e., we have

$$\Pi = \{u \in \mathcal{P} \mid \partial_x u \equiv 0\} \subset H^\infty_{\mathbb{C}}\,.$$

Using the real coordinates (I, ϕ) as new variables, we find that on the plane Π, the NLS equation restricts to the ODE

$$\dot{\phi} = 2(\Omega^2 - I)\,, \qquad \dot{I} = 0\,.$$

This equation shows that Π is foliated by periodic orbits for $I > 0$. For any nonzero value of the forcing frequency Ω, one of the periodic orbits becomes degenerate; i.e., $\dot{\phi}$ vanishes on it. This closed curve

$$\mathcal{C} = \{(\phi, I) \mid I = \Omega^2\}$$

is therefore a circle of equilibria (see Figure 2.1). Such equilibria correspond to unperturbed plane waves of (1.1) that are in a *one-to-one resonance* with the forcing.

2.3 The Flow near the Resonance

The stability type of the circle $\mathcal{C}$ is determined by the eigenvalues of the linearized flow near $\mathcal{C}$. As shown in Li et al. [14], in directions transverse to $\mathcal{C}$ the linearized flow admits the eigenvalues

$$\Omega_j^{\pm} = \pm j\sqrt{4\Omega^2 - j^2}\,, \quad j = 0, 1, 2, \ldots\,; \tag{2.8}$$

thus for any $\Omega > \frac{1}{2}$, the resonant circle is unstable. As the forcing frequency Ω increases, the number of linearly stable and unstable directions also increases. At the same time, for any fixed value of Ω, there will be infinitely many purely imaginary exponents $\Omega_j^{\pm}$ corresponding to sufficiently high values of j. As a result, in linear approximation any plane wave with $\Omega > \frac{1}{2}$ has finite-dimensional stable and unstable subspaces E^s and E^u and an infinite-dimensional center subspace. The center subspace is the direct sum of the plane $E^0 \equiv \Pi$ (corresponding to a double zero eigenvalue for $j = 0$) and an infinite-dimensional subspace E^c corresponding to the purely imaginary eigenvalues. The spaces E^s, E^u, and $E^0 \oplus E^c$ are stable, unstable, and center subspaces, respectively, for the linear operator $M\colon H^k_{\mathbb{C}} \to H^{k-2}_{\mathbb{C}}$ defined as

$$Mv = -iv_{xx} - 2i\Omega^2(v + \bar{v})\,. \tag{2.9}$$

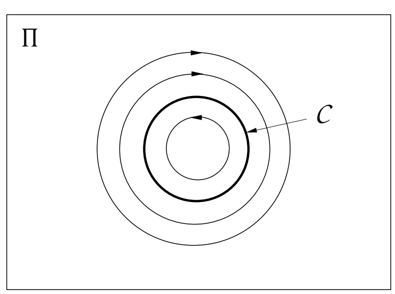

FIGURE 2.1. The resonant circle of fixed points.

We note that all these subspaces are independent of the plane wave under consideration, and hence the stable, unstable, zero, and center subbundles of the normal bundle of the circle are all trivial.

In this paper we study the global consequences of the simplest instability, which occurs for

$$\frac{1}{2} < \Omega < 1\,. \tag{2.10}$$

In that case, the stable and unstable subspaces are one-dimensional and hence E^c is of codimension 4. We now locally decompose the coordinate $u \in H^1_{\mathbb{C}}$ as $u \mapsto (y, z, I, \phi)$ with $y = (y_1, y_2) \in \mathbb{R}^2$, $z = (z_1, z_2) \in H^1 \times H^1$, $I \in \mathbb{R}$, $\phi \in S^1$. Here the y-coordinates are lined up with the linearly stable and unstable directions along the circle $\mathcal{C}$, the z-coordinates parametrize the center subspace E^c, and (I, ϕ) are action-angle variables on the plane Π in a vicinity of the resonant circle. The existence of this smooth change of coordinates follows from the triviality of the stable, unstable, and center subbundles of $N\mathcal{C}$, as noted above.

In a neighborhood of $\mathcal{C}$, the plane Π satisfies the equations $y = 0$ and $z = 0$, and equation (1.1) can be written locally as

$$\begin{aligned}
\dot{y} &= \Lambda y + \bar{Y}(y, z, I, \phi; \varepsilon)\,,\\
z_t &= Az + \bar{Z}(y, z, I, \phi; \varepsilon)\,,\\
\dot{I} &= \varepsilon \bar{E}(y, z, I, \phi; \varepsilon)\,,\\
\dot{\phi} &= \bar{F}_0(y, z, I, \phi) + \varepsilon \bar{F}_\varepsilon(y, z, I, \phi; \varepsilon)\,.
\end{aligned} \tag{2.11}$$

Here $\Lambda = \operatorname{diag}(-\lambda, \lambda)$ with $\lambda = \sqrt{4\Omega^2 - 1}$, and $A = M \mid E^c$ has a purely imaginary discrete spectrum. It follows from the definition of the operator M that A has a dense domain $D_A \subset H^1$ with $H^3 \subset D_A$, and by (2.5) we have

$$\|Az\|_{H^{k-2}} \le K_A \|z\|_{H^k} \tag{2.12}$$

for an appropriate constant $K_A > 0$.

Since (2.1) admits a (continuous) flow, the operator A necessarily generates a C^0-group on H^1. The flow satisfies the linearized system of equations

$$\begin{aligned}
\partial_t z_1 &= \partial_{xx} z_2\,,\\
\partial_t z_2 &= -\partial_{xx} z_1 - 4\Omega^2 z_1\,.
\end{aligned}$$

If $\mathbf{z}_j^k(t)$ with $k \ge 2$ denotes the k^{th}-order Fourier coefficient of the solution $z_j(t)$ of this linear system, then we have

$$\begin{pmatrix} \mathbf{z}_1^k(t) \\ \mathbf{z}_2^k(t) \end{pmatrix} = \begin{pmatrix} \cos \lambda_k t & 0 \\ -\lambda_k k^2 \sin \lambda_k t & \cos \lambda_k t \end{pmatrix} \begin{pmatrix} \mathbf{z}_1^k(0) \\ \mathbf{z}_2^k(0) \end{pmatrix}$$

with $\lambda_k = k\sqrt{k^2 - 4\Omega^2}$. This formula gives

$$\left|\mathbf{z}^k(t)\right|^2 = \left|\mathbf{z}_1^k(t)\right|^2 + \left|\mathbf{z}_2^k(t)\right|^2 \le \left(1 + \sqrt{1 - \frac{4\Omega^2}{k^2}}\right) \left|\mathbf{z}^k(0)\right|^2 < 2\left|\mathbf{z}^k(0)\right|^2,$$

where we used (2.10) and the fact that $k \geq 2$. This last inequality implies

$$\|z(t)\|_{L^2}^2 = \|z_1(t)\|_{L^2}^2 + \|z_2(t)\|_{L^2}^2 = \frac{1}{2\pi}\sum_{k=2}^{\infty}\left\{\left|\mathbf{z}_1^k(t)\right|^2 + \left|\mathbf{z}_2^k(t)\right|^2\right\}$$
$$< \frac{1}{2\pi}\sum_{k=2}^{\infty} 2\left|\mathbf{z}^k(0)\right|^2 = 2\|z(0)\|_{L^2}^2$$

and

$$\|\partial_x z(t)\|_{L^2}^2 = \|\partial_x z_1(t)\|_{L^2}^2 + \|\partial_x z_2(t)\|_{L^2}^2$$
$$= \frac{1}{2\pi}\sum_{k=2}^{\infty} k^2\left\{\left|\mathbf{z}_1^k(t)\right|^2 + \left|\mathbf{z}_2^k(t)\right|^2\right\}$$
$$< \frac{1}{2\pi}\sum_{k=2}^{\infty} 2k^2\left|\mathbf{z}^k(0)\right|^2 = 2\|\partial_x z(0)\|_{L^2}^2 ,$$

which in turn yield

$$\|z(t)\|_{H^1} = \|\exp At \cdot z(0)\|_{H^1} \leq C_A\|z(0)\|_{H^1} \tag{2.13}$$

with $C_A = \sqrt{2}$. Therefore, the C^0-group generated by the operator A is uniformly bounded.

2.4 Orbits Homoclinic to the Resonance

We have found that for the parameter range $\frac{1}{2} < \Omega < 1$, each fixed point on the circle $\mathcal{C}$ admits one linearly unstable and one linearly stable direction. Bäcklund transformations can be used to show that along these unstable and stable directions solutions leave and come back to the circle $\mathcal{C}$ (see, e.g., Ercolani and McLaughlin [4]). This means that heteroclinic orbits exist in the phase space $\mathcal{P}$ that connect different equilibria in $\mathcal{C}$ to one another. From any point on the resonant circle there are precisely two heteroclinic connections to another point of the circle. The heteroclinic solutions are of the form

$$u_\pm^h(x,t) = \Omega e^{i\phi_0}\frac{\cos 2p - i\sin 2p\tanh\tau \pm \sin p\,\mathrm{sech}\,\tau\cos x}{1 \mp \sin p\,\mathrm{sech}\,\tau\cos x},$$
$$p = \tan^{-1}\sqrt{4\Omega^2 - 1}\,,$$
$$\tau = \sqrt{4\Omega^2 - 1}(t + t_0)\,. \tag{2.14}$$

The $\pm$ index refers to two distinct families, which in turn are parametrized by the phase variable ϕ_0 and the initial time t_0. The two orbit families form two 2-dimensional homoclinic manifolds $W_0^\pm(\mathcal{C})$ to the circle $\mathcal{C}$.

Formula (2.14) also gives the two limit points of the heteroclinic connections:

$$\lim_{t\to+\infty} u^h_\pm(t) = \Omega e^{i\phi_0}\left(\cos 2p - i\sin 2p\right) = \Omega e^{i(\phi_0-2p)}\,,$$
$$\lim_{t\to-\infty} u^h_\pm(t) = \Omega e^{i\phi_0}\left(\cos 2p + i\sin 2p\right) = \Omega e^{i(\phi_0+2p)}\,.$$

There is a constant phase shift of

$$\Delta\phi = -4p = -4\tan^{-1}\sqrt{4\Omega^2-1} \tag{2.15}$$

between the two limit points of every heteroclinic orbit. The heteroclinic orbits also have the property that after perturbation, the leading-order change of energy along them is the same for all orbits. Using the notation $G=(g,\bar g)$ and the duality pairing

$$\langle\nabla H_0, G\rangle \equiv \langle\nabla_u H_0, g\rangle + \langle\nabla_{\bar u} H_0, \bar g\rangle\,,$$

we can write this change of energy as a Melnikov-type integral:

$$\begin{aligned}\mathcal{I} &= \int_{-\infty}^{\infty}\langle\nabla H_0, G\rangle\,|_{u^h_\pm(t)}\,dt\\ &= \int_{-\infty}^{\infty}\int_0^{2\pi}\left(\nabla_u H_0\, g + \nabla_{\bar u} H_0\,\bar g\right)|_{u^h_\pm(t)}\,dx\,dt\\ &= -2\,\mathrm{Re}\int_{-\infty}^{\infty}\int_0^{2\pi}\left(\bar u_{xx} + 2\left[|u|^2-\Omega^2\right]\bar u\right)\hat D u|_{u^h_\pm(t)}\,dx\,dt\\ &= -2\,\mathrm{Re}\int_{-\infty}^{\infty}\int_0^{2\pi}\left(\bar u_{xx} + 2\left[|u|^2-\Omega^2\right]\bar u\right)\hat D u|_{\tilde u^h_\pm(t)}\,dx\,dt\end{aligned} \tag{2.16}$$

with $\tilde u^h_\pm(t) = e^{-i\phi_0}u^h_\pm(t)$. Formula (2.14) shows that $\tilde u^h_\pm(t)$ is independent of the phase ϕ_0, and hence the integral $\mathcal{I}$ is certainly the same for all orbits in $W_0^+(\mathcal{C})$ and for all orbits in $W_0^-(\mathcal{C})$. To see that $\mathcal{I}$ is the same for both $W_0^+(\mathcal{C})$ and $W_0^-(\mathcal{C})$, note that $u^h_+(x,t) = u^h_-(x+\pi,t)$. Therefore, introducing the change of variables $x\mapsto x+\pi$ gives

$$\langle\nabla H_0, G\rangle|_{u^h_-(t)} = \langle\nabla H_0, G\rangle|_{u^h_+(t)}\ ,$$

and hence the integral $\mathcal{I}$ is the same for all heteroclinic connections in $W_0^+(\mathcal{C})\cup W_0^-(\mathcal{C})$.

3 Invariant Manifolds and Fenichel Coordinates

In our study of the perturbed NLS equation, we first recall two important invariant manifold results from Li et al. [14]. The first theorem below is concerned with the existence of a locally invariant center manifold for the circle $\mathcal{C}$ for $\varepsilon\geq 0$. (We recall that a manifold with boundary is locally invariant if solutions can only leave the manifold through its boundary.) The center manifold turns out to be a normally hyperbolic manifold that admits locally invariant stable and unstable manifolds.

THEOREM 3.1 *For $\varepsilon \geq 0$ small, the invariant plane Π is contained in a codimension-2, locally invariant manifold*

$$\mathcal{M}_\varepsilon = \left\{ (y, z, I, \phi) \mid y = y^\varepsilon(z, I, \phi),\ (z, I, \phi) \in V \subset H^1 \times \mathbb{R}^1 \times S^1 \right\} \subset \mathcal{P}\,,$$

where the function $y^\varepsilon(z, I, \phi)$ is of class C^r in its arguments as well as in ε, and $y^\varepsilon(0, I, \phi) = 0$. Furthermore, for any $r \geq 1$, $\mathcal{M}_\varepsilon$ admits codimension-1 local stable and unstable manifolds of class C^r, denoted $W^s_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ and $W^u_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$, which depend on ε in a C^r fashion.

The next theorem of Li et at. [14] gives the existence of invariant foliations for the local stable and unstable manifolds of $\mathcal{M}_\varepsilon$.

THEOREM 3.2 *The local unstable manifold $W^u_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ is foliated by a negatively invariant family $\mathcal{F}^u = \bigcup_{p \in \mathcal{M}_\varepsilon} f^u(p)$ of C^r-curves $f^u(p)$, i.e., $\mathcal{F}^u = W^u_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ and $F^{-t}\left(f^u(p)\right) \subset f^u\left(F^{-t}(p)\right)$ for any $t \geq 0$ and $p \in \mathcal{M}_\varepsilon \subset H^1$ (here F^t denotes the flow generated by system (1.1)). Moreover, the fibers $f^u(p)$ are of class C^r in ε and p, and $f^u(p) \cap f^u(p') = \varnothing$ unless $p = p'$. Finally, there exist $C_u, \lambda_u > 0$ such that if $q \in f^u(p)$, then*

$$\|F^{-t}(q) - F^{-t}(p)\|_{H^k} < C_u e^{-\lambda_u t}$$

for any $t \geq 0$ as long as $F^{-t}(p) \in \mathcal{M}_\varepsilon$. The local stable manifold $W^s_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$ admits a positively invariant foliation $\mathcal{F}^s = \bigcup_{p \in \mathcal{M}_\varepsilon} f^s(p)$ with similar properties.

REMARK 3.3 We note that the plane Π is a subset of H^∞, so there exist fibers $f^u(p)$ and $f^s(p)$ for any $p \in \Pi$ in any Sobolev space H^k. Since $H^{k+1} \subset H^k$ and the fibers emanating from a given base point p are unique in any H^k-space, we obtain a unique, $2m$-parameter family of fibers $f^u(p)$ and $f^s(p)$ emanating from points in Π. The union of these stable and unstable fibers immediately provides us with H^∞ local stable and unstable manifolds $W^s_{\mathrm{loc}}(\Pi)$ and $W^u_{\mathrm{loc}}(\Pi)$ for the plane Π.

To study the dynamics near $\mathcal{M}_\varepsilon$, we will use a normal form that is an infinite-dimensional version of the one first suggested by Fenichel [5] (see also Jones and Kopell [9]). This normal form can be obtained through local changes of coordinates that "straighten out" the stable and unstable manifolds of $\mathcal{M}_\varepsilon$ as well as their foliations.

As a preliminary step, we introduce the scaling

$$I = I_0 + \sqrt{\varepsilon}\eta \tag{3.1}$$

to blow up a neighborhood of the circle of equilibria $\mathcal{C}$. In terms of the coordinates (y, z, η, ϕ), we obtain the following result:

LEMMA 3.4 *There exists $\varepsilon_0 > 0$ such that for $0 \leq \varepsilon < \varepsilon_0$, a C^r local change of coordinates $\mathcal{T}_\varepsilon \colon (y, z, \eta, \phi) \mapsto (w, \zeta, \rho, \psi)$ (with a C^r-inverse) transforms system*

(2.1) *to the form*

$$
\begin{aligned}
\dot w_1 &= [-\lambda + (Y_1, w) + \langle Y_2, \zeta\rangle + \sqrt{\varepsilon} Y_3] w_1 \,,\\
\dot w_2 &= [\lambda + (Y_4, w) + \langle Y_5, \zeta\rangle + \sqrt{\varepsilon} Y_6] w_2 \,,\\
\zeta_t &= A\zeta + (Z_1\zeta, \zeta) + \sqrt{\varepsilon} Z_2 \zeta + Z_3 w_1 w_2 \,,\\
\dot\rho &= \sqrt{\varepsilon} E \,,\\
\dot\psi &= \langle F_1\zeta, \zeta\rangle + \sqrt{\varepsilon} F_2 + F_3 w_1 w_2 \,.
\end{aligned}
\tag{3.2}
$$

Here $(\cdot,\cdot)$ denotes the Euclidean inner product. The functions $Y_1, Y_4 : \mathcal{P} \times [0, \varepsilon_0] \to \mathbb{R}^2$, $Y_2, Y_5 : \mathcal{P} \times [0, \varepsilon_0] \to H^{-1}$, $Y_3, Y_6 : \mathcal{P} \times [0, \varepsilon_0] \to \mathbb{R}$, $E, F_2, F_3 : \mathcal{P} \times [0, \varepsilon_0] \to \mathbb{R}$, $Z_1 : \mathcal{P} \times [0, \varepsilon_0] \to (H^1)^{2\times 2}$, $Z_3 : \mathcal{P} \times [0, \varepsilon_0] \to H^1$, and the three-tensors $(Z_1 \cdot, \cdot)$ and $\langle F_1 \cdot, \cdot\rangle$ are all of class C^{r-4} in (y, z, η, ϕ) and ε. Moreover,

$$
D_w Z_1 = 0 \,, \quad D_w Z_2 = 0 \,, \quad D_w F_1 = 0 \,, \quad D_w F_2 = 0 \,. \tag{3.3}
$$

PROOF: The proof of the theorem closely follows the steps outlined in Fenichel [5] for finite-dimensional systems. Namely, we introduce the change of coordinates $y \mapsto w$ near the manifold $\mathcal{M}_\varepsilon$ in which $\mathcal{M}_\varepsilon$ is described by $w_1 = w_2 = 0$, and $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ satisfy $w_2 = 0$ and $w_1 = 0$, respectively. Next, we change the (z, η, ϕ)-coordinates appropriately to (ζ, ρ, ψ)-coordinates so that the stable and unstable fibers described in Theorem 3.2 satisfy $\zeta = \text{const}$, $\rho = \text{const}$, and $\psi = \text{const}$. The details of this construction can be found, e.g., in Tin [20] or Jones [10].

The only subtle point that arises in the infinite-dimensional case is the following: The changes of coordinates depend on the H^1-variable z; hence all components of the transformed equations will have terms arising from z_t on their right-hand sides. As a result, the unbounded term Az is not confined to the z equations anymore as in (2.11), and the system appears to become less tractable. However, the apparently unbounded terms are in fact always bounded due to cancellations. We show this for the first change of coordinates only, since later coordinate changes can be dealt with similarly.

Based on (3.1), we can rewrite (2.11) as

$$
\begin{aligned}
\dot y &= \Lambda y + Y(y, z, \eta, \phi; \sqrt{\varepsilon}) \,,\\
z_t &= Az + Z(y, z, \eta, \phi; \sqrt{\varepsilon}) \,,\\
\dot\eta &= \sqrt{\varepsilon} E(y, z, \eta, \phi; \sqrt{\varepsilon}) \,,\\
\dot\phi &= F_0(y, z, \eta) + \sqrt{\varepsilon} F_\varepsilon(y, z, \eta, \phi; \sqrt{\varepsilon}) \,,
\end{aligned}
\tag{3.4}
$$

where Y, Z, E, F_0, and F_ε are of class C^r nonlinear functions. The blowup construction implies that this system has a plane of equilibria for $\varepsilon = 0$ that satisfies $y = 0$ and $z = 0$. As a result,

$$
Y(0, 0, \eta, \phi; 0) = 0 \,, \qquad Z(0, 0, \eta, \phi; 0) = 0 \,, \qquad F_0(0, 0, \eta) = 0 \,,
$$

must hold, which imply

$$
Y = f_1 y + \langle f_2, z\rangle + \sqrt{\varepsilon} f_3 \,, \qquad Z = f_4 y + f_5 z + \sqrt{\varepsilon} f_6 \,, \\
F_0 = f_7 y + \langle f_8, z\rangle + \sqrt{\varepsilon} f_9 \,, \tag{3.5}
$$

for appropriate C^{r-1}-functions $f_1, \ldots, f_9$. Since the linearized flow of (3.4) at any point of Π leaves the y, z, and $\eta - \phi$ subspaces invariant, the functions in (3.5) can be rewritten as

$$
\begin{aligned}
Y &= (f_{10} y) y + \langle f_{11} y, z\rangle + \langle f_{12} z, z\rangle + \sqrt{\varepsilon} f_3 \,, \\
Z &= (f_{13} y, y) + (f_{14}, y) z + (f_{15} z, z) + \sqrt{\varepsilon} f_6 \,, \\
F_0 &= (f_{16} y) y + \langle f_{17} y, z\rangle + \langle f_{18} z, z\rangle + \sqrt{\varepsilon} f_9 \,,
\end{aligned} \tag{3.6}
$$

with appropriate C^{r-2}-functions $f_{10}, \ldots, f_{18}$. We recall that $(\cdot, \cdot)$ denotes the usual Euclidean scalar product of vectors, while $\langle \cdot, \cdot \rangle$ denotes the duality pairing between H^{-1} and H^1. From Theorem 3.1 we know that $\mathcal{M}_\varepsilon$ must satisfy an equation of the form

$$
y = y^\varepsilon(z, \eta, \phi) = y^0(z, I_0 + \sqrt{\varepsilon}\eta, \phi) + \varepsilon y^1(z, \eta, \phi; \sqrt{\varepsilon}) \,, \quad |\eta| + \|z\|_{H^1} < \delta \,,
$$

where y^ε is a C^r-function that depends on the parameter ε in a C^r fashion, and $\delta > 0$ is a sufficiently small number. We introduce the change of variables

$$
w = y - y^\varepsilon(z, \eta, \phi) \,. \tag{3.7}
$$

For initial data $z \in H^3$, the solution is a C^1-function of time. Then from equation (3.4) we obtain that the w-component of the transformed equations is of the form

$$
\begin{aligned}
\frac{d}{dt} w = {} & \Lambda(w + y^\varepsilon) + Y(w + y^\varepsilon, z, \eta, \phi) \\
& - D_z y^\varepsilon (Az + Z) - D_\eta y^\varepsilon \sqrt{\varepsilon} E - D_\phi y^\varepsilon F \,.
\end{aligned} \tag{3.8}
$$

As we indicated earlier, (3.8) suggests that in the new coordinates the right-hand side of the w-component of the evolution equation may not be a differentiable function any more. However, from the local invariance of $\mathcal{M}_\varepsilon$ we obtain that

$$
\begin{aligned}
D_z y^\varepsilon A z = {} & \Lambda y^\varepsilon + Y(y^\varepsilon, z, \eta, \phi; \sqrt{\varepsilon}) - D_z y^\varepsilon Z(y^\varepsilon, z, \eta, \phi; \sqrt{\varepsilon}) \\
& - \sqrt{\varepsilon} D_\eta y^\varepsilon E(y^\varepsilon, z, \eta, \phi) - D_\phi y^\varepsilon F(y^\varepsilon, z, \eta, \phi)
\end{aligned} \tag{3.9}
$$

for $z \in H^3$ and $|\eta| + \|z\|_{H^1} < \delta$. Substituting this last expression back into (3.8) and making use of the structure of the right-hand side in (3.6), we see that for $z \in H^3$, (3.8) can be written as

$$
\frac{d}{dt} w = \Lambda w + (\tilde{f}_{10} w) w + \langle \tilde{f}_{11} w, z\rangle + \langle \tilde{f}_{12} z, z\rangle + \sqrt{\varepsilon}\, \tilde{f}_3 \tag{3.10}
$$

with a C^{r-2} right-hand side. This last expression shows that for $z \in H^3$, the function $D_z y^\varepsilon A z$ is in fact of class C^{r-1}. But the space H^3 is dense in H^1; thus any point $z \in H^1$ can be approximated by a sequence $\{z_k\}_{k=1}^\infty \subset H^3$ that converges to z in the H^1-norm. Then the continuity of the right-hand side of (3.9) implies that in the limit $z_k \to z$, equation (3.8) remains of the form (3.10). Therefore, equation

(3.10) holds for any initial condition $z \in H^1$ in the sense of distributions provided z is small enough in the H^1-norm.

In the new coordinates the manifold $\mathcal{M}_\varepsilon$ satisfies $w = 0$. Consequently, by the local invariance of $\mathcal{M}_\varepsilon$, (3.10) implies

$$\langle \tilde{f}_{12} z, z\rangle + \sqrt{\varepsilon}\tilde{f}_3 = [\langle \tilde{f}_{19} z, z\rangle + \sqrt{\varepsilon}\tilde{f}_{20}]w\,,$$

where the matrix-valued functions $\tilde{f}_{19}$ and $\tilde{f}_{20}$ are of class C^{r-3} in their arguments. Therefore, in the new coordinate system, (3.4) can be rewritten as

$$\begin{aligned}
\frac{d}{dt}w &= [\Lambda + (\tilde{f}_{10} w) + \langle \tilde{f}_{19} z + \tilde{f}_{11}, z\rangle + \sqrt{\varepsilon}\tilde{f}_{20}]w\,,\\
z_t &= Az + (\tilde{f}_{13} w, w) + (\tilde{f}_{14}, w)z + (\tilde{f}_{15} z, z) + \sqrt{\varepsilon}\,\tilde{f}_6\,,\\
\frac{d}{dt}\eta &= \sqrt{\varepsilon}\tilde{E}(w, z, \eta, \phi; \sqrt{\varepsilon})\,,\\
\frac{d}{dt}\phi &= (\tilde{f}_{16} w)w + \langle \tilde{f}_{17} z, z\rangle + \langle \tilde{f}_{18} z, z\rangle + \sqrt{\varepsilon}\tilde{f}_9\,.
\end{aligned} \tag{3.11}$$

The remaining steps in the proof can be shown to yield bounded terms in the same fashion. Enforcing the invariance conditions after each step leads to further factorization of the right-hand side of (3.11), which finally gives the normal form (3.2). □

4 Local Estimates

We want to use the normal form (3.2) to study the behavior of trajectories in a neighborhood of the manifold $\mathcal{M}_\varepsilon$. The trajectories we are interested in are contained in the unstable manifold $W^u(\Pi)$, and they do not intersect the local stable manifold $W^s_{\text{loc}}(\mathcal{M}_\varepsilon)$ upon entering a small neighborhood of $\mathcal{M}_\varepsilon$. Since $\mathcal{M}_\varepsilon$ is of "saddle-type," such trajectories pass near the manifold and leave its neighborhood. We are interested in how the coordinates (w, ζ, ρ, ψ) change during this passage.

For $0 \le \varepsilon \le \varepsilon_0$, the normal form is related to the NLS equation (2.1) within some fixed open set

$$S = \{(w, \zeta, \rho, \psi) \mid |w| < K_w,\ \|\zeta\|_{H^1} < K_\zeta,\ \sqrt{\varepsilon}|\rho| < K_I,\ \psi \in S^1\}\,,$$

where K_w, K_ζ, and K_I are fixed positive constants (see Figure 4.1). We consider solutions $u(t) = (w(t), \zeta(t), \rho(t), \psi(t))$ of the normal form that enter a small, fixed "box"

$$\begin{aligned}
U_0 = \{(y, z, \rho, \psi) \in S \mid &|w_i| \le \delta_0 < \sqrt{2}K_w/4,\\
&\|\zeta\|_{H^1} \le \delta_0 < K_\zeta,\ |\rho| \le K_\rho < K_I/\sqrt{\varepsilon}\}
\end{aligned}$$

with positive constants δ_0 and K_ρ. Since the functions on the right-hand side of (3.2) are of class C^{r-4} on S for all $0 \le \varepsilon \le \varepsilon_0$ and for appropriate $B > 0$, we

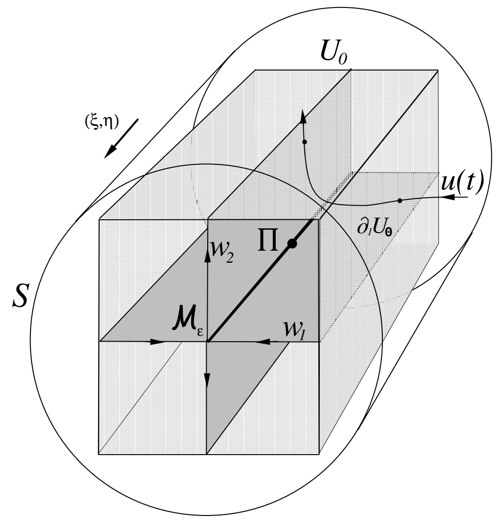

FIGURE 4.1. The sets U_0 and S.

have

$$
\begin{aligned}
&|Y_i|,\ \|Y_j\|_{H^1},\ \|Z_q\|_{H^1},\ |E|,\ \|F_1\|_{H^1},\ |F_l| < B, \\
&|DY_i|,\ \|DY_j\|_{H^1},\ \|DZ_q\|_{H^1},\ |DE|,\ \|DF_1\|_{H^1},\ |DF_l| < B,
\end{aligned} \tag{4.1}
$$

for $i = 1, 3, 4, 6$, $j = 2, 5$, $l = 2, 3$, $q = 1, 2, 3$, and $0 \le \varepsilon \le \varepsilon_0$. We want to follow a solution $u(t)$ that enters the set U_0 by intersecting its boundary ∂U_0 within the domain

$$\partial_1 U_0 = \{(w, \zeta, \rho, \psi) \in \partial U_0 \mid \|\zeta\|_{H^1} < \delta_0,\ |\rho| < K_\rho\}$$

at time $t = T^*$, as shown in Figure 4.1. For such a solution we have $w_1(0) = \delta_0$, and we assume that for $0 < \varepsilon \le \varepsilon_0$, the rest of the coordinates of the entry point $u(0)$ obey the *entry conditions*

$$\|\zeta(0)\|_{H^1} < c_1 \varepsilon^\beta, \qquad \frac{c_2 \varepsilon}{\delta_0} < |w_2(0)| < \frac{c_3 \varepsilon}{\delta_0}, \qquad |\rho(0)| < c_4 < K_\rho, \tag{4.2}$$

for fixed positive constants $c_1, \ldots, c_4$ and for some power $\frac{1}{2} < \beta < 1$.

The first condition in (4.2) restricts the set of initial conditions to those with small "oscillatory" components. These components give the fast-varying coordinates of trajectories passing close to $\mathcal{M}_\varepsilon$. The second condition in (4.2) means that the solution $u(t)$ enters U_0 close to the local stable manifold $W^s_{\text{loc}}(\mathcal{M}_\varepsilon)$ and hence stays near $\mathcal{M}_\varepsilon$ for a long time. We cannot track the fast oscillatory components on long time scales with great precision, but their norm turns out to remain small as long as it was small at $t = 0$. In particular, $u(t)$ exits U_0 through the domain $\partial_1 U_0$ of its boundary (see Figure 4.2).

LEMMA 4.1 *Suppose that for a solution $u(t) \in H^1$, the entry conditions in* (4.2) *are satisfied. Then for any fixed constant β with $\frac{1}{2} < \beta < 1$, there exist $\varepsilon_1 > 0$ and*

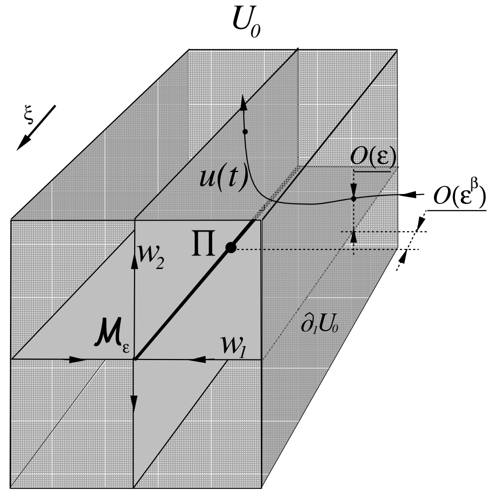

FIGURE 4.2. The geometry of the entry conditions.

$\delta_1 > 0$ *such that for all* $0 < \delta_0 < \delta_1$ *and* $0 < \varepsilon_0 < \varepsilon_1$ *there exists* $T^* > 0$ *with* $u(T^*) \in \partial_1 U_0$*. Moreover, the minimal such time* T^* *obeys the estimate*

$$T^* < T_\varepsilon = \frac{2}{\lambda} \log \frac{\delta_0^2}{c_2 \varepsilon} . \tag{4.3}$$

PROOF: The main ideas in the proof are the same as in Haller [7], where the same passage problem was addressed with finite-dimensional ζ-components. For this reason, we only sketch the proof and refer the reader to Haller [7, 8] for details.

We fix the constants B_ζ and α with $B_\zeta > c_1 > 0$ and $\beta < \alpha < 1$. By the continuity of the solution $u(t)$ in t, (4.2) implies the existence of a time $\bar{T} > 0$ such that for all $t \in [0, \bar{T})$, we have

$$\|\zeta(t)\|_{H^1} \le B_\zeta \varepsilon^\beta , \qquad |\rho(t)| \le K_\rho , \qquad |w_1(t) w_2(t)| \le \frac{c_3}{\delta_0} \varepsilon^\alpha . \tag{4.4}$$

Clearly, (4.4) implies $u(t) \in S$. For small enough $t > 0$, it also implies $u(t) \in U_0$ by the continuity of $u(t)$ in t. It is also clear that $\bar{T}$ can be slightly increased and (4.4) will still hold. Let $T^* > 0$ denote the time when $u(t)$ first intersects the boundary ∂U_0. One can easily see that $T^* < T_\varepsilon$ by assuming the contrary and observing that in that case $|w_2(T_\varepsilon)| > |w_{20}| \exp(\lambda T_\varepsilon / 2) > \delta_0$, which is a contradiction. We have to argue that $\bar{T}$ can in fact be increased up to T^* if we choose B_ζ, K_ρ, and α properly and keep ε small enough. We proceed by assuming that $\bar{T}$ cannot be increased to T^* for any choice of the constants in (4.4) and show that this leads to a contradiction.

Let us assume that for all fixed B_ζ, K_ρ, and α, there exists a time T_0 with $\bar{T} \le T_0 < T^*$ such that (4.4) holds for all $t < T_0$, but at least one of the inequalities is violated at $t = T_0$. We will consider these inequalities individually and argue that none of them can be violated at $t = T_0$. We note that since $T_0 < T^*$, we have $|w| < \sqrt{2}\delta_0$.

For all $0 \le t < T_0$, a Gronwall estimate on the third equation in (3.2) combined with (2.13) and (4.1) gives

$$\|\zeta(t)\|_{H^1} \le C_A\left[c_1\varepsilon^\beta + B\frac{c_3}{\delta_0}\varepsilon^\alpha T_\varepsilon\right]e^{2C_A B B_\zeta \sqrt{\varepsilon} T_\varepsilon} < 2ec_1C_A\varepsilon^\beta\,, \tag{4.5}$$

which, by the continuity of $\|\zeta(t)\|_{H^1}$, yields

$$\|\zeta(T_0)\|_{H^1} \le 2ec_1C_A\varepsilon^\beta < B_\zeta\varepsilon^\beta \tag{4.6}$$

for $B_\zeta = 7c_1C_A$. Hence the first inequality in (4.4) cannot be violated at $t = T_0$.

A direct estimate on the normal form (3.2) shows that the ρ-component of the solution obeys

$$|\rho(t)| < |\rho(0)| + \sqrt{\varepsilon}Bt < c_4 + \frac{2B}{\lambda}\sqrt{\varepsilon}\log\frac{\delta_0^2}{c_2\varepsilon} < c_4 + 1\,. \tag{4.7}$$

Thus, selecting $K_\rho = c_4 + 2$ and using the continuity of the function $\rho(t)$, we obtain that the second inequality in (4.4) cannot be violated either at $t = T_0$.

As far as the last inequality in (4.4), the normal form (3.2) yields the differential equation

$$\frac{d}{dt}(w_1w_2) = [(Y_1 + Y_4,\, w) + \langle Y_2 + Y_5,\, \zeta\rangle + \sqrt{\varepsilon}(Y_3 + Y_6)]w_1w_2. \tag{4.8}$$

Then a simple Gronwall estimate shows that

$$\begin{aligned}|w_1(t)w_2(t)| &\le c_3\varepsilon\exp\left\{2B\left[2\sqrt{2}\delta_0 + B_\zeta\varepsilon^\beta + \sqrt{\varepsilon}\right]T_\varepsilon\right\}\\ &< \frac{c_3\varepsilon}{\delta_0}\exp 2\delta_0 B[\sqrt{2}+1]T_\varepsilon\,,\end{aligned}$$

which implies that

$$|w_1(t)w_2(t)| < \frac{c_3}{\delta_0}\left(\frac{\delta_0^2}{c_2}\right)^{4B[\sqrt{2}+1]\frac{\delta_0}{\lambda}}\varepsilon^{1-4B[\sqrt{2}+1]\frac{\delta_0}{\lambda}} < \frac{c_3}{\delta_0}\varepsilon^\alpha \tag{4.9}$$

if we choose δ_0 small enough. Again, by continuity with respect to t, (4.9) implies

$$|w_1(T_0)w_2(T_0)| \le \frac{c_3\varepsilon^\alpha}{\delta_0}\,;$$

hence the last inequality in (4.4) cannot be violated at $t = T_0$ either. But this contradicts our original assumption on the time T_0 and proves the statement of the lemma. □

We are now in the position to study how the coordinates of passing trajectories change while they pass through the neighborhood U_0.

LEMMA 4.2 *Let us fix a constant* $\frac{1}{2} < \beta < 1$ *and assume that for* $0 < \varepsilon < \varepsilon_0$ *and* $0 < \delta_0 < \delta_1$, *the entry conditions* (4.2) *hold for a solution* $u(t) \in H^1$ *that enters the set* U_0 *at* $t = 0$ *and leaves it at* $t = T^*$. *Let us introduce the notation* $a = (w_{20}, \zeta_0, \rho_0, \psi_0)$ *and let* $u_0 = (\delta_0, a)$ *and* $u^* = u(T^*) = (w_1^*, \delta_0, \zeta^*, \rho^*, \psi^*)$ *define the coordinates of the solution at entry and departure, respectively. Then there exist constants* $K > 0$, $0 < \mu < \frac{1}{2}$, *and* $\delta_0^* > 0$, *and for any* $0 < \delta_0 < \delta_0^*$ *there exists* $\varepsilon_0^* > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0^*$ *the following estimates hold*:

(i)

$$|w_1^*| < K\varepsilon^\beta\,, \qquad \|\zeta^* - \zeta_0\|_{H^1} < K\varepsilon^\beta\,,$$
$$|\rho^* - \rho_0| < K\sqrt{\varepsilon}^\beta\,, \qquad |\psi^* - \psi_0| < K\sqrt{\varepsilon}^\beta\,.$$

(ii)

$$|D_a\, w_1^*| < K\varepsilon^\beta\,, \qquad \|D_a\,\zeta^* - (0,1,0,0)\|_{H^{-1}} < K\varepsilon^\mu\,,$$
$$|D_a\,\rho^* - (0,0,1,0)| < K\varepsilon^\mu\,, \qquad |D_a\,\psi^* - (0,0,0,1)| < K\varepsilon^\mu\,.$$

(iii)

$$|D_{\varepsilon^\mu} w_1^*| < K\varepsilon^\beta\,, \qquad \|D_{\varepsilon^\mu}\zeta^*\|_{H^{-1}} < K\varepsilon^\mu\,,$$
$$|D_{\varepsilon^\mu}\,\rho^*| < K\varepsilon^\mu\,, \qquad |D_{\varepsilon^\mu}\psi^*| < K\varepsilon^\mu\,.$$

PROOF: Again, most of the proof is similar to that of the analogous finite-dimensional results in Haller [7], so we only outline the main steps. From the normal form (3.2), we easily obtain the bounds

$$T_1 = \frac{1}{\lambda + 3\delta_0 B}\log\frac{\delta_0^2}{c_2\varepsilon} < T^* < T_2 = \frac{1}{\lambda - 3\delta_0 B}\log\frac{\delta_0^2}{c_2\varepsilon} \tag{4.10}$$

for any solution obeying conditions in (4.2). The normal form also provides us with the estimate

$$|w_1^*| = |w_1(T^*)| < |w_1(T_1)| < |w_{10}|\, e^{-(\lambda-3\delta_0 B)T_1} < \delta_0\left(\frac{\delta_0^2}{c_2}\right)^{\frac{\lambda-3\delta_0 B}{\lambda+3\delta_0 B}}\varepsilon^\beta \tag{4.11}$$

provided

$$\delta_0 < \frac{\lambda(1-\beta)}{3B(1+\beta)}\,. \tag{4.12}$$

Since we have shown in the proof of Lemma 4.1 that all the inequalities in (4.4) hold for $t \in [0, T^*]$, selecting $B_\zeta = 7c_1C_A$ (as in the proof of that lemma) and setting $t = T^*$, we obtain

$$\|\zeta^*\|_{H^1} < B_\zeta\varepsilon^\beta\,.$$

This inequality and (4.2) imply that

$$\|\zeta^* - \zeta_0\|_{H^1} \le \|\zeta^*\|_{H^1} + \|\zeta_0\|_{H^1} < (B_\zeta + 1)\varepsilon^\beta\,. \tag{4.13}$$

From the third equation of the normal form (3.2), we see that

$$|\rho^* - \rho_0| \leq \sqrt{\varepsilon} B T_\varepsilon < \frac{2B}{\lambda} \sqrt{\varepsilon}^\beta . \tag{4.14}$$

Finally, the last equation in (3.2), (4.1), and (4.4) yield the estimate

$$\begin{aligned} |\psi^* - \psi_0| &< \left[B_\zeta^2 B \varepsilon^{2\beta} + \sqrt{\varepsilon} B + \frac{B c_3}{\delta_0} \varepsilon^\alpha \right] T_\varepsilon \\ &< \frac{2B}{\lambda} \left[B_\zeta^2 + \frac{c_3}{\delta_0} + 1 \right] \sqrt{\varepsilon}^\beta , \end{aligned} \tag{4.15}$$

where we used (4.3). But then (4.11), (4.13), (4.14), and (4.15) show that statement (i) of the lemma is satisfied for a large enough constant K.

To prove statement (ii), we first need the variational equation associated with the normal form (3.2). We shall only sketch the estimates in (ii) for the derivatives of u^* with respect to ρ_0 that satisfy the equations

(4.16)

$$\begin{aligned} \frac{d}{dt} (D_{\rho_0} w_1) &= \left[-\lambda + (Y_1, w) + \langle Y_2, \zeta \rangle + \sqrt{\varepsilon} Y_3 \right] D_{\rho_0} w_1 \\ &+ \left[(DY_1 D_{\rho_0} u, w) + (Y_1, D_{\rho_0} w) + \langle DY_2 D_{\rho_0} u, \zeta \rangle \right. \\ &\quad \left. + \langle Y_2, D_{\rho_0} \zeta \rangle + \sqrt{\varepsilon} DY_3 D_{\rho_0} u \right] w_1 , \\ \frac{d}{dt} (D_{\rho_0} w_2) &= \left[\lambda + (Y_4, w) + \langle Y_5, \zeta \rangle + \sqrt{\varepsilon} Y_6 \right] D_{\rho_0} w_2 \\ &+ \left[(DY_4 D_{\rho_0} u, w) + (Y_4, D_{\rho_0} w) + \langle DY_5 D_{\rho_0} u, \zeta \rangle \right. \\ &\quad \left. + \langle Y_2, D_{\rho_0} \zeta \rangle + \sqrt{\varepsilon} DY_3 D_{\rho_0} u \right] w_2 , \\ \frac{d}{dt} (D_{\rho_0} \zeta) &= A D_{\rho_0} \zeta + (DZ_1 D_{\rho_0} u \, \zeta, \zeta) + (Z_1 D_{\rho_0} \zeta, \zeta) + (Z_1 \zeta, D_{\rho_0} \zeta) \\ &+ \sqrt{\varepsilon} \, DZ_2 D_{\rho_0} u \zeta + \sqrt{\varepsilon} Z_2 \, D_{\rho_0} \, \zeta + DZ_3 D_{\rho_0} u w_1 w_2 \\ &+ Z_3 D_{\rho_0} (w_1 w_2) , \\ \frac{d}{dt} (D_{\rho_0} \rho) &= \sqrt{\varepsilon} DE_3 D_{\rho_0} u , \\ \frac{d}{dt} (D_{\rho_0} \psi) &= \langle DF_1 D_{\rho_0} u \, \zeta, \zeta \rangle + \langle F_1 D_{\rho_0} \zeta, \zeta \rangle + \langle F_1 \zeta, D_{\rho_0} \zeta \rangle \\ &+ \sqrt{\varepsilon} DF_2 D_{\rho_0} u + DF_3 D_{\rho_0} u w_1 w_2 + F_3 D_{\rho_0} (w_1 w_2) . \end{aligned}$$

We select constants α, γ, μ, and ν with

$$0 < \mu < \nu < \frac{1}{2} < \gamma < \beta < \alpha < 1 . \tag{4.17}$$

Then, by the continuity of $D_{u_0}u(t)$ with respect to t, there exists a time $T_0 \leq T^*$ such that for all $t \in [0, T_0)$ and for $\varepsilon > 0$ sufficiently small,

$$\begin{gathered} \|D_{\rho_0}\zeta(t)\|_{H^1} \leq B_\zeta^{'}\varepsilon^\gamma\,, \qquad |D_{\rho_0}\rho(t) - 1| \leq K_\rho^{'}\varepsilon^\mu\,, \\ |D_{\rho_0}\psi(t)| \leq K_\psi^{'}\varepsilon^\mu, \end{gathered} \tag{4.18}$$

and

$$\begin{aligned} &|D_{\rho_0}[w_1(t)w_2(t)]| \leq K_0^{'}\varepsilon^\beta\,, \qquad &&|D_{\rho_0}w_1(t)| \leq K_w^{'}\varepsilon^\beta\,, \\ &|D_{\rho_0}w_2(t)| \leq K_w^{'}\varepsilon^{-\nu}\,, &&\|D_{\rho_0}u(t)\|_{H^1} \leq 2K_w^{'}\varepsilon^{-\nu}, \end{aligned} \tag{4.19}$$

with appropriate positive constants.

As in the proof of Lemma 4.1, we can show that none of the inequalities in (4.18) and (4.19) can be violated at $t = T_0 = T^*$ if we choose the constants properly. In analogy with the finite-dimensional case, the necessary estimates can be obtained by setting, e.g.,

$$\alpha = \frac{\beta+1}{2}\,, \qquad \gamma = \frac{2\beta+1}{4}\,, \qquad \nu = \beta(1-\beta)\,, \qquad \mu = \frac{1-\beta}{2}\,,$$

and selecting $\delta_0 > 0$ small enough (see Haller [7, 8] for details).

The proof of statement (iii) is based on similar estimates for the variational equation for the derivatives of $u(t)$ with respect to ε^μ. In particular, we can show that the inequalities

$$\|D_\varepsilon\zeta(t)\|_{H^{-1}} \leq \bar{B}_\zeta^{'}\varepsilon^\gamma\,, \qquad |D_\varepsilon\rho(t)| \leq \bar{K}_\rho^{'}\varepsilon^\mu\,, \qquad |D_\varepsilon\psi(t)| \leq \bar{K}_\psi^{'}\varepsilon^\mu\,,$$

and

$$\begin{aligned} &|D_\varepsilon[w_1(t)w_2(t)]| \leq \bar{K}_0^{'}\varepsilon^\beta\,, \qquad &&|D_\varepsilon w_1(t)| \leq \bar{K}_w^{'}\varepsilon^\beta\,, \\ &|D_\varepsilon w_2(t)| \leq \bar{K}_w^{'}\varepsilon^{-\nu}\,, &&\|D_\varepsilon u(t)\|_{H^{-1}} \leq 2\bar{K}_w^{'}\varepsilon^{-\nu}\,. \end{aligned}$$

continue to hold up to $t = T^*$, which then imply statement (iii) of the lemma. □

The last local estimate we need in our construction is concerned with the integral of the norm of the coordinates over the time of passage within the set U_0. Such quantities will be essential in our later estimates for the change of the energy of solutions.

LEMMA 4.3 *Let us fix the constant $\frac{1}{2} < \beta < 1$ and assume that for $0 < \varepsilon < \varepsilon_0$ and $\delta_0 < \delta_1$, the entry conditions* (4.2) *hold for a solution $u(t)$ that enters the set U_0 at $t = 0$ and leaves it at $t = T^*$. Then there exist constants $L > 0$ and $\delta_0^* > 0$, and for any $\delta_0 < \delta_0^*$ there exists $\varepsilon_0^* > 0$ such that for all $0 < \varepsilon < \varepsilon_0^*$ we have*

$$\begin{aligned} &\int_0^{T^*} \|\zeta(t)\|_{H^1}\,dt < L\sqrt{\varepsilon}\,, \qquad &&\int_0^{T^*} |w_1(t)|\,dt < L\delta_0\,, \\ &\int_0^{T^*} |w_2(t)|\,dt < L\delta_0\,, &&\int_0^{T^*} |\rho(t)|\,dt < L\varepsilon^\mu\,, \end{aligned} \tag{4.20}$$

where $\mu = (1-\beta)/2$.

PROOF: From the normal form (3.2), (4.9), and (4.10) we obtain that for $t \in [0, T^*]$,

$$\begin{aligned} |\zeta(t)| &\leq \left|e^{At}\zeta_0\right| + \int_0^t \left|e^{A(t-\tau)}\left((Z_1\zeta,\zeta) + \sqrt{\varepsilon}Z_2\zeta + Z_3w_1w_2\right)\right| d\tau \\ &< C_A\left(B_\zeta\varepsilon^\beta + B_0c_3\varepsilon^\alpha\frac{1}{\lambda - 3\delta_0B_0}\log\frac{\delta_0}{c_2\varepsilon}\right) \\ &\quad + \int_0^t 2\delta_0C_AB_0\,|\zeta(\tau)|\,d\tau\,, \end{aligned}$$

which, by Gronwall's inequality, implies

$$\begin{aligned} |\zeta(t)| &< C_A\left(B_\zeta\varepsilon^\beta + B_0c_3\varepsilon^\alpha\frac{1}{\lambda - 3\delta_0B_0}\log\frac{\delta_0}{c_2\varepsilon}\right)e^{2\delta_0C_AB_0t} \\ &< 2C_AB_\zeta\varepsilon^\beta e^{2\delta_0C_AB_0t}\,, \end{aligned}$$

since $\alpha > \beta$. Consequently, we have

$$\begin{aligned} \int_0^{T^*} |\zeta(t)|\ dt &< \frac{B_\zeta\varepsilon^\beta}{\delta_0B_0}\left[\exp\left(\frac{2\delta_0C_AB_0}{\lambda - 3\delta_0B_0}\log\frac{\delta_0}{c_2\varepsilon}\right) - 1\right] \\ &< \frac{2B_\zeta}{\delta_0B_0}\varepsilon^{\beta - \frac{2\delta_0C_AB_0}{\lambda - 3\delta_0B_0}} < \frac{2B_\zeta}{\delta_0B_0}\sqrt{\varepsilon} \end{aligned}$$

for $\delta_0, \varepsilon > 0$ small enough, which proves the first inequality in (4.20). The remaining three estimates follow similarly from the normal form (3.2) (see Haller [7] for more details). □

5 Local and Global Maps

Lemma 4.2 shows that the "local map" $u_0 \mapsto u^*(u_0)$, as well as its partial derivatives with values in H^{-1}, remain bounded as $\varepsilon \to 0$. This enables us to extend the local map to the limit $\varepsilon = 0$ so that the extension is differentiable in ε^μ at $\varepsilon = 0$. This extension will be useful later when we construct multi-pulse orbits for the perturbed NLS equation using an implicit function theorem argument near $\varepsilon = 0$. The smooth extension of the local map is needed as we will need smoothness of the constructed solutions in ε.

To elaborate on the above idea of extension, we introduce the set

$$\begin{aligned} \mathcal{L}_\varepsilon = \{(w,\zeta,\rho,\psi) \in \partial_1U_0 \cap W^u(\Pi) \mid |w_1| = \delta_0\,, \\ \frac{c_2\varepsilon}{\delta_0} \leq |w_2| \leq \frac{c_3\varepsilon}{\delta_0},\ \|\zeta\|_{H^1} \leq c_1\varepsilon^\beta,\ |\rho| \leq c_4\}\,. \end{aligned} \tag{5.1}$$

Note that $\mathcal{L}_\varepsilon$ is a subset of the unstable manifold $W^u(\Pi)$ whose points satisfy the entry conditions in (4.2).

For positive ε, $\mathcal{L}_\varepsilon$ is the disjoint union of two-dimensional manifolds, and these manifolds degenerate into the single manifold

$$\mathcal{L}_0 = \partial_1U_0 \cap W^s_{\text{loc}}(\Pi)$$

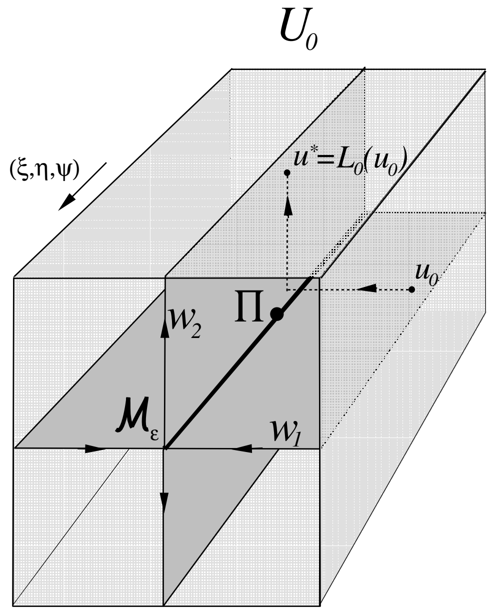

FIGURE 5.1. The meaning of the extension L_0 of the local map.

for $\varepsilon = 0$. For $\varepsilon > 0$, we define the *local map* $L_\varepsilon \colon \mathcal{L}_\varepsilon \to \partial_1 U_0$ as

$$L_\varepsilon(\delta_0, w_{20}, \zeta_0, \rho_0, \psi_0) = (w_1^*, \delta_0, \zeta^*, \rho^*, \psi^*) \tag{5.2}$$

(see Lemma 4.2 for notation). The map L_ε is of class C^1 with values in H^{-1}. For $\varepsilon \geq 0$ we now define the extension map $L_0 : \mathcal{L}_\varepsilon \to \partial_1 U_0$ as

$$L_0(\delta_0, w_{20}, \zeta_0, \rho_0, \psi_0) = (0, \delta_0, \zeta_0, \rho_0, \psi_0)\,.$$

This map projects any point to the manifold $W^s_{\mathrm{loc}}(\mathcal{C})$ and then maps the projection along an unstable fiber to the intersection of the fiber with $\partial_1 U_0$, as shown in Figure 5.1. Clearly, L_0 is a smooth map with values in H^1. In addition, we have the following result:

COROLLARY 5.1 *For $\varepsilon > 0$ small enough and for fixed $\frac{1}{2} < \beta < 1$ in the entry conditions* (4.2)*, there exists $0 < \mu < \frac{1}{2}$ such that the local map can be written as*

$$L_\varepsilon(u_0) = L_0(u_0) + \varepsilon^\mu L_1(u_0, \varepsilon^\mu)\,,$$

where L_1 is C^1 in its arguments with values in H^{-1} and $L_1(u_0; 0) = 0$.

The statement of this corollary follows directly from Lemma 4.2, since the manifold $\mathcal{L}_\varepsilon$ is finite-dimensional; hence the solution-dependent constants K and μ appearing in the statement of the lemma can be chosen uniformly for $u_0 \in \mathcal{L}_\varepsilon$.

REMARK 5.2 It is also easy to see from (5.2) that the map L_0 is C^1 in δ_0 in a neighborhood of $\delta_0 = 0$. In this limit, the domain of L_0 becomes $\mathcal{L}_0 = \Pi$.

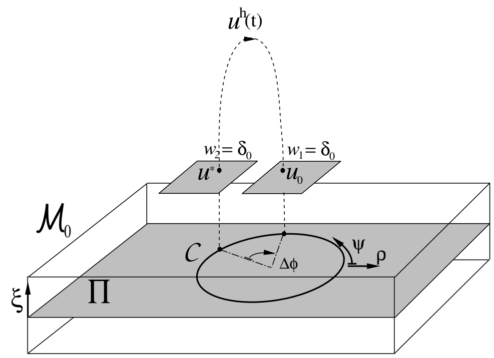

FIGURE 5.2. The definition of the extension G_0 of the global map.

We also want to follow initial conditions as they leave the box U_0 and then return. To this end, we define the domain

(5.3) $$\mathcal{G}_\varepsilon = \{(w, \zeta, \rho, \psi) \in \partial_1 U_0 \cap W^u(\Pi) \mid |w_2| = \delta_0,\ \|\zeta\|_{H^1} \leq K\varepsilon^\beta\}$$

and the *global map* $G_\varepsilon \colon \mathcal{G}_\varepsilon \to \partial_1 U_0$ as

(5.4) $$G_\varepsilon(w_1^*, \delta_0, \zeta^*, \rho^*, \psi^*) = (\delta_0, w_{20}, \zeta_0, \rho_0, \psi_0)\,.$$

The constant $K > 0$ in the definition of $\mathcal{G}_\varepsilon$ is the same as in statement (i) of Lemma 4.2. An expression for the global map is given below.

LEMMA 5.3 *For $\varepsilon \geq 0$ and for all sufficiently small $\delta_0 \geq 0$, the global map can be written as*

$$G_\varepsilon(u^*) = u^* + \Delta\phi + \delta_0 G_1(u^*, \delta_0) + \sqrt{\varepsilon}G_2(u^*, \varepsilon)\,,$$

where G_j are C^1 in their arguments with values in H^{-1}, and the phase shift $\Delta\phi$ is defined in (2.15).

PROOF: Note that the map $G_0 : \mathcal{G}_0 \to \Pi$ remains well-defined in the limit $\delta_0 = 0$ with domain $\mathcal{G}_0 = \Pi$. This map maps the α-limit points of unperturbed heteroclinic orbits in $W^u(\mathcal{C}) \equiv W^s(\mathcal{C})$ to their ω-limit points. Therefore, for $\delta_0 = 0$, we have $G_0(u^*) = u^* + \Delta\phi$. For $\delta_0 > 0$, G_0 maps the first intersections of solutions in the homoclinic manifolds $W_0^\pm(\mathcal{C})$ with ∂U_0 to their second intersections with ∂U_0 (see Figure 5.2). But these solutions are just the unperturbed fibers in $W_{\text{loc}}^{s,u}(\mathcal{C})$, and fibers depend smoothly on their base points; thus we obtain that

(5.5) $$G_0(u^*) = u^* + \Delta\phi + \delta_0 G_1(u^*, \delta_0)\,.$$

By the properties of the underlying flow, the global map $G_\varepsilon(u^*)$ is smooth in the initial condition u^* and the parameter $\sqrt{\varepsilon}$ with values in H^{-1}. We finally observe that initial conditions in the domain of G_ε are at most $\mathcal{O}(\varepsilon^\beta)$ (with $\beta > \frac{1}{2}$) away from $\mathcal{G}_0$, and the magnitude of the perturbation in the normal (3.2) is of order $\mathcal{O}(\sqrt{\varepsilon})$. These facts together with (5.5) complete the proof of the lemma. □

6 Energy Estimates

In this section we study how the value of the Hamiltonian $H = H_0 + \varepsilon H_1$ changes on solutions as they repeatedly approach and leave a vicinity of the manifold $\mathcal{M}_\varepsilon$. Later we will use the "energy" H and the variables (w_2, ζ, ρ, ψ) to track solutions that visit the neighborhood U_0. This tracking of solutions will be the key tool in constructing multipulse solutions homoclinic to $\mathcal{M}_\varepsilon$. Since the Hamiltonians H_0 and H_1 are defined for the original complex evolution equation (2.1), we need to evaluate them at complex conjugate pairs of points. In our notation, we only emphasize the dependence of these functions on $u \in H^1_{\mathbb{C}}$; i.e., we use the notation $F(u) \equiv F(u, \bar{u})$ for functions of u and $\bar{u}$. Our notation for the derivative and the gradient of such functions is, respectively,

$$DF \equiv (D_u F, D_{\bar{u}} F), \qquad \nabla F \equiv (\nabla_u F, \nabla_{\bar{u}} F).$$

Accordingly, for a vector $A = (a, \bar{a}) \in H^1_{\mathbb{C}} \times H^1_{\mathbb{C}}$, we use the shorthand notation

$$\langle \nabla F, A \rangle = \langle \nabla_u F, a \rangle + \langle \nabla_{\bar{u}} F, \bar{a} \rangle .$$

We now prove our main energy estimate for solutions that lie in the unstable manifold $W^u(\Pi)$ and obey the entry conditions (4.2).

LEMMA 6.1 *Suppose that $u(t)$ is a solution of the normal form* (3.2) *that lies in the unstable manifold of the invariant plane Π. Let q_0 be the first intersection of $u(t)$ with the surface $\partial_1 U_0$, and let $b_\varepsilon = b_0 + (0, \sqrt{\varepsilon}\eta) \in \Pi$ with $b_0 = (\phi_0, 0) \in \mathcal{C}$ be the base point of the unstable fiber $f^u(b_\varepsilon)$ that contains the point q_0. Suppose that the solution returns to $\partial_1 U_0$ N times to intersect it at the points $p_1, \dots, p_N$ and to leave it again at the points $q_1, \dots, q_{N-1}$. Assume further that, for some constants $\frac{1}{2} < \beta < 1$, $0 < \varepsilon < \varepsilon_0$, and $\delta_0 < \delta_1$, the entry conditions* (4.2) *hold for the solution $u(t)$ at each entry point p_k.* (*For $N = 1$, $c_2 = 0$ is allowed in* (4.2).)

Then, for $\delta_0, \varepsilon > 0$ sufficiently small, we have

$$H(p_N) = H_0 \mid \mathcal{C} + \varepsilon \left[\mathcal{H}(b_0) + N\mathcal{I} + \mathcal{O}(\delta_0, \varepsilon^\mu) \right],$$

where $0 < \mu < \frac{1}{2}$ and the quantity $\mathcal{I}$ is defined in (2.16). *The "slow" Hamiltonian $\mathcal{H}$ is the first-order term in the expansion of $(H_0 + \varepsilon H_1) \mid \Pi$ near the circle $\mathcal{C}$, i.e.,*

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} D_I^2 H_{01}(\Pi)\Big|_{\mathcal{C}} \eta^2 + H_1 \mid \mathcal{C} \\ &= -\eta^2 + 2\Gamma\Omega \sin\phi . \end{aligned} \tag{6.1}$$

PROOF: We can write the energy $H(p_N)$ as

$$\begin{aligned} H(p_N) = H(b_\varepsilon) + [H(q_0) - H(b_\varepsilon)] + \sum_{l=1}^{N-1} H(q_l) - H(p_l) \\ + \sum_{l=1}^{N} H(p_l) - H(q_{l-1}) . \end{aligned} \tag{6.2}$$

The first sum in this expression is the change in the energy of $u(t)$ during local passages near $\mathcal{M}_\varepsilon$, while the second sum is the change of energy outside U_0. We now estimate the four main terms in (6.2) separately.

The first term can be written as

$$H(b_\varepsilon) = (H_0 + \varepsilon H_1)\,|_{b_\varepsilon} = H_0 \mid \mathcal{C} + \varepsilon \mathcal{H}(b_0) + \mathcal{O}(\varepsilon^{3/2}), \tag{6.3}$$

since $\nabla H_0 = 0$ on the circle $\mathcal{C}$ for $\varepsilon = 0$.

To estimate the second term in (6.2), we consider the "Hamiltonian" unstable fiber $f^u_{g=0}(b_\varepsilon)$ (i.e., a fiber for the case of $g = 0$), which intersects the surface $\partial_1 U_0$ at a point q_H. By the continuity of the Hamiltonian H, for zero dissipation, orbits asymptoting to each other must have the same energy. As a result, by Theorem 3.2, we must have $H(q_H) = H(b_\varepsilon)$ for $g \equiv 0$. If $q_c \in \mathcal{C}$ is the projection of the point q_H on the circle $\mathcal{C}$, then we have $\nabla H_0(q_c) = 0$, and the mean value theorem gives

$$\begin{aligned}
|H(q_0) - H(b_\varepsilon)| &= |H(q_0) - H(q_H)| = |DH(q_*) \cdot (q_0 - q_H)| \\
&= |(DH_0(q_*) + \varepsilon DH_1(q_*)) \cdot (q_0 - q_H)| \\
&= |(DH_0(q_* - q_c) + \varepsilon DH_1(q_*)) \cdot (q_0 - q_H)| \\
&\leq \|DH_0(q_* - q_c) + \varepsilon DH_1(q_*)\|_{H^{-1}} \|q_0 - q_H\|_{H^1} \\
&\leq \big(\,(1 + K_{01})\,\|q_* - q_c\|_{H^1} + \varepsilon K'\big)\|q_0 - q_H\|_{H^1}\,,
\end{aligned} \tag{6.4}$$

where the point q_* lies on the line connecting q_0 and q_H, $K' > 0$ is an upper bound for $\|DH_1\|_{H^1}$ within the cylinder S, $K_{01} > 0$ is an upper bound on $\|DH_{01}\|_{H^1}$, and we used the inequality (2.5). Since the unstable fibers are of class C^r in the parameter ε, and the H^1-distance of the point q_* from the circle $\mathcal{C}$ is less than δ_0, we have the estimates

$$\|q_0 - q_H\|_{H^1} < K_1 \varepsilon\,, \qquad \|q_* - q_c\|_{H^1} < \delta_0\,,$$

for some constant K_1. Therefore, the inequality (6.4) can be rewritten as

$$|H(q_0) - H(b_\varepsilon)| < (1 + K_{01} + K')K_1 \delta_0 \varepsilon\,. \tag{6.5}$$

To estimate the third term in (6.2), we recall that the solution $u(t)$ is of class C^1 in t for initial data $p_l \in H^3_{\mathbb{C}}$. For such initial values we can write

$$\begin{aligned}
\sum_{l=1}^{N-1} H(q_l) - H(p_l) &= \sum_{l=1}^{N-1} \int_0^{T_l^*} \dot{H}(u(t))dt \\
&= \sum_{l=1}^{N-1} \int_0^{T_l^*} DH \cdot (iJ\nabla H + \varepsilon G)_{u(t)}\, dt \\
&= \varepsilon \sum_{l=1}^{N-1} \int_0^{T_l^*} \langle \nabla H_0 + \varepsilon \nabla H_1, G\rangle_{u(t)}\; dt \\
&= \varepsilon \sum_{l=1}^{N-1} \int_0^{T_l^*} \langle \nabla H_0, G\rangle_{u(t)}\; dt + \mathcal{O}\Big(\varepsilon^2 \log \frac{1}{\varepsilon}\Big).
\end{aligned} \tag{6.6}$$

Here T_l^* denotes the time of flight for the solution $u(t)$ from the point p_l to q_l, and hence it obeys the estimate (4.3). We also used the relation $\langle \nabla H, iJ\nabla H\rangle = 0$ with the matrix J defined as

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now observe that $\sum_{l=1}^{N-1} H(q_l) - H(p_l)$ is a continuous function of the initial conditions p_1, and hence by the denseness of $H^3_{\mathbb{C}}$ in the phase space $H^1_{\mathbb{C}}$, (6.6) holds for any solution $u(t)$ with initial data in $H^1_{\mathbb{C}}$ (see the proof of Lemma 3.4 for the details of a similar argument).

We now estimate the terms in the integrand on the right-hand side of (6.6). Noting that $\nabla H_0 \mid \mathcal{C} = 0$, we obtain that if (w, ζ, ρ, ψ) are the coordinates of a point $p \in S$, then

$$\begin{aligned} \nabla H_0(p) = A_1(w,\zeta,\rho,\psi)w_1 + A_2(w,\zeta,\rho,\psi)w_2 + A_3(w,\zeta,\rho,\psi)\zeta \\ + A_4(w,\zeta,\rho,\psi)\rho \end{aligned} \tag{6.7}$$

for appropriate C^{r-1} functions A_i. Using (2.5), Lemma 4.3, and (6.7), we obtain

$$\sum_{l=1}^{N-1} \int_0^{T_l^*} \langle \nabla H_0, G\rangle_{u(t)}\, dt = \mathcal{O}(\delta_0) + \mathcal{O}(\varepsilon^{\mu}). \tag{6.8}$$

This last equation and the energy expression (6.6) shows

$$\sum_{l=1}^{N-1} H(q_l) - H(p_l) = \mathcal{O}(\varepsilon\delta_{0,}\varepsilon^{1+\mu}). \tag{6.9}$$

To complete the proof of the lemma, it remains to estimate the last sum in the expression (6.2). Standard "finite-time-of-flight" Gronwall estimates imply that outside the fixed neighborhood U_0 of the manifold $\mathcal{M}_\varepsilon$, the perturbed solutions remain close to a chain of unperturbed solutions $u^l(t)$, $l = 1, \ldots, N$, with

$$\lim_{t\to-\infty} u^1(t) = b_0\,, \quad \lim_{t\to+\infty} u^{l-1}(t) = \lim_{t\to-\infty} u^l(t)\,, \quad l = 2, \ldots, N. \tag{6.10}$$

(The uniform upper bound for these flight times can be obtained by restricting to compact subsets of $W^u(\Pi)$.) Since the size of U_0 is of order $\mathcal{O}(\delta_0)$, we can compute the change in energy between the points q_{l-1} and p_l in the same way as in equation (6.6) for initial conditions in $H^1_{\mathbb{C}}$. We then obtain

$$\sum_{l=1}^{N} H(p_l) - H(q_{l-1}) = \varepsilon \sum_{l=1}^{N} \int_{-\infty}^{\infty} \langle \nabla H_0, G\rangle\,|_{u^l(t)}\, dt + \mathcal{O}(\varepsilon\delta_0). \tag{6.11}$$

Again, the denseness of $H^3_{\mathbb{C}}$ in $H^1_{\mathbb{C}}$ and the continuity of the above expression allows us to conclude that (6.11) holds for arbitrary initial conditions. But (6.2), (6.3), (6.9), and (6.11) together imply the statement of the lemma since $\mathcal{I} = \int_{-\infty}^{\infty} \langle \nabla H_0, G\rangle\,|_{u^l(t)}\, dt$ is independent of the choice of the solution $u^l(t)$, as we observed after formula (2.16). □

We now derive an estimate for the energy of a point $s_N \in W^s_{\text{loc}}(\mathcal{M}_\varepsilon) \cap \partial_1 U_0$ that has the same (ζ, ρ, ψ)-coordinates as the point p_N on the incoming solution $u(t)$ (see Figure 6.1). This estimate will be important when we compute the en-

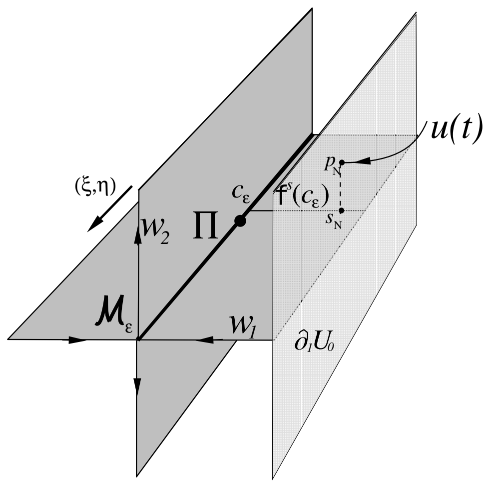

FIGURE 6.1. The definition of the points s_N and c_ε.

ergy difference between the entry point p_N and its projection s_N on the unstable manifold of $\mathcal{M}_\varepsilon$.

LEMMA 6.2 *Suppose that $u(t)$ is a solution of the normal form, and let the points $p_1, \dots, p_N$ and $q_0, \dots, q_{N-1}$ be defined as in Lemma 6.1. Suppose that the assumptions of that lemma hold and $c_\varepsilon \in \mathcal{M}_\varepsilon$ is the base point of a stable fiber $f^s(c_\varepsilon)$ such that for the point $s_N = f^s(c_\varepsilon) \cap \partial_1 U_0$,*

$$(\zeta_{p_N}, \rho_{p_N}, \psi_{p_N}) = (\zeta_{s_N}, \rho_{s_N}, \psi_{s_N}) \,. \tag{6.12}$$

Then, for the energy of the point s_N, we have the expression

$$H(s_N) = H_0 \mid \mathcal{C} + \varepsilon \mathcal{H}(b_0 + N\Delta\phi) + \mathcal{O}\Big(\varepsilon\delta_0, \varepsilon^{1+\frac{\beta}{2}}\Big) \,, \tag{6.13}$$

where $\Delta\phi$ is defined in (2.15) *and $\mathcal{H}$ is defined in* (6.1).

PROOF: Since the entry conditions (4.2) are assumed to hold for the incoming solution $u(t)$, equation (6.12) implies that for the stable fiber $f^s(c_\varepsilon)$ containing s_N we have

$$\|\zeta_{c_\varepsilon}\|_{H^1} < K_7 \varepsilon^\beta. \tag{6.14}$$

Since s_N lies at a distance of order $\mathcal{O}(\delta_0)$ from the invariant manifold $\mathcal{M}_\varepsilon$, by the smoothness of individual stable fibers we have

$$(\eta_{c_\varepsilon}, \phi_{c_\varepsilon}) = (\eta_{s_N}, \phi_{s_N}) + \mathcal{O}(\delta_0) \,. \tag{6.15}$$

We now relate the energy of the base point c_ε to the energy of the point s_N. Let the point s_H be the intersection of the "Hamiltonian" fiber $f^s_{g=0}(c_\varepsilon)$ with the surface $\partial_1 U_0$. Then, applying the mean value inequality as in (6.4) with some point s_* lying on the line connecting s_N and s_H, we can write

$$\begin{aligned}|H(s_N) - H(c_\varepsilon)| &= |H(s_N) - H(s_H)| \\ &= |DH(s^*)\cdot(s_N - s_H)| < (1 + K_{01} + K')\delta_0 K_8 \varepsilon\,.\end{aligned}$$

From this inequality we obtain that

$$H(s_N) = H(c_\varepsilon) + \mathcal{O}(\delta_0\varepsilon)\,. \tag{6.16}$$

Hence, to find an approximation for the energy of the point s_N, we have to compute the energy of the fiber base point c_ε. The restricted Hamiltonian $\mathcal{H}_\varepsilon = H \mid \mathcal{M}_\varepsilon$ is easily found to be of the form

$$\mathcal{H}_\varepsilon = H \mid \mathcal{M}_\varepsilon = H_0 \mid \mathcal{C} + \varepsilon\mathcal{H} + \mathcal{O}\Big(\varepsilon\|z\|_{H^1}, \|z\|^2_{H^1}, \varepsilon^{\frac{3}{2}}\Big) \tag{6.17}$$

with the slow Hamiltonian $\mathcal{H}$ defined in (6.1).

Since the solution $u(t)$ travels for an $\mathcal{O}(1)$ amount of time near the set of trajectories described in (6.10), we know that the point q_0 is $\mathcal{O}(\sqrt{\varepsilon})$-close to the unperturbed solution $u^1(t)$, and the point p_N is $\mathcal{O}(\sqrt{\varepsilon}^\beta)$-close to the unperturbed solution $u^N(t)$. Since $u^N(t)$ locally coincides with an unperturbed stable fiber, the smoothness of fibers implies that the base point c_ε of the fiber containing q_N is H^1 $\mathcal{O}(\sqrt{\varepsilon}^\beta)$-close to the unperturbed fiber base point $\lim_{t\to\infty} u^N(t)$ (see Figure 6.1). As a result, we obtain

$$c_\varepsilon = b_0 + N\Delta\phi + \mathcal{O}\Big(\sqrt{\varepsilon}^\beta\Big). \tag{6.18}$$

But equation (6.18) with (6.14) and (6.17) gives

$$H(c_\varepsilon) = \mathcal{H}_\varepsilon(c_\varepsilon) = H_0 \mid \mathcal{C} + \varepsilon\mathcal{H}(b_0 + N\Delta\phi) + \mathcal{O}\Big(\varepsilon^{1+\frac{\beta}{2}}, \varepsilon^{2\beta}, \varepsilon^{\frac{3}{2}}\Big),$$

which implies the statement of the lemma. It remains to note that all the constants in the above estimates can be chosen uniformly if we restrict to initial conditions in a compact subset of the finite-dimensional manifold $W^u(\Pi)$. ☐

We are now in a position to strengthen Lemma 4.2 on the coordinates of the solution $u(t)$ upon its exit from the set U_0. The improvement is the fact that the local map L_ε is actually C^0 $\mathcal{O}(\varepsilon)$-close to L_0 if we just consider the w_1- and ζ-coordinates of the image.

LEMMA 6.3 *If the solution $u(t)$ is contained in the manifold $W^u(\Pi)$, then statement* (i) *of Lemma 4.2 can be strengthened to*

$$\begin{aligned}&|w_1^*| < K\varepsilon\,, && \|\zeta^*\|_{H^1} < K\varepsilon^\beta\,,\\ &|\rho^* - \rho_0| < K\sqrt{\varepsilon}^\beta\,, && |\psi^* - \psi_0| < K\sqrt{\varepsilon}^\beta\,.\end{aligned}$$

PROOF: Consider the point $q^* \in W^u_{\rm loc}(\Pi)$ for which $w_{1q^*} = 0$, $\zeta_{q^*} = 0$, and $(\rho_{q^*}, \psi_{q^*}) = (\rho_{u^*}, \psi_{u^*})$ hold. By the first inequality in (i) of Lemma 4.2, the points q^* and u^* are $\mathcal{O}(\varepsilon^\beta)$-close. To determine the energy of the point q^*, we consider the unstable fiber $f^u(b^*)$ that contains q^*. For zero dissipation ($g \equiv 0$), the energy of the base point b^* of the fiber $f^u_{g=0}(b^*)$ can be written in the form $H(b^*) = H_0 \mid \mathcal{C} + \mathcal{O}(\varepsilon)$, where we used the expression (6.17). Since the energy is constant on fibers for $g \equiv 0$, we immediately obtain

$$H(q^*) = H_0 \mid \mathcal{C} + \mathcal{O}(\varepsilon)\,. \tag{6.19}$$

Equation (6.19) remains unchanged for nonzero dissipation, since unstable fibers are perturbed by an amount of order $\mathcal{O}(\varepsilon)$ when we add the dissipative terms. For $u^* = q_1$, we obtain from Lemma 6.1 that

$$H(u^*) = H_0|\mathcal{C} + \mathcal{O}(\varepsilon)\,. \tag{6.20}$$

Then (6.19) and (6.20) together with the mean value theorem give

$$\begin{aligned} K_{10}\varepsilon > |H(q^*) - H(u^*)| &= \left| DH(\hat{q}(u^*,\varepsilon)) \cdot \frac{q^* - u^*}{\|q^* - u^*\|_{H^1}} \right| \|q^* - u^*\|_{H^1} \\ &= |\langle \nabla H(\hat{q}(u^*,\varepsilon)), E(u^*,\varepsilon)\rangle| \, \|q^* - u^*\|_{H^1}\,, \end{aligned} \tag{6.21}$$

where $\hat{q}(u^*,\varepsilon)$ is a point on the line connecting the points q^* and u^*, and $E(u^*,\varepsilon) = (e(u^*,\varepsilon), \bar{e}(u^*,\varepsilon))$ with $e(u^*,\varepsilon)$ being a unit vector on that line. Since the representation of $E(u^*,\varepsilon)$ in the $(w_1, w_2, \zeta, \rho, \psi)$-coordinates is just $(0,1,0,0)$, using equation (1.1), we have

$$\begin{aligned} |\langle \nabla H(\hat{q}), E(u^*,\varepsilon)\rangle| &= |\langle -i\,(u_t|_{s_\varepsilon} - \varepsilon g)\,, E(u^*,\varepsilon)\rangle| \\ &\geq |\dot{w}_1|\,|_{\hat{q}} - \varepsilon \sup_{U_k} \|g\|_{H^1} \\ &\geq |\dot{w}_1|\,|_{\hat{q}} - \varepsilon K_g\,, \end{aligned} \tag{6.22}$$

where K_g is an upper bound on g on the set U_0. Here we used the real coordinate representation of u_t from equation (3.2). From (3.2) we also have the estimate

$$\begin{aligned} |\dot{w}_1|\,|_{\hat{q}} &\geq \left[|\lambda - B_0(|w| + \|\zeta\|_{H^1} + \sqrt{\varepsilon})|\,|w_1|\right]|_{\hat{q}} \\ &\geq \left|\lambda - B_0\left(\delta_0 \frac{\sqrt{2}}{2} + \delta_0 + \sqrt{\varepsilon}\right)\right| \delta_0 \\ &\geq \delta_0 \frac{\lambda}{2}\,. \end{aligned}$$

Therefore, we can find a uniform lower bound $K_{11} > 0$ such that (6.21) can be rewritten as

$$|\langle \nabla H(\hat{q}(u^*,\varepsilon), E(u^*,\varepsilon)\rangle| > K_{11} - \varepsilon^\mu K > \frac{K_{11}}{2}\,.$$

This inequality and (6.21) show that

$$\|q^* - u^*\|_{H^1} < \frac{2K_{10}}{K_{11}}\varepsilon\,, \tag{6.23}$$

which in turn gives

$$|w_1^*| < K_{12}\varepsilon\,, \tag{6.24}$$

since $w_{1q^*} = 0$. The remaining inequalities are just restatements of the results listed in Lemma 4.2. □

A solution $u(t) \in W^u(\Pi)$ is homoclinic to the manifold $\mathcal{M}_\varepsilon$ if the points $p_N \in W^u(\Pi)$ and $s_N \in W^s_{\text{loc}}(\mathcal{M}_\varepsilon)$ coincide. By construction, these points have the same w_1-, ζ-, ρ-, and ψ-coordinates, so they coincide if their w_2-coordinates are equal. However, following the evolution of the w_2-coordinate along solutions is not possible since w_2 is only defined near $\mathcal{M}_\varepsilon$. Instead, we show that the w_2-coordinate of p_N can be uniquely determined as a function of the other coordinates and $H(p_N)$. This fact will enable us to find orbits by solving the equation

$$H(p_N) - H(s_N) = 0\,. \tag{6.25}$$

LEMMA 6.4 *Suppose that the conditions of Lemma 6.1 are satisfied. Then for* $\varepsilon > 0$ *small enough there exists a* C^1*-function* $f_\varepsilon : H^1 \times \mathbb{R} \times S^1 \times \mathbb{R} \mapsto \mathbb{R}$ *such that for any* $l = 1, \dots, N$,

$$w_{2p_l} = f_\varepsilon\left(\zeta_{p_l}, \rho_{p_l}, \psi_{p_l}, H(p_l)\right).$$

PROOF: The surface $\{w_1 = \delta_0\}$ satisfies $u = s_\varepsilon(w_2, \zeta, \rho, \psi)$, where s_ε is a C^r-embedding into the phase space $\mathcal{P}$. The intersection of the energy surface $\{H(u, \bar{u}) = h\}$ with $\{w_1 = \delta_0\}$ satisfies the equation

$$H(S_\varepsilon(w_2, \zeta, \rho, \psi)) - h = 0$$

where $S_\varepsilon = (s_\varepsilon, \bar{s}_\varepsilon)$. On this intersection set, the coordinate w_2 is a C^1-function of the rest of the coordinates and the energy h provided

$$\langle \nabla H(S_\varepsilon(w_2, \zeta, \rho, \psi)), D_{w_2} S_\varepsilon(w_2, \zeta, \rho, \psi)\rangle \neq 0 \tag{6.26}$$

holds at the points of intersection. We want to see if this equation is satisfied at the point p_l. Since $p_l \to s_l$ as $\varepsilon \to 0$, and p_l is contained in a compact subset of $W^u(\Pi)$, it is enough to verify that

$$|\langle \nabla H_0(s_l), D_{w_2} S_0(w_{2s_l}, \zeta_{s_l}, \rho_{s_l}, \psi_{s_l})\rangle| > c_l \tag{6.27}$$

for some constant $c_l > 0$. But $D_{w_2} S_0(w_{2s_l}, \zeta_{s_l}, \rho_{s_l}, \psi_{s_l})$ lies in the tangent space of $\partial_1 U_0$, so (6.27) follows from the same argument that we used to give a lower bound for the expression in (6.21). □

7 The Energy Function

As we noted in the previous section, a solution $u(t)$ that approaches the plane Π and the manifold $\mathcal{M}_\varepsilon$ in backward and forward time, respectively, must necessarily satisfy the energy equation (6.25). Lemmas 6.1 and 6.2 show that at leading order, the left-hand side of this equation is given by the N^{th}-order *energy function*

$$\begin{aligned}\Delta^N \mathcal{H}(\phi) &= \mathcal{H}(\eta, \phi + N\Delta\phi) - \mathcal{H}(\eta, \phi) - N\mathcal{I} \\ &= 2\Omega\Gamma[\sin(\phi + N\Delta\phi) - \sin\phi] - N\mathcal{I}\,. \end{aligned} \tag{7.1}$$

This function $\Delta^N\mathcal{H}$ is C^1, which can be seen as follows: The solution $u^h(t)$ is C^r with respect to the ϕ-coordinate of its backward limit point u. This follows by picking the initial condition $u_0^h \in \partial U_0 \cap W^u_{\text{loc}}(\Pi)$ so that $u^h(0) = u_0^h$ and recalling the smoothness of unperturbed unstable fibers in $W^u_{\text{loc}}(\Pi)$ with respect to their base points. It remains to point out that $\nabla_u H_{00}(\cdot)$ is a linear map that is continuous with values in H^{-1}; hence it is C^1 with values in H^{-1}. Since G is C^1 with values in H^1, we obtain that $D_u \langle \nabla H_{00}, G\rangle$ is continuous; i.e., $\mathcal{I}$ is of class C^1.

One expects that nondegenerate zeros of the energy function give rise to zeros of the energy equation (6.25) and hence can be used to construct orbits homoclinic to the manifold $\mathcal{M}_\varepsilon$. To describe the properties of such homoclinic orbits, we now introduce some definitions.

DEFINITION 7.1 Let us consider a point $b_0 \in \mathcal{C}$, and let $j = \{j_l\}_{l=1}^N$ be a sequence of $+1$'s and -1's. An orbit u_ε of system (2.1) is called an *N-pulse homoclinic orbit with base point b_0 and jump sequence j* if for some $\mu > 0$ and for $\varepsilon > 0$ sufficiently small,

(i) u_ε intersects an unstable fiber $f^u(b_\varepsilon)$ with base point $b_\varepsilon = b_0 + \mathcal{O}(\varepsilon^\mu) \in \Pi$,

(ii) u_ε intersects a stable fiber $f^s(c_\varepsilon)$ with base point $c_\varepsilon = b_0 + N\Delta\phi + \mathcal{O}(\varepsilon^\mu) \in \mathcal{M}_\varepsilon$ such that $\operatorname{dist}_{H^1}(c_\varepsilon, \Pi) = \mathcal{O}(\varepsilon)$, and

(iii) outside a small fixed neighborhood of the manifold $\mathcal{M}_\varepsilon$, the orbit u_ε is order $\mathcal{O}(\varepsilon^\mu)$ H^1-close to a chain of unperturbed heteroclinic solutions $u^l(t)$, $l = 1, \dots, N$, such that

$$\lim_{t\to-\infty} u^1(t) = b_0\,, \qquad \lim_{t\to+\infty} u^{l-1}(t) = \lim_{t\to-\infty} u^l(t)\,, \quad l = 2, \dots, N,$$

and

$$u^l(t) \in \begin{cases} W_0^+(\mathcal{C}) & \text{if } j_l = +1, \\ W_0^-(\mathcal{C}) & \text{if } j_l = -1. \end{cases}$$

A three-pulse homoclinic orbit is shown in Figure 7.1.

Let us now consider a point p^+ that is on the unperturbed stable manifold $W^{s+}_{\text{loc}}(\mathcal{M}_0)$. Here the superscript + refers to the component of $W^s_{\text{loc}}(\mathcal{M}_0)$ that contains points of the homoclinic manifold $W_0^+(\mathcal{C})$. Since $W^{s+}_{\text{loc}}(\mathcal{M}_0)$ is a hypersurface, it makes sense to define the vector $\mathbf{n}(p^+) \in H^1$ as the unit normal to $W^{s+}_{\text{loc}}(\mathcal{M}_0)$ that points in the direction of the other unperturbed homoclinic manifold $W_0^-(\mathcal{C})$. (See Figure 7.2 for a schematic picture.)

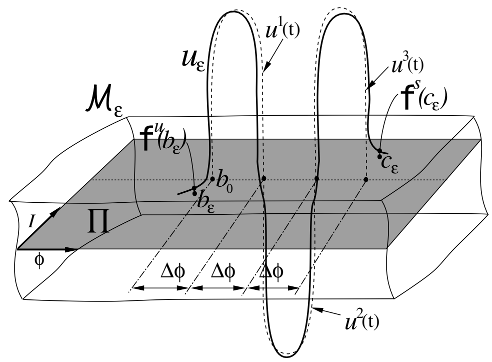

FIGURE 7.1. 3-pulse homoclinic orbit with jump sequence $\{+1,-1,+1\}$.

This allows us to introduce the number

$$\sigma = \operatorname{sign}\langle \nabla H_0, \mathbf{N}(p^+)\rangle \tag{7.2}$$

with $\mathbf{N} = (n, \bar{n})$. Note that σ is independent of the choice of the point p^+ by the normal hyperbolicity of the unperturbed manifold $\mathcal{M}_0$. Furthermore, σ remains the same if we interchange the roles of the homoclinic manifolds $W_0^+(\mathcal{C})$ and $W_0^-(\mathcal{C})$ in this construction. Using the results in Haller and Wiggins [6] for the two-mode truncation of the NLS equation, we find the value of σ to be

$$\sigma = +1$$

(see Haller [8] for the details of the calculation). This means that the energy H_0 is locally higher on the side of $W_{\text{loc}}^{s+}(\mathcal{M}_0)$ that contains $W_0^-(\mathcal{C})$. This meaning of σ is clearly preserved under small perturbations.

Our next definition uses σ to build sign sequences that will turn out to yield jump sequences for multipulse orbits.

DEFINITION 7.2 For any value $\phi_0 \in S^1$, the *positive sign sequence* $\chi^+(\phi_0) = \{\chi_l^+(\phi_0)\}_{l=1}^N$ is defined as

$$\chi_1^+(\phi_0) = +1, \;\; \chi_{l+1}^+(\phi_0) = \sigma \operatorname{sign}(\Delta^l \mathcal{H}(\phi_0))\chi_l^+(\phi_0)\,, \quad l = 1, \ldots, N-1.$$

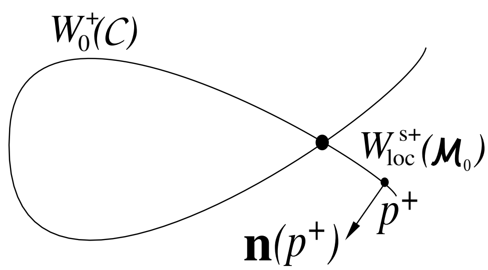

FIGURE 7.2. The definition of the vector $\mathbf{n}(p^+)$.

The *negative sign sequence* $\chi^-(\phi_0) = \{\chi_l^-(\phi_0)\}_{l=1}^N$ is defined as

$$\chi^-(\phi_0) = -\chi^+(\phi_0)\,.$$

The main result of this section is formulated in the following theorem, which establishes a connection between the zeros of the energy function and N-pulse homoclinic orbits:

THEOREM 7.3 *Suppose that for some positive integer $N \geq 1$, $\phi_0 \in S^1$ is a transverse zero of the function $\Delta^N \mathcal{H}$; i.e., we have*

$$\Delta^N \mathcal{H}(\phi_0) = 0\,, \quad D_\phi \Delta^N \mathcal{H}(\phi_0) \neq 0\,.$$

Suppose further that $\Delta^l \mathcal{H}(\phi_0) \neq 0$ holds for all integers $l = 1, \ldots, N-1$.

Then there exist constants $0 < \mu < \frac{1}{2}$ and $C_\eta > 0$ such that for any small enough $\varepsilon > 0$, the NLS equation (2.1) *admits two one-parameter families of N-pulse homoclinic orbits $u_\varepsilon^\pm(\phi, \eta_0)$ with base points $b_\varepsilon^\pm(\phi, \eta_0) \in \Pi$ such that*

$$b_\varepsilon^\pm(\phi, \eta_0) = (\phi_0 + \mathcal{O}(\varepsilon^\mu), \sqrt{\varepsilon}\,\eta_0)\,.$$

Here $|\eta_0| < C_\eta$ is an arbitrary localized action value. The jump sequences of the orbits are given by $\chi^\pm(\phi_0)$, respectively. Furthermore, the base points $b_\varepsilon^\pm$ depend on ϕ and ε^μ in a C^1 fashion.

PROOF: Consider a point $b_\varepsilon = (\phi_0, \sqrt{\varepsilon}\,\eta_0)$ on the plane Π and the unstable fiber $f^u(b_\varepsilon)$ based at b_ε that lies in the manifold $W_\varepsilon^{u+}(\Pi)$. (Here $W_\varepsilon^{u+}(\Pi)$ denotes the connected component of $W_\varepsilon^u(\Pi)$ that contains the homoclinic manifold $W_0^+(\mathcal{C})$ for $\varepsilon = 0$.) The fiber $f^u(b_\varepsilon)$ intersects the surface $|w_2| = \delta_0$ at a point q_0, as shown in Figure 7.3. Let us consider a solution $u(t)$ with initial condition $u(0) = q_0$. By Remark 3.3, we know that $u(t) \in H^\infty$ for all $t \in \mathbb{R}$. This solution leaves the neighborhood U_0 of the manifold $\mathcal{M}_\varepsilon$ and, by standard Gronwall estimates, returns and intersects the face $|w_1| = \delta_0$ of the surface $\partial_1 U$ at a point p_1 (see Figure 7.3). Since the unstable fibers are straight in the (w, ζ, ρ, ψ)-coordinates, we have $\|\zeta_{q_0}\|_{H^1} = 0$, and hence q_0 lies in the domain $\mathcal{G}_\varepsilon$ of the global map G_ε (see (5.3)) and we can write $p_1 = G_\varepsilon(q_0)$.

Since the manifold $W_{\text{loc}}^{s+}(\mathcal{M}_\varepsilon)$ is a graph over the variables (w_1, ζ, ρ, ψ), there exists a unique point $s_1 \in W_{\text{loc}}^{s+}(\mathcal{M}_\varepsilon) \cap \partial_1 U_0$ with

$$(\zeta_{s_1}, \rho_{s_1}, \psi_{s_1}) = (\zeta_{p_1}, \rho_{p_1}, \psi_{p_1})\,,$$

as shown in Figure 7.3. According to Lemma 6.4, $p_1 \equiv s_1$ holds if and only if

$$H(p_1) - H(s_1(p_1)) = 0\,, \tag{7.3}$$

where we view s_1 as a function of p_1. Since $p_1 \in H^\infty$ and the projection $p_1 \mapsto s_1$ clearly maps H^k into H^k for any $k \geq 1$, we know that $s_1 \in H^\infty$ is a smooth function of p_1. As a result, the right-hand side of equation (7.3) is C^1 in the variable p_1.

By standard Gronwall estimates, the point p_1 of the solution $u(t)$ is $\mathcal{O}(\varepsilon)$ H^1-close to a stable fiber $f^s(b_1)$ with base point $b_1 = b_\varepsilon + \Delta\phi \in \Pi$ (see Figure 7.3).

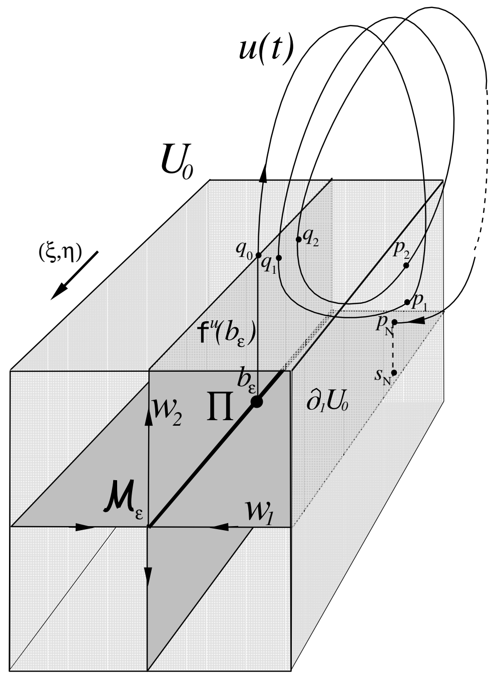

FIGURE 7.3. The construction of the proof of Theorem 7.1.

As a result, it satisfies the entry conditions listed in (4.2) with $\beta = 1$ and $c_2 = 0$. Consequently, Lemma 6.1 applies with $n = 1$ and gives

(7.4) $$H(p_1(b_\varepsilon)) = H_0 \mid \mathcal{C} + \varepsilon \left[\mathcal{H}(b_0) + N\mathcal{I} + \mathcal{O}(\delta_0, \varepsilon^\mu)\right]$$

for an appropriate constant $0 < \mu < \frac{1}{2}$. Furthermore, Lemma 6.2 with $n = 1$ gives

(7.5) $$H(s_1(b_\varepsilon)) = H_0 \mid \mathcal{C} + \varepsilon\mathcal{H}(b_0 + \Delta\phi) + \mathcal{O}\Big(\varepsilon\delta_0, \varepsilon^{\frac{3}{2}}\Big).$$

Since $b_\varepsilon = b_0 + \mathcal{O}(\sqrt{\varepsilon}) = (\phi_0, \sqrt{\varepsilon}\eta_0)$, for any $\varepsilon > 0$ we can use (7.4) and (7.5) to rewrite the energy equation (7.3) as

(7.6) $$\Delta^1\mathcal{H}(\phi_0) + \delta_0\mathcal{F}_1(p_1(b_\varepsilon); \delta_0, \varepsilon^\mu) + \varepsilon^\mu\mathcal{G}_1(p_1(b_\varepsilon); \delta_0, \varepsilon^\mu) = 0$$

with $p_1 = (0, w_{2p_1}, \zeta_{p_1}, \rho_{p_1}, \psi_{p_1}) = G_\varepsilon(q_0)$. The relationship between b_0 and p_1 is given by

(7.7) $$p_1(b_0) = G_\varepsilon \circ P_\varepsilon^u(b_0)\,,$$

where $P_\varepsilon^u : W_{\text{loc}}^{u+}(\Pi) \cap \partial_1 U_0 \to \Pi$ is the fiber projection map that maps the intersection points of unstable fibers in $W_{\text{loc}}^{u+}(\Pi)$ with the surface $\partial_1 U_0$ to their base points. By Theorem 3.2, the function P_ε^u is a C^r-map on H^1. By Lemma 5.3, G_ε is a C^1-map from $\mathcal{G}_\varepsilon$ to H^{-1}. As a result, equation (7.7) shows that p_1 is a C^1-function of $b_0 \in H^\infty$ with values in H^∞. This in turn implies that the right-hand side of the energy equation (7.6) is of class C^1 with respect to b_0 because the functions $\mathcal{F}_1$ and $\mathcal{G}_1$ are C^1 in p_1, as we observed after formula (7.3), and $\Delta^1\mathcal{H}$ is a C^1-function.

Suppose that $N = 1$ in the statement of the theorem. Then, by assumption, (ϕ_0, η_0) with any $0 < |\eta_0| \leq C_\eta$ is a solution of equation (7.6) for $\delta_0 = \varepsilon = 0$. We want to apply the implicit function theorem to argue that this solution can be continued for $\varepsilon, \delta_0 > 0$. Setting $\varepsilon = 0$ and differentiating (7.6) with respect to ϕ yields

$$D_\phi \left[\Delta^1\mathcal{H}(\phi_0) + \delta_0\mathcal{F}_1(p_1(b_0); \delta_0, 0)\right] = D_\phi\Delta^1\mathcal{H}(\phi_0) + \delta_0 \left\langle \nabla_{p_1}\mathcal{F}, DG_0 DP_0^u D_{\phi_0}\mathcal{T}_0^{-1} \right\rangle \Big|_{(\eta_0,\phi_0)}, \tag{7.8}$$

where $\mathcal{T}_\varepsilon$ is the normal form transformation constructed in Lemma 3.4. Now $D_\phi\Delta^1\mathcal{H}$ is a continuous function, and we have $D_\phi\Delta^1\mathcal{H}(\phi_0) \neq 0$ by assumption. Hence for sufficiently small $\delta_0 > 0$, (7.8) is nonzero. (This follows by recalling that the right-hand side of (7.8) continuous in (η_0, ϕ_0) and the term

$$\langle \nabla_{p_1}\mathcal{F}, DG_0 DP_0^u D_{\phi_0}\mathcal{T}_0^{-1}\rangle|_{(\eta_0,\phi_0)}$$

remains bounded as $\delta_0 \to 0$ by Lemma 5.3.) Thus (7.6) admits a solution $\bar\phi(\eta_0, \delta_0) = \phi_0 + \mathcal{O}(\delta_0)$ for $\delta_0 > 0$ small and $\varepsilon = 0$. We fix δ_0 sufficiently small and substitute the solution $\bar\phi$ back into equation (7.6). We observe that the derivative of the left-hand side of the resulting equation with respect to ϕ is given by

$$D_\phi\Delta^1\mathcal{H}\,(\bar\phi) + \delta_0 \left\langle \nabla_{p_1}\mathcal{F}_1, DG_\varepsilon DP_\varepsilon^u D_{\phi_0}\mathcal{T}_\varepsilon^{-1} \right\rangle + \varepsilon^\mu \left\langle \nabla_{p_1}\mathcal{G}_1, DG_\varepsilon DP_\varepsilon^u D_\phi\mathcal{T}_\varepsilon^{-1} \right\rangle .$$

By Lemma 5.3, this derivative is continuous at $\varepsilon = 0$ and is also nonzero by assumption. Thus equation (7.6) admits a solution $\hat\phi(\eta_0, \delta_0, \varepsilon) = \phi_0 + \mathcal{O}(\delta_0, \varepsilon^\mu)$ for $\varepsilon > 0$ sufficiently small. For any fixed ε, the solution cannot depend on δ_0, since δ_0 is just an auxiliary parameter to measure the size of the neighborhood U_0 that we have worked in. Therefore, we must have $d\hat\phi/d\delta_0 = 0$, implying $\hat\phi(\eta_0, \varepsilon) = \phi_0 + \mathcal{O}(\varepsilon^\mu)$. This proves the existence of the orbit family $u_\varepsilon^+(\eta_0)$ for $N = 1$. The smoothness of $u_\varepsilon^+(\eta_0)$ with respect to ε^μ follows from Lemma 5.3.

Assume now that $N > 1$ in the statement of the theorem. Then, by the assumptions of the theorem, for ε and δ_0 sufficiently small, the energy equation (7.6) cannot be satisfied, so the solution $u(t)$ does not intersect the local stable manifold of $\mathcal{M}_\varepsilon$ upon its first return to the neighborhood U_0. Using (7.4), (7.5), and the compactness of $[-C_\eta, C_\eta] \times S^1$, there exist positive constants $K_1^{(1)}$ and $K_2^{(1)}$ such that

$$K_1^{(1)}\varepsilon < |H(p_1) - H(s_1)| < K_2^{(1)}\varepsilon . \tag{7.9}$$

Now the mean value theorem implies that

$$|H(p_1) - H(s_1)| = \left| \left\langle \nabla H(p_1^*), \frac{p_1 - s_1}{\|p_1 - s_1\|_{H^1}} \right\rangle \right| \|p_1 - s_1\|_{H^1}$$

$$> C_2^{(1)} \|p_1 - s_1\|_{H^1} \,, \tag{7.10}$$

where p_1^* is a point on the line connecting p_1 and s_1, and the existence of $C_2^{(1)} > 0$ follows from an argument similar to that leading to estimate (6.27). At the same time, the mean value theorem and (2.5) imply that

$$|H(p_1) - H(s_1)| < C_1^{(1)} \|p_1 - s_1\|_{H^1} \tag{7.11}$$

for some constant $C_1^{(1)} > 0$, so it follows from (7.9)–(7.11) that

$$\frac{K_1^{(1)}\varepsilon}{C_1^{(1)}} < \|p_1 - s_1\|_{H1} < \frac{K_2^{(1)}\varepsilon}{C_2^{(1)}} \,. \tag{7.12}$$

This last expression in (7.12) immediately shows that the coordinates $(w_{2p_1}, \zeta_{p_1}, \rho_{p_1}, \psi_{p_1})$ satisfy the entry conditions in (4.2) (because the normal form coordinates of the point s_1 satisfy $w_{1s_1} = \delta_0$, $w_{2s_1} = 0$, and $\|\zeta_{s_1}\|_{H^1} = \mathcal{O}(\varepsilon)$). Consequently, the point p_1 is contained in the domain $\mathcal{L}_\varepsilon$ of the local map L_ε, and we can write $q_1 = L_\varepsilon(p_1)$ where q_1 is the next intersection of the solution $u(t)$ with the surface $\partial_1 U_0$.

Let p_2 denote the intersection of the solution $u(t)$ with the surface $\partial_1 U_0$ upon its second return to the neighborhood U_0. (The existence of p_2 is guaranteed by the usual Gronwall estimates for $\varepsilon > 0$ small enough.) We again have a point $s_2 \in W^s_{\rm loc}(\mathcal{M}_\varepsilon) \cap \partial_1 U_0$ such that

$$(\zeta_{s_2}, \rho_{s_2}, \psi_{s_2}) = (\zeta_{p_2}, \rho_{p_2}, \psi_{p_2}) \,.$$

Again, the solution $u(t)$ gives rise to a 2-pulse homoclinic orbit if

$$H(p_2) - H(s_2(p_2)) = 0$$

or, alternatively,

$$\Delta^2 \mathcal{H}(\phi_0) + \delta_0 \mathcal{F}_2(p_2(b_\varepsilon); \delta_0, \varepsilon^\mu) + \varepsilon^\mu \mathcal{G}_2(p_2(b_\varepsilon); \delta_0, \varepsilon^\mu) = 0 \,, \tag{7.13}$$

where we used Lemmas 6.1 and 6.2. As in equation (7.6), the functions $\mathcal{F}_2$ and $\mathcal{G}_2$ are C^1 in their arguments. Since

$$p_2(b_\varepsilon) = G_\varepsilon \circ L_\varepsilon \circ G_\varepsilon \circ P^u_\varepsilon(b_\varepsilon) \,,$$

we see that for $\varepsilon \geq 0$, p_2 is a C^1-function of b_ε and ε^μ by Corollary 5.1 and Lemma 5.3 with values in H^∞ (recall that $u(t) \in H^\infty$). Then, just as in the case of $N = 1$, the implicit function theorem applied to (7.13) implies the existence of the orbit family $u_\varepsilon^+(\eta_0)$ for $N = 2$.

The proof for any $N > 2$ is identical to the case of $N = 2$, and the existence of the other N-pulse homoclinic orbit family $u_\varepsilon^-(\eta_0)$ for any $N \geq 1$ can be obtained from an identical construction for solutions contained in $W^{u-}(\Pi)$. Therefore, it remains to show that the jump sequences of the two families $u_\varepsilon^\pm(\eta_0)$ are indeed given by the sign sequences $\chi^\pm(\phi_0)$, respectively. We sketch the argument only for u_ε^+ since the argument for u_ε^- is identical.

Consider an N-pulse homoclinic orbit u_ε^+. By construction, it makes its first pulse in the vicinity of the unperturbed manifold $W_0^+(\mathcal{C})$; hence the first element of its jump sequence is indeed $\chi_1^+(\phi_0) = +1$. For small $\varepsilon, \delta_0 > 0$, at the first re-entry point p_1 we have

$$\begin{aligned} &\operatorname{sign}(H(s_1) - H(p_1)) \\ &\quad = \operatorname{sign}[\varepsilon(\Delta^1\mathcal{H}(\phi_0 + \mathcal{O}(\delta_0, \varepsilon^\mu)) + \delta_0 \mathcal{F}_N(p_N(b_\varepsilon^+); \delta_0, \varepsilon^\mu) \\ &\qquad\quad + \varepsilon^\mu \mathcal{G}_N(p_N(b_\varepsilon^+); \delta_0, \varepsilon^\mu))] \\ &\quad = \operatorname{sign}(\Delta^1\mathcal{H}(\phi_0)) \,. \end{aligned} \tag{7.14}$$

If this quantity is positive, then at the point p_1 the solution $u(t)$ has higher energy than nearby points in the hypersurface $W_{\text{loc}}^{s+}(\mathcal{M}_\varepsilon)$. Recalling the meaning of the constant σ (see (7.2)), we can conclude that $\sigma \operatorname{sign}(\Delta^1\mathcal{H}(\phi_0)) = +1$ implies that the solution $u(t)$ stays near the homoclinic manifold $W_0^+(\mathcal{C})$, whereas $\sigma \operatorname{sign}(\Delta^1\mathcal{H}(\phi_0)) = -1$ causes the solution to perform its second jump in the vicinity of the manifold $W_0^-(\mathcal{C})$. Therefore, the second element in the jump sequence of u_ε^+ is given by $\chi_2^+(\phi_0)$ as defined in Definition 7.2. The remaining elements of the jump sequence of u_ε^+ are constructed recursively in the same fashion; hence they coincide with the corresponding elements of the sign sequence $\chi^+(\phi_0)$ in Definition 7.2. This completes the proof of the theorem. ☐

The above theorem gives the basic tool for constructing multipulse orbits that backward-asymptote to the plane Π and intersect the locally invariant manifold $W_{\text{loc}}^s(\mathcal{M}_\varepsilon)$ in forward time. To find the asymptotic behavior of multipulse orbits, one has to have some approximate knowledge of the dynamics on the two-dimensional plane Π. A Taylor expansion shows that near the resonant circle $\mathcal{C}$, the flow on Π satisfies the equations

$$\begin{aligned} \dot{\phi} &= \sqrt{\varepsilon} D_\eta \mathcal{H}_g(\eta, \phi) + \mathcal{O}(\varepsilon) \,, \\ \dot{\eta} &= -\sqrt{\varepsilon} D_\phi \mathcal{H}_g(\eta, \phi) + \mathcal{O}(\varepsilon) \,, \end{aligned} \tag{7.15}$$

with

$$\begin{aligned} \mathcal{H}_g(\eta, \phi) &= \mathcal{H}(\eta, \phi) - \int_0^\phi (\hat{D}u)_I|_{\mathcal{C}}(u)\, du \\ &= -\eta^2 + 2\Gamma\Omega \sin\phi - \int_0^\phi (\hat{D}u)_I|_{\mathcal{C}}(u)\, du \end{aligned} \tag{7.16}$$

where $(\hat{D}u)_I$ is the I-component of the perturbation term $\hat{D}u$ in equation (2.1). As seen from (7.15), for finite times solutions on the manifold Π are approximated with an error of order $\mathcal{O}(\sqrt{\varepsilon})$ by the level curves of the function $\mathcal{H}_g$. In general, the flow generated by $\mathcal{H}_g$ is only locally Hamiltonian since $\mathcal{H}_g$ is not necessarily periodic in ϕ.

8 Homoclinic Tree in the Forced NLS Equation

Without the dissipative term $\hat{D}u$, equation (2.1) is a near-integrable Hamiltonian system. We are interested in finding multipulse orbits for this system that exhibit jumping behavior around the plane Π of spatially independent solutions. These solutions may ultimately leave $W^s_{\rm loc}(\mathcal{M}_\varepsilon)$ through its boundary, but they remain $\mathcal{O}(\varepsilon)$ H^1-close to Π on time scales of order $\mathcal{O}(1/\sqrt{\varepsilon})$. As a result, *in numerical simulations they appear as solutions homoclinic to* Π.

For $\hat{D} \equiv 0$, the constant $\mathcal{I}$ computed in (2.16) vanishes, and hence the energy function studied in the previous section simplifies to

$$\Delta^N \mathcal{H}(\phi) = 2\Omega\Gamma \left[\sin\left(\phi + N\Delta\phi\right) - \sin\phi\right]. \tag{8.1}$$

If we use the nondimensionalized variables, we find that the energy function obtained in (8.1) for the partial differential equation is the same as that obtained for its modal truncation in Haller and Wiggins [6]. Since the existence of multipulse homoclinic orbits is fully determined by the energy function, we can directly use the finite-dimensional study carried out in Haller and Wiggins [6] to construct multipulse solutions for the full, forced NLS equation.

Although the sign constants $\sigma^{\rm PDE} = +1$ and $\sigma^{\rm trunc} = -1$ differ in sign, the angular variable ϕ used in Haller and Wiggins [6] also differs in sign from that used in this paper. Therefore, if ϕ_0 is a transverse zero of $\Delta^N \mathcal{H}^{\rm trunc}$, then $\tilde{\phi}_0 = -\phi_0$ is a transverse zero for $\Delta^N \mathcal{H}^{\rm PDE}$ (given in (8.1)), and

$$\begin{aligned}
\sigma^{\rm trunc}\left[\Delta^k \mathcal{H}^{\rm trunc}(\phi_0)\right] &= \left(-\sigma^{\rm PDE}\right)\left[-\Delta^k \mathcal{H}^{\rm trunc}(-\phi_0)\right] \\
&= \left(-\sigma^{\rm PDE}\right)\left[-\Delta^k \mathcal{H}^{\rm PDE}(\tilde{\phi}_0)\right] \\
&= \sigma^{\rm PDE}\left[\Delta^k \mathcal{H}^{\rm PDE}(\tilde{\phi}_0)\right].
\end{aligned}$$

Consequently, *any N-pulse orbit for the modal truncation yields an N-pulse orbit for the forced NLS equation with the same jump sequence.* Since the flow on the plane Π is close to the pendulum flow generated by the slow Hamiltonian $\mathcal{H}$, any zero line $\phi = \phi_0$ will intersect families of slow periodic orbits that are created near the resonance by the perturbation. As a result, for any transverse zero of $\Delta^N \mathcal{H}$, there exist two families of multipulse orbits that are backward-asymptotic to a slow periodic orbit on Π. (These orbits are quasi-periodic in the original coordinates used in (1.1)). Since equation (8.1) gives the transverse zeros

$$\phi_1^N = \frac{\pi}{2} - \frac{N\Delta\phi\left(\Omega\right)}{2}, \qquad \phi_2^N = \frac{3\pi}{2} - \frac{N\Delta\phi\left(\Omega\right)}{2}, \tag{8.2}$$

a direct application of Theorem 7.3 yields the following result:

THEOREM 8.1 *Let us fix the forcing frequency* $\frac{1}{2} < \Omega < 1$*, and consider any integer* $N \geq 1$ *for which* $N\Delta\phi(\Omega) \neq 2j\pi$ *for all integers* j*.*

Then, for $\varepsilon > 0$ *sufficiently small,*

(i) *The forced NLS equation admits four* 1*-parameter families of* N*-pulse homoclinic orbits, which are backward-asymptotic to slow periodic solutions on the invariant plane* Π*. The coordinates of the base points of the* N*-pulse homoclinic orbits are of the form*

$$u^{l,\pm} = \left(\Omega + \mathcal{O}(\sqrt{\varepsilon})\right) e^{i(\phi_l^N + \mathcal{O}(\sqrt{\varepsilon}))}, \quad l = 1, 2.$$

(ii) *For* $N > 1$*, the jump sequences of the orbit families are given by the recursive formula*

$$j_{k+1}^{l,\pm} = \pm \operatorname{sign}\left[\Gamma\left[\sin\left(\phi_l^N + k\Delta\phi\right) - \sin\phi_l^N\right]\right] j_k^{l,\pm}, \\ k = 1, \dots, N-1, \quad l = 1, 2,$$

where $j_1^{l,\pm} = \pm 1$*. Furthermore, for any* $l_1, l_2 \in \{+1, -1\}$*,* $l_1 \neq l_2$*, the following holds: Every time the jump sequence* $j^{l_1,\pm}$ *changes sign, the jump sequence* $j^{l_2,\pm}$ *with* $l_1 \neq l_2$ *will not change sign.*

The multipulse orbits described above necessarily exhibit the same type of homoclinic bifurcations as the analogous orbits for the modal truncation, since their appearance and disappearance is governed by the same equation. As discussed in detail in Haller and Wiggins [6], we can classify the slow periodic orbits created in the resonance band into layers. Crossing the boundaries of these layers, the homoclinic orbit with the lowest number of pulses undergoes a bifurcation that changes its pulse number. As a result, slow periodic orbits in different layers have different types of "primary" homoclinic orbits, all of which were shown in [6] to be alternating in terms of their jump sequences. Plotting the half-widths of the above layers as a function of the phase shift $\Delta\phi$, we obtain the *homoclinic tree* shown in Figure 8.1. This diagram shows how the width of the layers containing slow periodic orbits with the same type of primary homoclinic orbits changes as the phase shift is varied. Any fixed $\Delta\phi = \mathrm{const}$ slice of the diagram therefore gives the widths of the layers that exist for that value of $\Delta\phi$. The primary pulse numbers corresponding to these layers are shown in Figure 8.2. This diagram indicates that homoclinic orbits with higher pulse numbers are easier to destroy by a change in the system parameters. It is highly surprising that such a complicated structure can be shown to exist in a partial differential equation. The construction of these diagrams is entirely based on the analysis of the zeros of the one-variable real function $\Delta^N\mathcal{H}$ and can be found in Haller and Wiggins [6].

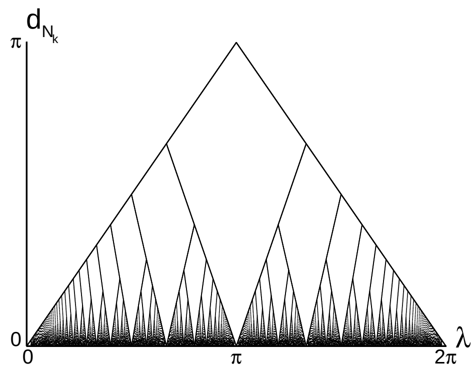

FIGURE 8.1. The homoclinic tree.

9 N-Pulse Jumping Orbits in the Damped-Forced NLS Equation

The numerical experiments of Bishop et al. [2] on the perturbed NLS equation were performed with the mode-independent damping term

$$g(u, \bar{u}) = \hat{D}u = -\alpha u \tag{9.1}$$

with damping coefficient $\alpha > 0$. The irregular jumping of solutions in the time domain was already noted for this simple damping term.

According to Theorem 7.3, the existence of multipulse orbits is determined by the zeros of the energy function

$$\begin{aligned} \Delta^N \mathcal{H}(\phi) &= \mathcal{H}(\eta, \phi + N\Delta\phi) - \mathcal{H}(\eta, \phi) - N\mathcal{I} \\ &= 2\Omega\Gamma \left[\sin\left(\phi + N\Delta\phi\right) - \sin\phi\right] - \alpha\Omega N \mathcal{I}_\alpha(\Omega) \\ &= \Omega\Gamma \cos\left(\phi + N\Delta\phi/2\right) \sin\left(N\Delta\phi/2\right) - \alpha\Omega N \mathcal{I}_\alpha(\Omega) \end{aligned} \tag{9.2}$$

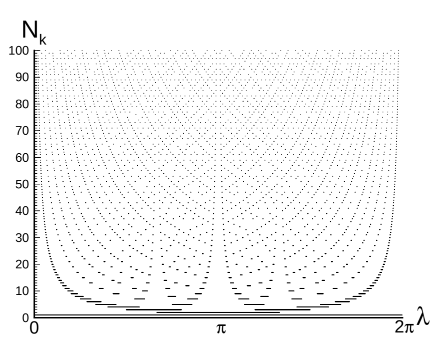

FIGURE 8.2. The pulse numbers as a function of the phase shift.

where

$$
\begin{aligned}
\mathcal{I}_\alpha(\Omega) &= \frac{1}{\Omega}\int_{-\infty}^{\infty} \langle \nabla H_0, G\rangle \left.\right|_{u^h_\pm(t)} dt \\
&= \frac{2}{\Omega}\,\mathrm{Re}\int_{-\infty}^{\infty}\int_0^{2\pi} \left(u\bar{u}_{xx} + 2\,|u|^2\left[|u|^2-\Omega^2\right]\right)\Big|_{u^h_\pm(t)}\, dx\, dt\,.
\end{aligned} \tag{9.3}
$$

Note that $\mathcal{I}_\alpha(\Omega)$ only depends on the forcing frequency Ω. From (9.2) we obtain that if

$$
\Delta\phi \neq \frac{2j\pi}{N}\,, \quad j\in\mathbb{Z}, \tag{9.4}
$$

and

$$
\chi_\alpha \leq \left|\frac{\sin\frac{N\Delta\phi(\Omega)}{2}}{N\mathcal{I}_\alpha(\Omega)}\right| \tag{9.5}
$$

holds with $\chi_\alpha = \alpha/\Gamma$, then the zeros of the function $\Delta^N\mathcal{H}(\phi)$ are given by

$$
\begin{aligned}
\phi_1^N &= \frac{\pi}{2} - \frac{N\Delta\phi\,(\Omega)}{2} - \cos^{-1}\frac{N\chi_\alpha\mathcal{I}_\alpha(\Omega)}{\sin\frac{N\Delta\phi(\Omega)}{2}}\,,\\
\phi_2^N &= \frac{3\pi}{2} - \frac{N\Delta\phi\,(\Omega)}{2} - \cos^{-1}\frac{N\chi_\alpha\mathcal{I}_\alpha(\Omega)}{\sin\frac{N\Delta\phi(\Omega)}{2}}\,.
\end{aligned}
$$

These zeros are also easily seen to be transverse; thus each gives rise to a multipulse homoclinic orbit in the sense of Definition 7.1.

The flow on the invariant plane obeys the equations

$$
\begin{aligned}
\dot{\phi} &= -\sqrt{\varepsilon}2\eta + \mathcal{O}(\varepsilon)\,,\\
\dot{\eta} &= -\sqrt{\varepsilon}\left(2\Gamma\Omega\cos\phi + 2\alpha\Omega^2\right) + \mathcal{O}(\varepsilon)\,,
\end{aligned} \tag{9.6}
$$

which, as described in (7.15) and (7.16), are *locally* Hamiltonian at leading order with the Hamiltonian

$$
\mathcal{H}_g(\eta,\phi) = -\eta^2 + 2\Gamma\Omega\sin\phi + 2\alpha\Omega^2\phi\,. \tag{9.7}
$$

For $|\chi_\alpha\Omega| < 1$, this local Hamiltonian has two critical points: a saddle $s_0(\chi_\alpha)$ and a center $c_0(\chi_\alpha)$ given by

$$
s_0(\chi_\alpha) = \left(0, \pi + \cos^{-1}(-\chi_\alpha\Omega)\right), \qquad c_0(\chi_\alpha) = \left(0, \cos^{-1}(-\chi_\alpha\Omega)\right)\,. \tag{9.8}
$$

The level curves of $\mathcal{H}_g$ are shown in Figure 9.1(a), and the corresponding phase portrait of (9.6) is shown in Figure 9.1(b). Note that the unstable manifold of the saddle point is intersected transversely by any $\phi = \mathrm{const}$ line and hence by the lines $\phi_i^N = \mathrm{const}$. This is also true for the stable manifold of the actual saddle point $s_\varepsilon(\chi_\alpha)$ of equation (9.6). Therefore, a direct application of Theorem 7.3 leads to the following result:

THEOREM 9.1 *Let us fix the forcing frequency $\frac{1}{2} < \Omega < 1$ and consider any integer $N \geq 1$ for which conditions* (9.4) *and* (9.5) *hold. Then, for $\varepsilon > 0$ sufficiently small,*

(i) *The perturbed NLS equation* (2.1) *admits four one-parameter families of N-pulse homoclinic orbits, which are backward-asymptotic to the invariant plane Π and forward-asymptotic to a codimension-2 invariant manifold $\mathcal{M}_\varepsilon$ that contains Π. In each orbit family, at least one orbit is backward-asymptotic to a saddle fixed point of the plane Π. The coordinates of the base points of the N-pulse homoclinic orbits are of the form*

$$u^{l,\pm} = \left(\Omega + \mathcal{O}(\sqrt{\varepsilon})\right) e^{i(\phi_l^N + \mathcal{O}(\sqrt{\varepsilon}))}, \quad l = 1, 2.$$

(ii) *For $N > 1$, the jump sequences of the orbit families are given by the recursive formula*

$$j_{k+1}^{l,\pm} = \pm \operatorname{sign}\left[\Gamma\left[\sin\left(\phi_l^N + k\Delta\phi\right) - \sin\phi_l^N\right] - \alpha k \mathcal{I}_\alpha(\Omega)\right] j_k^{l,\pm}, \\ k = 1, \ldots, N-1, \quad l = 1, 2,$$

where $j_1^{l,\pm} = \pm 1$. For $\alpha > 0$ sufficiently small and for any $l_1, l_2 \in \{+1, -1\}$, $l_1 \neq l_2$, the following holds: Every time the jump sequence $j^{l_1,\pm}$ changes sign, the jump sequence $j^{l_2,\pm}$ with $l_1 \neq l_2$ will not change sign.

10 *N*-Pulse Šilnikov-Type Orbits in the Damped-Forced NLS Equation

In a series of papers by McLaughlin et al. (see the introduction), the NLS equation is considered with the dissipative term

$$g(u, \bar{u}) = \hat{D}u \equiv -\alpha u + \beta \hat{B}_K u, \tag{10.1}$$

where the operator $\hat{B}_K$ is the smoothed diffusion operator: It acts as the operator ∂_x^2 for low wave numbers but vanishes on higher Fourier modes of the function $u(x)$. If $\tilde{b}(k)$ denotes the Fourier transform of $\hat{B}_K u(x)$ and $\tilde{u}(k)$ is the Fourier

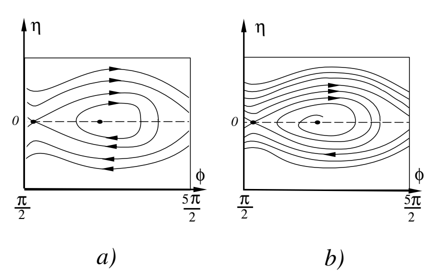

FIGURE 9.1. The levels of $\mathcal{H}_g$ and the flow on the plane Π.

transform of $u(x)$, then

$$\tilde{b}(k) = \begin{cases} -k^2 \tilde{u}(k) & \text{if } k < K, \\ 0 & \text{if } k \geq K, \end{cases}$$

with some fixed, large integer $K > 0$. Using this dissipative term, Kovačič and Wiggins [11] found orbits homoclinic to a saddle point in the two-mode truncation of the NLS. These Šilnikov-type orbits exist for a codimension-1 set of the (α, β)-parameter plane. Similar results were recently obtained by Li et al. [14] for the full PDE (see also McLaughlin and Shatah [18]).

However, the set of parameter values for Šilnikov-type orbits is rather small, so one cannot expect to observe chaos for generic parameter values. In this section we show that multipulse analogs of the Šilnikov-type orbits also exist for the full partial differential equation, and their domain of existence in the parameter space is a fairly large set. Consequently, complicated dynamics exists for much larger sets of parameter values than those obtained from Melnikov-type calculations for single-pulse homoclinic orbits.

For the dissipative term (10.1), the energy function takes the form

$$\begin{aligned} \Delta^N \mathcal{H}(\phi) = 2\Omega\Gamma\left[\sin\left(\phi + N\Delta\phi\right) - \sin\phi\right] \\ - N\Omega\left[\alpha \mathcal{I}_\alpha(\Omega) - \beta \mathcal{I}_\beta(\Omega, K)\right], \end{aligned} \tag{10.2}$$

with $\mathcal{I}_\beta$ defined as

$$\mathcal{I}_\beta(\Omega, K) = -\frac{2}{\Omega} \operatorname{Re} \int_{-\infty}^{\infty} \int_0^{2\pi} \left(\bar{u}_{xx} + 2\left[|u|^2 - \Omega^2\right]\bar{u}\right) \hat{B}_K u \Big|_{u_\pm^h(t)} \, dx \, dt \, .$$

Since the homoclinic solutions $u_\pm^h(t)$ are given by H^∞-functions, the duality pairing

$$\langle \nabla H_0, \beta u_{xx} \rangle \big|_{u_\pm^h(t)} = -\frac{2\beta}{\Omega} \operatorname{Re} \int_0^{2\pi} \left(\bar{u}_{xx} + 2\left[|u|^2 - \Omega^2\right]\bar{u}\right) u_{xx} \Big|_{u_\pm^h(t)} \, dx$$

is bounded, and we have

$$\lim_{K\to\infty} \mathcal{I}_\beta(\Omega, K) = \mathcal{I}_0(\Omega) \tag{10.3}$$

with

$$\mathcal{I}_0(\Omega) = -\frac{2\beta}{\Omega} \operatorname{Re} \int_{-\infty}^{\infty} \int_0^{2\pi} \left(\bar{u}_{xx} + 2\left[|u|^2 - \Omega^2\right]\bar{u}\right) u_{xx} \Big|_{u_\pm^h(t)} \, dx \, dt \, . \tag{10.4}$$

Since the integrand in this expression is an analytic function of x, its Fourier coefficients decay exponentially with the wave number k. This fact enables us to write

$$\mathcal{I}_\beta(\Omega, K) = \mathcal{I}_0(\Omega) + \mathcal{O}(e^{-\gamma K})$$

for an appropriate constant $\gamma > 0$ and K sufficiently large.

In the vicinity of the resonant circle $\mathcal{C}$, trajectories are still close to those of the restricted system (9.6). We would like to construct orbits homoclinic to the saddle point $s_\varepsilon(\chi_\alpha)$. As shown in Li et al. [14], $s_\varepsilon(\chi_\alpha)$ has infinitely many eigenvalues with negative real parts that perturb from the purely imaginary eigenvalues on the linearized equation. Li et al. also showed the existence of a codimension-1 stable manifold $W^s_{\rm loc}(s_\varepsilon(\chi_\alpha))$ for $\varepsilon > 0$ small enough. The intersection of this stable manifold with $\mathcal{M}_\varepsilon$ is a codimension-1 submanifold of $\mathcal{M}_\varepsilon$ whose tangent space at $s_\varepsilon(\chi_\alpha)$ is close to the product of the center subspace E^c with the stable subspace of the saddle $s_0(\chi_\alpha)$. Furthermore, the "height" of this stable manifold $W^s(s_\varepsilon(\chi_\alpha))$ is $\mathcal{O}(\varepsilon^{3/4})$.

In our construction, first we want to ensure that the energy-difference function (10.2) has a transverse zero ϕ_0. This always holds if

$$\Delta\phi \neq \frac{2j\pi}{N}\,, \quad j \in \mathbb{Z}\,, \qquad |\alpha\mathcal{I}_\alpha(\Omega) - \beta\mathcal{I}_\beta(\Omega, K)| < \frac{\Gamma}{N}\left|\sin\frac{N\Delta\phi}{2}\right|. \tag{10.5}$$

As in the proof of Theorem 7.3, this means that the equation

$$\Delta^N\mathcal{H}(\phi;\chi_\alpha) + \delta_0\mathcal{F}_N(p_N(\phi);\delta_0,\varepsilon^\mu,\chi_\alpha) + \varepsilon^\mu\mathcal{G}_N(p_N(\phi);\delta_0,\varepsilon^\mu,\chi_\alpha) = 0$$

has a solution $\bar\phi(\chi_\alpha,\varepsilon) = \phi_0(\chi_\alpha) + \mathcal{O}(\varepsilon^\mu)$. By the C^1-dependence of $\bar\phi$ on ε^μ (cf. Theorem 7.3), the curve $\{\phi = \bar\phi(\chi_\alpha,\varepsilon)\}$ intersects the unstable manifold of the fixed point $s_\varepsilon(\chi_\alpha)$ transversely in a point

$$\bar p(\chi_\alpha,\varepsilon) = (\eta_0(\chi_\alpha) + \mathcal{O}(\varepsilon^\mu), \phi_0(\chi_\alpha) + \mathcal{O}(\varepsilon^\mu)) \in \Pi\,.$$

This means that there exists an N-pulse homoclinic orbit in the sense of Definition 7.1 with base point $\bar p(\chi_\alpha,\varepsilon)$. This orbit intersects a stable fiber $f^s(\hat p(\chi_\alpha,\varepsilon))$ whose base point has the (y,z,η,ϕ)-coordinates

$$\begin{aligned}\hat p(\chi_\alpha,\varepsilon) = (0, \mathcal{O}(\varepsilon), \eta_0(\chi_\alpha) + \mathcal{O}(\varepsilon^\mu), \phi_0(\chi_\alpha) + \Delta\phi(\chi_\alpha) + \mathcal{O}(\varepsilon^\mu))\\ \in \mathcal{M}_\varepsilon\,.\end{aligned} \tag{10.6}$$

We would like to find conditions under which this base point lies in the stable manifold of the fixed point $s_\varepsilon(\chi_\alpha)$, and hence the N-pulse orbit is homoclinic to $s_\varepsilon(\chi_\alpha)$.

In a vicinity of the invariant plane Π, the stable manifold of $s_\varepsilon(\chi_\alpha)$ can be written as a graph over either the (ϕ,z)- or the (η,z)-variables. Considering the former case (the latter can be dealt with in the same way), we obtain that a compact subset of $W^s(s_\varepsilon(\chi_\alpha))$ satisfies an equation of the form

$$\eta = m_1(\phi,\chi_\alpha) + zm_2(\phi,z,\chi_\alpha,\varepsilon) \tag{10.7}$$

where m_j are of class C^r and $\eta = m_1(\phi,\chi_\alpha)$ defines locally the stable manifold of s_0 on the plane Π. As shown in Li et al. [14], the "height" of the manifold $W^s(s_\varepsilon(\chi_\alpha))$ is $\mathcal{O}(\varepsilon^{3/4})$; i.e., the representation (10.7) is valid for $\|z\|_{H^1} < C\varepsilon^{3/4}$. But from (10.6) we see that

$$\mathrm{dist}_{H^1}(\hat p(\chi_\alpha,\varepsilon),\Pi) = \mathcal{O}(\varepsilon)\,;$$

therefore for $\varepsilon > 0$ small enough, $\hat p(\chi_\alpha,\varepsilon)$ lies in a domain where the representation (10.7) is valid. Then by (10.6) and (10.7), $\hat p(\chi_\alpha,\varepsilon) \in W^s(s_\varepsilon(\chi_\alpha))$ holds if

(10.8)
$$\begin{aligned}&\eta_0(\chi_\alpha)+\varepsilon^\mu h_\eta(\chi_\alpha,\varepsilon)-m_1(\phi_0(\chi_\alpha)+\Delta\phi(\chi_\alpha)+\varepsilon^\mu h_\phi(\chi_\alpha,\varepsilon),\chi_\alpha)\\&-\varepsilon h_z(\chi_\alpha,\varepsilon)m_2(\phi_0(\chi_\alpha)+\Delta\phi(\chi_\alpha)+\varepsilon^\mu h_\phi(\chi_\alpha,\varepsilon),\varepsilon h_z(\chi_\alpha,\varepsilon),\chi_\alpha,\varepsilon)=0\,,\end{aligned}$$

where the functions h_η, h_ϕ, and h_z are C^1 in χ_α and ε^μ. Assume now that the approximate projection $(0,0,\eta_0(\chi_\alpha),\phi_0(\chi_\alpha)+\Delta\phi(\chi_\alpha))$ of $\hat{p}(\chi_\alpha,\varepsilon)$ crosses the stable manifold of s_0 transversely for a parameter value $\chi_\alpha=\chi_\alpha^0$. Then, using (10.7), we can write

$$\begin{aligned}\eta_0(\chi_\alpha^0)-m_1(\phi_0(\chi_\alpha^0)+\Delta\phi(\chi_\alpha^0),\chi_\alpha^0)=0\,,\\D_{\chi_\alpha}[\eta_0(\chi_\alpha)-m_1(\phi_0(\chi_\alpha)+\Delta\phi(\chi_\alpha),\chi_\alpha)]_{\chi_\alpha=\chi_\alpha^0}\neq0\,;\end{aligned}\qquad(10.9)$$

thus the implicit function theorem guarantees a solution $\bar{\chi}_\alpha(\varepsilon)=\chi_\alpha^0+\mathcal{O}(\varepsilon^\mu)$ to equation (10.8). Consequently, for the parameter value $\bar{\chi}_\alpha(\varepsilon)$, the perturbed NLS equation admits an N-pulse homoclinic orbit that connects the fixed point $s_\varepsilon(\chi_\alpha)$ to itself.

It remains to find parameter values χ_α^0 for which $(\eta_0,\phi_0+\Delta\phi)$ does cross the one-dimensional stable manifold of the saddle $s_0(\chi_\alpha)$ transversely. If such a crossing occurs, then both (η_0,ϕ_0) and $(\eta_0,\phi_0+(N\Delta\phi)\bmod 2\pi)$ must lie on the same level curve of the slow Hamiltonian $\mathcal{H}_g$. (We have to take the modulus of the angle difference between the two points, since $\mathcal{H}_g$ is only a local Hamiltonian that is not globally constant on the unstable manifold of $s_0(\chi_\alpha)$ for $\phi\in\mathbb{R}$.) We therefore require

$$\mathcal{H}_g(\eta_0,\phi_0+N\Delta\phi)=\mathcal{H}_g(\eta_0,\phi_0)+4\pi L\alpha\Omega^2\qquad(10.10)$$

for some integer L. This condition is obtained from (9.7) by observing that for $\phi\in\mathbb{R}$, the values of the Hamiltonian on the infinitely many copies of the saddle $s_0(\chi_\alpha)$ differ by integer multiples of $2\alpha\Omega^2\cdot2\pi$. Since ϕ_0 is a zero of the energy-difference function (10.2), equation (10.10) can be rewritten in the form

$$\beta=\frac{\alpha}{\mathcal{I}_\beta(\Omega,K)}\left[2\Omega\,\Delta\phi(\Omega)+\mathcal{I}_\alpha(\Omega)-4\pi\Omega\frac{L}{N}\right].\qquad(10.11)$$

For any fixed N and (Ω,Γ,α,K), this last expression defines the set of β-values for which the first equation in (10.9) is satisfied. Since the expression is linear in α, the derivative $d\beta/d\alpha$ is nonzero whenever the condition gives a nonzero β. As a result, the crossing is transversal, and hence the second crossing condition in (10.9) is also satisfied. For fixed (Ω,Γ,α,K), we obtain a β-value from equation (10.11) for each value of the integer L. However, only those L-values give meaningful results for which the condition

$$\left|L-\frac{N\Delta\phi(\Omega)}{2\pi}\right|<\frac{\Gamma}{4\pi\alpha\Omega}\left|\sin\frac{N\Delta\phi(\Omega)}{2}\right|\qquad(10.12)$$

holds. This last inequality is obtained by combining the second inequality in (10.5) with equation (10.11). Using formula (10.3), we obtain the following result:

THEOREM 10.1 *Let N be an arbitrary but fixed positive integer, and let the forcing frequency Ω with $\frac{1}{2} < \Omega < 1$ be such that condition* (9.4) *is satisfied. Assume that L is an integer satisfying* (10.12) *and*

$$M_0^L = \left\{ (\alpha, \beta, \Gamma, \varepsilon) \Big| \; \beta = \frac{\alpha}{\mathcal{I}_0\,(\Omega)} \left[2\Omega\, \Delta\phi + \mathcal{I}_\alpha(\Omega) - 4\pi\Omega \frac{L}{N} \right] \right\} \tag{10.13}$$

*is a nonempty, codimension-*1 *surface of the $(\alpha, \beta, \Gamma, \varepsilon)$ parameter space.*

*Then there exists $\varepsilon_0 > 0$, $K_0 > 0$, $\gamma > 0$, and for all $0 < \varepsilon < \varepsilon_0$ and $K > K_0$, there exist two codimension-*1 *surfaces $M_\varepsilon^{L\pm} \in \mathbb{R}^4$ with the following properties*:

(i) *$M_\varepsilon^{L\pm}$ is $\mathcal{O}(\varepsilon^\mu, e^{-\gamma K})$ C^0-close to the surface $M_0^{L\pm}$ in the $(\alpha, \beta, \Gamma, \varepsilon)$ parameter space for an appropriate constant $0 < \mu < \frac{1}{2}$.*

(ii) *For every $(\alpha, \beta, \Gamma, \varepsilon) \in M_\varepsilon^{L\pm}$, system* (2.1) *admits an N-pulse homoclinic orbit that is doubly asymptotic to the fixed point*

$$s_\varepsilon(\chi_\alpha) = (\eta_0(\chi_\alpha) + \mathcal{O}(\varepsilon^\mu), \phi_0(\chi_\alpha) + \mathcal{O}(\varepsilon^\mu)) \in \Pi\,.$$

The jump sequence of the orbits in $M_\varepsilon^{L\pm}$ is given by

$$\begin{aligned} j_{k+1}^\pm = &\pm \operatorname{sign}[\Gamma[\sin(\phi_0(\chi_\alpha) + k\Delta\phi(\chi_\alpha)) - \sin\phi_0(\chi_\alpha)] \\ &- \alpha k \mathcal{I}_\alpha(\Omega)] j_k^\pm\,, \quad k = 1, \ldots, N-1, \end{aligned}$$

where $j_1^\pm = \pm 1$.

11 Disintegration of the Unstable Manifold of Π

The previous sections were concerned with the existence of multipulse homoclinic orbits that are doubly asymptotic to the manifold $\mathcal{M}_\varepsilon$. Individual multipulse orbits are in general difficult to observe, so they cannot fully account for the jumping behavior of the perturbed NLS equation. Yet the significance of homoclinic orbits is great: They separate open sets in the manifold $W^u(\Pi)$ that exhibit different behaviors. Namely, every time an N-pulse homoclinic orbit returns to the manifold $\mathcal{M}_\varepsilon$, the unstable manifold $W^u(\Pi)$ is intersected transversely by the stable manifold $W^s(\mathcal{M}_\varepsilon)$. As a result, $W^u(\Pi)$ is divided into subsets in which solutions will perform jumps near different components of the unperturbed homoclinic structure. This implies observable, irregular, transient behavior near the broken homoclinic structure, even if there are no chaotic invariant sets created by the perturbation (see Rom-Kedar et al. [19] for related numerical results).

The methods we developed in earlier sections can in fact be used to follow any solution in the unstable manifold $W^u(\Pi)$ on time scales that are of order $\mathcal{O}(\log 1/\sqrt{\varepsilon})$. This fact enables us to "track" pieces of the unstable manifold of Π as they depart from each other and perform different "jumps." We use the following definition to distinguish between different types of jumping orbits within the unstable manifold of Π:

DEFINITION 11.1 Let us consider a point $b_0 \in \mathcal{C}$ and let $j = \{j_l\}_{l=1}^N$ be a sequence of $+1$'s and -1's. An orbit u_ε of system (2.1) is called an *N-pulse orbit with base* *point* b_0 *and jump sequence* j if for some $0 < \mu < \frac{1}{2}$ and for $\varepsilon > 0$ sufficiently small,

(i) u_ε intersects an unstable fiber $f^u(b_\varepsilon)$ with base point $b_\varepsilon = b_0 + \mathcal{O}(\varepsilon^\mu) \in \Pi$, and

(ii) outside a small fixed neighborhood of the manifold $\mathcal{M}_\varepsilon$, the orbit u_ε is order $\mathcal{O}(\sqrt{\varepsilon})$ H^1-close to a chain of unperturbed heteroclinic solutions $u^l(t)$, $l = 1, \ldots, N$, such that

$$\lim_{t\to-\infty} u^l(t) = b_0\,, \quad \lim_{t\to+\infty} u^{l-1}(t) = \lim_{t\to-\infty} u^l(t)\,, \quad l = 2, \ldots, N.$$

Furthermore, for $l = 1, \ldots, N$ and for all $t \in \mathbb{R}$, we have

$$u^l(t) \in \begin{cases} W_0^+(\mathcal{C}) & \text{if } j_l = +1 \\ W_0^-(\mathcal{C}) & \text{if } j_l = -1. \end{cases}$$

We have the following result for the existence of such N-pulse orbits:

THEOREM 11.2 *Suppose that for some positive integer* N *and for some* $\phi_0 \in S^1$ *we have*

$$2\Gamma\left[\sin\left(\phi_0 + l\Delta\phi\right) - \sin\phi_0\right] - l\left[\alpha\mathcal{I}_\alpha(\Omega) - \beta\mathcal{I}_\beta(\Omega, K)\right] \neq 0\,, \qquad l = 1, \ldots, N-1.$$

Then, for $\varepsilon > 0$ *sufficiently small, there exist constants* $0 < \mu < \frac{1}{2}$ *and* $C_\eta > 0$ *such that for* any $0 \leq |\eta_0| < C_\eta$*, the system* (2.1) *admits two* N*-pulse orbits* $u_\varepsilon^\pm$ *with base point* $b_\varepsilon \in \Pi$ *such that* $\phi_{b_\varepsilon} = \phi_0 + \mathcal{O}(\varepsilon^\mu)$ *and* $\eta_{b_\varepsilon} = \eta_0$*. The jump sequences of the orbits are given by*

$$j_{k+1}^\pm = \pm\operatorname{sign}[\Gamma[\sin\left(\phi_0 + k\Delta\phi\right) - \sin\phi_0] - \alpha k\mathcal{I}_\alpha(\Omega)]j_k^\pm\,, \qquad k = 1, \ldots, N-1,$$

where $j_1^\pm = \pm 1$*. In particular, if* N *is an integer satisfying the assumptions of Theorem 9.1, then for* $\varepsilon > 0$ *small, both* $W^{u+}(\Pi)$ *and* $W^{u-}(\Pi)$ *disintegrate into at least* $2N$ *disjoint components, all of which have different jump sequences.*

PROOF: Using the assumption of the theorem and the arguments from the proof of Theorem 7.3, we immediately conclude that for $\varepsilon > 0$ small enough, the inequalities

$$\Delta^l\mathcal{H}(\phi_0) + \delta_0\mathcal{F}_l(p_l(b_\varepsilon); \delta_0, \varepsilon^\mu) + \varepsilon^\mu\mathcal{G}_l(p_l(b_\varepsilon); \delta_0, \varepsilon^\mu) \neq 0$$

hold for $l = 1, \ldots, N-1$. As a result, the unstable manifold $W^u(\Pi)$ contains two N-pulse orbits. The jump sequences of these orbits can be found in exactly the same way as in the proof of Theorem 7.3. □

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G. HALLER
Brown University
Division of Applied Mathematics
Box F
Providence, RI 02912
E-mail: haller@cfm.brown.edu

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