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# Quasi-objective coherent structure diagnostics from single trajectories

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## ABSTRACT

We derive measures of local material stretching and rotation that are computable from individual trajectories without reliance on other trajectories or on an underlying velocity field. Both measures are quasi-objective: they approximate objective (i.e., observer-independent) coherence diagnostics in frames satisfying a certain condition. This condition requires the trajectory accelerations to dominate the angular acceleration induced by the spatial mean vorticity. We illustrate on examples how quasi-objective coherence diagnostics highlight elliptic and hyperbolic Lagrangian coherent structures even from very sparse trajectory data.

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**A vast amount of discrete tracer trajectory data is available for fluid flows in the laboratory and in nature. All features of tracer trajectories, however, including their shape, velocities, and looping, depend on the reference frame of the observer even though the coherent flow structures one often tries to infer from tracers are observer indifferent. Here, we develop quasi-objective coherent structure diagnostics that can be computed from individual single tracer trajectories without reliance on other trajectories. A quasi-objective diagnostic approximates an objective diagnostics in all frames of reference that satisfy certain conditions we derive. We show how the single-trajectory quasi-objective scalar fields highlight material coherent structures even from sparse trajectory data sets on which multi-trajectory objective diagnostics tend to perform poorly.**

## I. INTRODUCTION

We seek objective (i.e., observer-indifferent) diagnostics for material stretching and rotation that can be evaluated on individual trajectories without reliance on other trajectories or on the underlying velocity field. Applications calling for such diagnostics include the analysis of observational drifter data, balloon data, and particle tracks from particle tracking velocimetry (PTV).

All elementary features of particle paths are non-objective, i.e., depend on the observer. For instance, in a frame traveling with any given particle, the particle becomes just a fixed point. All features of

the trajectory, such as velocity, acceleration, looping number, curvature, and trajectory length, therefore, vanish in this frame. In contrast, Lagrangian coherent structures (LCS), the persistent features of the material deformation field of the fluid as a moving continuum, are objective by definition.<sup>27</sup> Accordingly, any self-consistent identification of LCS should be carried out using objective quantities. Indeed, objectivity as a minimal requirement for flow-feature identification was already identified in the 1970s<sup>4,10,11,34</sup> and a number of recent approaches conform to this requirement (see Refs. 17, 22, 27, 29, and 39 for reviews).

Objective geometric quantities used in LCS identification are, however, based on the gradient of the flow map whose computation requires a large number of trajectories to be released from an initial grid. Examples of such quantities include the finite-time Lyapunov exponent (or FTLE<sup>21</sup>) for hyperbolic LCS and the polar rotation angle (or PRA<sup>12</sup>) for elliptic LCS in two dimensions. The computation of these diagnostics is only feasible if the velocity field is known over, or can at least be interpolated onto, a regular grid over a full flow domain.<sup>27</sup> For the special case of closed (or elliptic) LCSs, alternative probabilistic methods have been developed that seek regions enclosed by such LCSs as coherent sets.<sup>13,14</sup> These objective methods do not technically require differentiation of the flow map but their implementations utilize box-counting methods that only converge for large numbers of densely spaced trajectories. More recent clustering methods for coherent set detection are, in principle, applicable to sparse trajectory data, but still require a reasonably high number of evenly placed trajectories to give meaningful

results.<sup>15,18,44</sup> None of these methods offer a coherence indicator that is computable on a single trajectory. On the contrary, the coherence measures assigned to individual trajectories by probabilistic and statistical methods will change if the position and number of the other trajectories used in the algorithm change—even though the underlying fluid flow and its coherent structures remain the same.

Notable exceptions to this trend are two objective indicators of the complexity of individual trajectories: the correlation dimension and the ergodicity defect proposed in Ref. 43 for LCS extraction. While these objective diagnostics can be computed from single trajectories, their connection to material attributes of fluid trajectory motion, such as material stretching and rotation, is unknown. Absolute dispersion,<sup>40</sup> trajectory length,<sup>36</sup> and maximal trajectory extent<sup>37</sup> are also attractively simple single-trajectory diagnostics, but they are not objective and their relationship to material stretching and rotation is also unknown. In contrast, the Lagrangian-averaged vorticity deviation (LAVD) is an objective single-trajectory diagnostic<sup>25</sup> but requires the knowledge of the full velocity field for the purposes of computing the vorticity along trajectories. A further non-objective, single-particle diagnostics for LCS identification is the Lagrangian spin parameter,<sup>47</sup> which relies on the calibration of an additional stochastic model for the velocity field. Yet, another approach adapts the wavelet ridge analysis method of Ref. 8 to Lagrangian trajectory analysis.<sup>32,33</sup> While physically insightful, this technique is non-objective either, as it fits a time-varying ellipse model to identify signatures of coherent eddies based on the looping characteristics of trajectories.

While all identifiable features of individual trajectories are non-objective, trajectory data are often the only available information about fluid flows. It is, therefore, essential to assess what trajectory-based quantities (if any) can still be tied mathematically, at least in qualifying frames, to established and objective LCS diagnostics. Answering this question would enable, for instance, a mathematically justifiable and physically self-consistent extraction of Lagrangian jets, fronts, and eddies from NOAA’s Global Drifter Program, which comprises more than 20 000 trajectories.<sup>35</sup> Such an extraction should then lead to a systematic assessment of the performance of promising nonobjective trajectory-based methods, such as Ref. 33, in avoiding false positives and negatives in elliptic LCS detection.

Here, we address this challenge by deriving objective coherence diagnostics for material stretching and rotation that are computable solely from single-particle trajectory data. Each diagnostic approximates a related objective, material coherence measure in frames that satisfy certain computable conditions. We refer to the approximate coherence diagnostics developed here as quasi-objective under those conditions. Physically, the quasi-objectivity condition we obtain requires the trajectory accelerations to dominate the angular acceleration induced by the spatial mean vorticity in the given frame. We illustrate the power of quasi-objective stretching and rotation measures on two- and three-dimensional examples.

## II. OBJECTIVE LAGRANGIAN STRETCHING AND ROTATION MEASURES

Consider particle motions generated by a differential equation

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t), \tag{1}$$

with a continuously differentiable velocity field  $\mathbf{v}$  defined for times  $t \in [t_0, t_N]$  on a spatial domain  $U \subset \mathbb{R}^n$  of dimension  $n \geq 1$ . Let  $\mathbf{x}(t; t_0, \mathbf{x}_0)$  be a trajectory with initial condition  $\mathbf{x}(t_0; t_0, \mathbf{x}_0) = \mathbf{x}_0$ . The flow map induced by trajectories is  $\mathbf{F}_{t_0}^t: \mathbf{x}_0 \rightarrow \mathbf{x}(t; t_0, \mathbf{x}_0)$ , mapping initial conditions at time  $t_0$  to their current positions at time  $t$ . When  $t_0$  and  $\mathbf{x}_0$  are fixed and hence do not need to be carried in our arguments, we will also use the shorthand notation  $\mathbf{x}(t) := \mathbf{x}(t; t_0, \mathbf{x}_0)$ .

A Lagrangian scalar field  $\mathcal{P}(\mathbf{x}_0)$  is called objective<sup>46</sup> if it remains invariant under all Euclidean observer changes of the form

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{y} + \mathbf{b}(t), \tag{2}$$

with a proper orthogonal tensor  $\mathbf{Q}(t) \in \text{SO}(n)$  and a vector  $\mathbf{b}(t)$ . Specifically, under all such transformations, the transformed scalar field  $\tilde{\mathcal{P}}$  in the  $\mathbf{y}$ -frame satisfies  $\tilde{\mathcal{P}}(\mathbf{y}_0) \equiv \mathcal{P}(\mathbf{x}_0)$ .

Consider a material curve,  $\boldsymbol{\gamma}(t; s) = \mathbf{F}_{t_0}^t(\boldsymbol{\gamma}(t_0; s)) \subset U$ , at time  $t$ , which has evolved from an initial curve  $\boldsymbol{\gamma}(t_0; s)$  parameterized by the scalar parameter  $s \in \mathbb{R}$ . Any tangent vector,

$$\boldsymbol{\xi}(t; s) = \partial_s \boldsymbol{\gamma}(t; s), \tag{3}$$

of  $\boldsymbol{\gamma}$  then satisfies the evolution equation

$$\begin{aligned} \dot{\boldsymbol{\xi}}(t; s) &= \partial_t \boldsymbol{\xi}(t; s) = \partial_t \partial_s \boldsymbol{\gamma}(t; s) = \partial_s \partial_t \boldsymbol{\gamma}(t; s) = \partial_s \partial_t \mathbf{F}_{t_0}^t(\boldsymbol{\gamma}(t_0; s)) \\ &= \partial_s \mathbf{v}(\boldsymbol{\gamma}(t; s), t) = \nabla \mathbf{v}(\boldsymbol{\gamma}(t; s), t) \partial_s \boldsymbol{\gamma}(t; s). \end{aligned}$$

Therefore, the tangent vector  $\boldsymbol{\xi}(t; s)$  of the material curve  $\boldsymbol{\gamma}(t; s)$  satisfies the classic equation of variations<sup>3</sup>

$$\dot{\boldsymbol{\xi}} = \nabla \mathbf{v}(\mathbf{x}(t), t) \boldsymbol{\xi} \tag{4}$$

along the trajectory  $\mathbf{x}(t)$  with  $\mathbf{x}(t_0) = \boldsymbol{\gamma}(t_0; s)$ .

### A. Objective stretching measures

We now fix the parameter  $s$  along the material curve  $\boldsymbol{\gamma}(t; s)$  such that  $\boldsymbol{\gamma}(t_0; s) = \mathbf{x}_0$ . We then seek to characterize the time-evolution of the tangent vector  $|\boldsymbol{\xi}(t)|$  at  $\boldsymbol{\gamma}(t; s)$  with an exponent  $\lambda$  as

$$|\boldsymbol{\xi}(t)| \sim e^{\lambda(t-t_0)} |\boldsymbol{\xi}_0| \tag{5}$$

for  $\boldsymbol{\xi}_0 \neq \mathbf{0}$ . This leads us to define the *averaged stretching exponent*

$$\lambda_{t_0}^{t_N}(\mathbf{x}_0, \boldsymbol{\xi}_0) := \frac{1}{t_N - t_0} \log \frac{|\boldsymbol{\xi}(t_N)|}{|\boldsymbol{\xi}_0|}, \tag{6}$$

sometimes called the finite-time Lyapunov exponent associated with the initial vector  $\boldsymbol{\xi}_0$  at the initial location  $\mathbf{x}_0$  (see, e.g., Ref. 38). Under any Euclidean observer change (2), we obtain

$$\tilde{\boldsymbol{\xi}} = \mathbf{Q}^T \boldsymbol{\xi},$$

whose substitution into (6) shows that the Lagrangian scalar field  $\lambda_{t_0}^{t_N}(\mathbf{x}_0, \boldsymbol{\xi}_0)$  is objective, i.e., remains invariant under arbitrary observer changes of the form (2). We recall that the maximum of  $\lambda > 0$  with respect to the direction  $\boldsymbol{\xi}_0$  is the finite-time Lyapunov exponent (FTLE) associated with the initial point  $\mathbf{x}_0$  by definition.<sup>27</sup>

If the instantaneous growth exponent satisfies  $\lambda < 0$  at time  $t$  in a two-dimensional incompressible flow, then a normal vector  $\boldsymbol{\xi}^\perp(t; s)$  to  $\boldsymbol{\xi}(t; s)$  at the same point of the trajectory will have a

growth exponent equal to  $|\lambda| > 0$ . Similarly, in three-dimensional incompressible flows,  $\lambda < 0$  implies that the instantaneous growth exponent of the *area* in a plane normal to the material line  $\boldsymbol{\gamma}(t; s)$  is equal to  $|\lambda| > 0$ . Therefore,  $|\lambda|$  is a measure of the strength of hyperbolicity experienced by a trajectory in an incompressible flow. To measure the temporal average of  $|\lambda|$  along the trajectory  $\boldsymbol{x}(t)$ , we define the *averaged hyperbolicity strength*

$$\bar{\lambda}_{t_0}^{t_N}(\boldsymbol{x}_0, \boldsymbol{\xi}_0) := \frac{1}{t_N - t_0} \int_{t_0}^{t_N} \left| \frac{d}{dt} \log \frac{|\boldsymbol{\xi}(t)|}{|\boldsymbol{\xi}_0|} \right| dt, \quad (7)$$

where  $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ . While the quantity  $\lambda_{t_0}^{t_N}$  measures the average net stretching (including shrinkage) in the direction of  $\boldsymbol{\xi}(t)$  over the time interval  $[t_0, t_N]$ , the quantity  $\bar{\lambda}_{t_0}^{t_N}$  accounts for all (positive) stretching experienced either parallel or normal to  $\boldsymbol{\xi}(t)$ . For this reason,  $\bar{\lambda}_{t_0}^{t_N}$  is a measure of the average strength of hyperbolicity experienced by a trajectory of an incompressible flow. By the same considerations that we have applied to  $\lambda_{t_0}^{t_N}(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ , the scalar field  $\bar{\lambda}_{t_0}^{t_N}(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$  is also objective.

### B. Objective rotation measures

To introduce our second class of trajectory-based diagnostic tool, we normalize the material tangent vector  $\boldsymbol{\xi}(t)$  defined in (3) to the unit tangent vector

$$\boldsymbol{e}(t) = \frac{\boldsymbol{\xi}(t)}{|\boldsymbol{\xi}(t)|}$$

of the material line  $\boldsymbol{\gamma}(t; s)$ ; we will denote by  $\boldsymbol{e}_0 := \boldsymbol{e}(t_0) = \boldsymbol{\xi}_0 / |\boldsymbol{\xi}_0|$  the initial position of  $\boldsymbol{e}(t)$  at time  $t_0$ .

Under any Euclidean observer change (2), we then obtain

$$\tilde{\boldsymbol{e}}(t) = \mathbf{Q}^T(t)\boldsymbol{e}(t). \quad (8)$$

Over the time interval  $[t_0, t_N]$ , the net rotation angle  $\alpha_{t_0}^{t_N}(\boldsymbol{x}_0, \boldsymbol{e}_0)$  per unit time experienced by  $\boldsymbol{\xi}_0$  along the trajectory starting from  $\boldsymbol{x}_0 = \boldsymbol{\gamma}(t_0; s)$  can be computed as

$$\alpha_{t_0}^{t_N}(\boldsymbol{x}_0, \boldsymbol{e}_0) = \frac{1}{t_N - t_0} \cos^{-1}(\boldsymbol{e}(t_0), \boldsymbol{e}(t_N)). \quad (9)$$

Formula (8) shows that  $\alpha_{t_0}^{t_N}$  is not an objective scalar because  $\boldsymbol{e}_0 = \boldsymbol{e}(t_0)$  is rotated by  $\mathbf{Q}^T(t_0)$  under an observer change (2), whereas  $\boldsymbol{e}(t_N)$  is rotated by  $\mathbf{Q}^T(t_N)$ . Still, rotations commute in two dimensions; therefore, if we have  $\alpha_{t_0}^t(\boldsymbol{x}_{01}, \boldsymbol{e}_{01}) = \alpha_{t_0}^t(\boldsymbol{x}_{02}, \boldsymbol{e}_{02})$  for two different initial positions  $\boldsymbol{x}_{01}$  and  $\boldsymbol{x}_{02}$  with corresponding initial material tangent unit vectors  $\boldsymbol{e}_{01}$  and  $\boldsymbol{e}_{02}$ , then we also have  $\alpha_{t_0}^t(\boldsymbol{y}_{01}, \tilde{\boldsymbol{e}}_{01}) = \alpha_{t_0}^t(\boldsymbol{y}_{02}, \tilde{\boldsymbol{e}}_{02})$  for their transformed counterparts. This is because the angle between  $\boldsymbol{e}_1(t_0)$  and  $\boldsymbol{e}_1(t_N)$  will be altered by the spatially independent rotation tensors  $\mathbf{Q}^T(t_0)$  and  $\mathbf{Q}^T(t_N)$  in the new frame by the same amount as the angle between  $\boldsymbol{e}_2(t_0)$  and  $\boldsymbol{e}_2(t_N)$  will be altered by the same rotation tensors. We conclude that the set of level curves of (9) is objective (i.e., invariant under observer changes), even though the value of  $\alpha_{t_0}^t(\boldsymbol{x}_0, \boldsymbol{e}_0)$  on them changes from one frame to the other. We stress that the same statement does not hold in three dimensions unless the vector field  $\boldsymbol{v}$  itself is objective. These properties mimic the objectivity properties of the PRA.<sup>12</sup>

We now seek to construct a quantity that objectively characterizes the rotation of unit vectors in both two and three dimensions. To this end, we consider the deviation of  $\dot{\boldsymbol{e}}$  from the domain-average of the local mean velocities of all unit material tangent vectors emanating from the same point. We first recall the result from Ref. 23 that the local mean angular velocity of material fibers at a point  $\boldsymbol{x}$  at time  $t$  equals

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{1}{2} \boldsymbol{\omega}(\boldsymbol{x}, t), \quad (10)$$

with  $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$  denoting the vorticity field. We also recall that under a frame change (2), vorticity transforms into

$$\tilde{\boldsymbol{\omega}} = \mathbf{Q}^T(\boldsymbol{\omega} - \dot{\boldsymbol{q}}), \quad (11)$$

where  $\dot{\boldsymbol{q}}$  is the vorticity of the frame change defined by the identity

$$\frac{1}{2} \dot{\boldsymbol{q}} \times \boldsymbol{e} = \dot{\mathbf{Q}} \mathbf{Q}^T \boldsymbol{e}$$

for all vectors  $\boldsymbol{e} \in \mathbb{R}^3$ .

By Eq. (10), the deviation of  $\dot{\boldsymbol{e}}$  from the spatially averaged local velocities of material unit vectors, or *relative tangent velocity*, is

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{e}} - \frac{1}{2} \bar{\boldsymbol{\omega}}(t) \times \boldsymbol{e} = \dot{\boldsymbol{e}} - \bar{\mathbf{W}}(t) \boldsymbol{e}, \quad (12)$$

with  $\bar{\boldsymbol{\omega}}(t)$  denoting the spatial average of the vorticity over the domain  $U$  and with  $\bar{\mathbf{W}}(t)$  denoting the spatial average of the spin tensor

$$\mathbf{W} = \frac{1}{2} [\nabla \boldsymbol{v} - [\nabla \boldsymbol{v}]^T] = -\mathbf{W}^T \quad (13)$$

over the same domain. Under the frame change (2), formula (11) shows that  $\dot{\boldsymbol{\epsilon}}$  transforms as

$$\begin{aligned} \dot{\tilde{\boldsymbol{\epsilon}}} &= \dot{\boldsymbol{\epsilon}} - \frac{1}{2} \tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{e}} = \mathbf{Q}^T [\dot{\boldsymbol{\epsilon}} - \dot{\mathbf{Q}} \mathbf{Q}^T \boldsymbol{e}] - \frac{1}{2} \mathbf{Q}^T (\tilde{\boldsymbol{\omega}} - \dot{\boldsymbol{q}}) \times \mathbf{Q}^T \boldsymbol{e} \\ &= \mathbf{Q}^T \left[ \dot{\boldsymbol{\epsilon}} - \frac{1}{2} \dot{\boldsymbol{q}} \times \boldsymbol{e} \right] - \frac{1}{2} \mathbf{Q}^T (\tilde{\boldsymbol{\omega}} - \dot{\boldsymbol{q}}) \times \mathbf{Q}^T \boldsymbol{e} \\ &= \mathbf{Q}^T \left[ \dot{\boldsymbol{\epsilon}} - \frac{1}{2} \dot{\boldsymbol{q}} \times \boldsymbol{e} - \frac{1}{2} (\tilde{\boldsymbol{\omega}} - \dot{\boldsymbol{q}}) \times \boldsymbol{e} \right] \\ &= \mathbf{Q}^T \left[ \dot{\boldsymbol{\epsilon}} - \frac{1}{2} \tilde{\boldsymbol{\omega}} \times \boldsymbol{e} \right] = \mathbf{Q}^T \dot{\boldsymbol{\epsilon}}, \end{aligned}$$

implying that the relative tangent velocity  $\dot{\boldsymbol{\epsilon}}$  of  $\boldsymbol{\xi}$  is an objective vector. Consequently,

$$\begin{aligned} \bar{\alpha}_{t_0}^{t_N}(\boldsymbol{x}_0, \boldsymbol{\xi}_0) &:= \frac{1}{t_N - t_0} \int_{t_0}^{t_N} |\dot{\boldsymbol{\epsilon}}(t)| dt \\ &= \frac{1}{t_N - t_0} \int_{t_0}^{t_N} \left| \dot{\boldsymbol{\epsilon}}(t) - \frac{1}{2} \tilde{\boldsymbol{\omega}}(t) \times \boldsymbol{e}(t) \right| dt, \quad (14) \end{aligned}$$

with  $\boldsymbol{e}(t) = \boldsymbol{\xi}(t) / |\boldsymbol{\xi}(t)|$ , is an objective measure for the average rotation speed experienced by the tangent vector  $\boldsymbol{\xi}$  during its evolution along the trajectory starting from  $\boldsymbol{x}_0$ . Indeed, this rate of change is entirely due to the rotation of  $\boldsymbol{\xi}$ , unaffected by any change in the length of  $\boldsymbol{\xi}$ .

We have used the  $\bar{\alpha}_{t_0}^{t_N}$  notation in (14) to point out that this quantity (the averaged norm of an angular velocity) has the same

relation to  $\alpha_{t_0}^{t_N}$  (an average angular velocity) as the averaged norm of the stretching rate  $\bar{\lambda}_{t_0}^{t_N}$  has to the average stretching rate  $\lambda_{t_0}^{t_N}$ . Unlike  $\lambda_{t_0}^{t_N}$ , however,  $\alpha_{t_0}^{t_N}$  is not objective and that is why the definition of  $\bar{\alpha}_{t_0}^{t_N}$  contains a subtracted mean rotation rate that makes  $\bar{\alpha}_{t_0}^{t_N}$  objective.

### III. QUASI-OBJECTIVE TRAJECTORY STRETCHING AND ROTATION IN STEADY FLOWS

We will call a Lagrangian scalar field  $P(\mathbf{x}_0)$  computed from  $\{\mathbf{x}(t_i)\}_{i=0}^N$  quasi-objective under a condition (A) if there exists an objective Lagrangian scalar field  $\mathcal{P}(\mathbf{x}_0)$  such that

$$P(\mathbf{x}_0) \approx \mathcal{P}(\mathbf{x}_0) \tag{15}$$

in any  $\mathbf{x}$ -frame in which condition (A) holds. The accuracy of the approximation indicated by the symbol  $\approx$  in (15) depends on the extent to which the condition (A) is satisfied.

In other words, a frame-dependent scalar field  $P(\mathbf{x}_0)$  is quasi-objective under a condition (A) if it happens to approximate the same objective scalar field  $\mathcal{P}(\mathbf{x}_0)$  in any frame in which condition (A) is satisfied.  $P(\mathbf{x}_0)$  will generally not approximate an objective field in all frames, as condition (A) is generally frame-dependent and hence will not hold in all frames. In Secs. III A and III B, we will derive quasi-objective measures of material stretching and rotation with computable formulas for their corresponding quasi-objectivity condition.

Our first quasi-objectivity conditions for our upcoming derivations will be the assumption that the velocity field is steady in the current frame, i.e.,

$$(A1) \quad |\partial_t \mathbf{v}(\mathbf{x}, t)| = 0,$$

holds. We will remove this assumption in our later extension of our results to unsteady flows.

Assumption (A1) enables us to rewrite (1) as the autonomous dynamical system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}). \tag{16}$$

Trajectories of two-dimensional steady flows ( $n = 2$ ) coincide with streamlines and already provide detailed information about coherent structures without further analysis. For this reason, the present discussion is mostly of interest for three-dimensional flows ( $n = 3$ ). Although the flow map and trajectories of the autonomous differential equation depend on the elapsed time  $t - t_0$  and hence  $t_0 = 0$  could be fixed, we still keep the explicit dependence on the current time  $t$  and initial time  $t_0$  in our formulas to facilitate their later extension to the unsteady case in which (A1) will no longer be assumed.

#### A. Trajectory-based approximations of stretching measures

Consider now the Lagrangian velocity vector

$$\mathbf{v}(t) := \mathbf{v}(\mathbf{x}(t)),$$

which satisfies the homogeneous linear evolution equation

$$\dot{\mathbf{v}} = \nabla \mathbf{v}(\mathbf{x}(t)) \mathbf{v}. \tag{17}$$

Note that this formula relies heavily on our assumption (A1); otherwise, the extra term  $\partial_t \mathbf{v}(\mathbf{x}(t), t)$  would appear on the right-hand side, making (17) an inhomogeneous linear system of differential equations.

A comparison of Eqs. (4) and (17) shows that  $\mathbf{v}(t)$  evolves as a material tangent vector in the given coordinate frame. Therefore, for the choice of the initial material tangent vector  $\boldsymbol{\xi}_0 = \mathbf{v}_0 := \mathbf{v}(\mathbf{x}_0) \neq \mathbf{0}$ , the averaged stretching exponent and averaged hyperbolicity strength can be written as

$$\begin{aligned} \lambda_{t_0}^{t_N}(\mathbf{x}_0, \mathbf{v}_0) &= \frac{1}{t_N - t_0} \log \frac{|\mathbf{v}(\mathbf{x}(t_N))|}{|\mathbf{v}_0|}, \\ \bar{\lambda}_{t_0}^{t_N}(\mathbf{x}_0, \mathbf{v}_0) &= \frac{1}{t_N - t_0} \int_{t_0}^{t_N} \left| \frac{d}{dt} \log \frac{|\mathbf{v}(\mathbf{x}(t))|}{|\mathbf{v}_0|} \right| dt. \end{aligned} \tag{18}$$

The right-hand sides of the formulas in (18) give close approximation for the objective expressions (6) and (7) with  $\boldsymbol{\xi}_0 = \mathbf{v}_0$  in the frame where assumption (A1) holds for the velocity field. In other words,  $\lambda_{t_0}^{t_N}(\mathbf{x}_0, \mathbf{v}_0)$  and  $\bar{\lambda}_{t_0}^{t_N}(\mathbf{x}_0, \mathbf{v}_0)$  are quasi-objective under condition (A1).

If discretized trajectory data  $\{\mathbf{x}(t_i)\}_{i=0}^N$  is available for the trajectory  $\mathbf{x}(t)$ , then formulas (18) yield the following result, with the trajectory velocities  $\dot{\mathbf{x}}(t_j)$  computed from finite differencing or spline interpolation.

**Theorem 1.** Under assumption (A1), the trajectory stretching exponents (TSEs), defined as

$$\begin{aligned} \text{TSE}_{t_0}^{t_N}(\mathbf{x}_0) &= \frac{1}{t_N - t_0} \log \frac{|\dot{\mathbf{x}}(t_N)|}{|\dot{\mathbf{x}}(t_0)|}, \\ \bar{\text{TSE}}_{t_0}^{t_N}(\mathbf{x}_0) &= \frac{1}{t_N - t_0} \sum_{i=0}^{N-1} \left| \log \frac{|\dot{\mathbf{x}}(t_{i+1})|}{|\dot{\mathbf{x}}(t_i)|} \right| \end{aligned} \tag{19}$$

for all  $\dot{\mathbf{x}}(t_i) \neq \mathbf{0}$ ,  $i = 0, \dots, N$ , are quasi-objective measures of trajectory stretching and hyperbolicity strength.

The following remarks are in order regarding Theorem 1:

1. While two-dimensional steady flows can already be fully understood from streamlines generated by trajectory plots, the analysis of three-dimensional steady flows benefits from the TSE diagnostics.
2. The TSE metrics are only quasi-objective, not objective. One manifestation of this is that they depend on trajectory velocities that are not objective. That said, the TSE metrics give the correct stretching of velocity vectors in case those vectors evolve materially in a given frame, which is guaranteed by condition (A1).
3. The TSE metrics can generally be used to highlight hyperbolic invariant manifolds of an autonomous vector field. That vector field does not have to be a fluid velocity field, and hence, unlike fluid velocity fields, it may be objective. For such objective autonomous vector fields, the TSE metrics are objective, not only quasi-objective, given that assumption (A1) is satisfied in any observer frame. Examples of objective autonomous vector fields include the active barrier vector fields  $\mathbf{x}' = \mathbf{w}(\mathbf{x}; t)$  derived

in Ref. 26, with the prime denoting differentiation with respect to a parameter along a barrier curve. With respect to this parameter, the vector field  $\mathbf{w}(\mathbf{x}; t)$  is autonomous. Under observer changes of the form (2),  $\mathbf{w}(\mathbf{x}; t)$  remains autonomous and transforms objectively as  $\mathbf{y}' = \tilde{\mathbf{w}}(\mathbf{y}; t) = \mathbf{Q}^T(t)\mathbf{w}(\mathbf{x}; t)$ . Invariant manifolds of  $\mathbf{w}(\mathbf{x}; t)$  are shown in Ref. 26 to coincide with time  $t$  positions of objective material barriers to vorticity and momentum in the underlying fluid flow. In Sec. V C, we show the application of the TSE metrics to the computation of objective momentum transport barriers in the unsteady ABC flow.

4. By its construction, the TSE field is expected to reproduce features of the FTLE field but without the computational burdens associated with the FTLE field. Indeed, unlike the FTLE, the TSE can be computed even for a single trajectory.
5. For  $N = 1$ , ridges (or trenches) of the TSE yield single-trajectory-based approximation for hyperbolic (or parabolic) objective Eulerian coherent structures (or OECS<sup>45</sup>), which are the instantaneous limits of hyperbolic (or parabolic) LCSs.

### B. Trajectory-based approximations of rotation measures

Consider now the normalized Lagrangian velocity vector

$$\mathbf{e}_v(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

Equation (12) shows that if  $\frac{1}{2}\bar{\boldsymbol{\omega}}(t) \times \mathbf{e}$  is much smaller in norm than  $\dot{\mathbf{e}}$ , then  $\dot{\mathbf{e}}$  is approximately equal to  $\dot{\mathbf{e}}$ . Therefore, if assumption (A1) holds [thus  $\mathbf{e}_v(t)$  evolves as a material unit vector  $\mathbf{e}(t)$ ] and, in addition,

(A2)

$$\left| \frac{1}{2}\bar{\boldsymbol{\omega}}(t) \times \mathbf{e}_v(t) \right| \ll |\dot{\mathbf{e}}_v(t)|$$

holds in the given frame, then the relative tangent velocity

$$\dot{\mathbf{e}}_v = \dot{\mathbf{e}}_v - \frac{1}{2}\bar{\boldsymbol{\omega}} \times \mathbf{e}_v$$

of  $\mathbf{v}(t)$  satisfies

$$|\dot{\mathbf{e}}_v| \approx |\dot{\mathbf{e}}_v|.$$

Note that assumption (A2) requires the Lagrangian accelerations to dominate the angular acceleration of the trajectory induced by the spatial mean vorticity. Consequently, if assumptions (A1) and (A2) hold in the frame in which the trajectory data  $\mathbf{x}(t)$  are given, then

$$\bar{\alpha}_0^{t_N}(\mathbf{x}_0, \mathbf{v}_0) := \frac{1}{t_N - t_0} \int_{t_0}^{t_N} |\dot{\mathbf{e}}_v(t)| dt \quad (20)$$

approximates the average of  $|\dot{\mathbf{e}}_v|$ .

To see the physical meaning of  $\bar{\alpha}_0^{t_N}(\mathbf{x}_0, \mathbf{v}_0)$  more specifically, we set  $t_0 = t$  and  $t_N = t + \delta$  in formula (9) to obtain

$$\cos(\alpha_t^{t+\delta}) = \langle \mathbf{e}_v(t), \mathbf{e}_v(t + \delta) \rangle \quad (21)$$

for the angle  $\alpha_t^{t+\delta}$  between  $\mathbf{e}_v(t)$  and  $\mathbf{e}_v(t + \delta)$ . Differentiating (21) with respect to  $\delta$  leads to the expression

$$-|\alpha_t^{t+\delta}|^2 \delta + \mathcal{O}(\delta^2) = \langle \mathbf{e}_v(t), \dot{\mathbf{e}}_v(t + \delta) \rangle,$$

where we have assumed that  $\lim_{\delta \rightarrow 0} |\alpha_t^{t+\delta}|$  exists and is bounded. Dividing both sides by  $(-\delta)$  and taking the  $\delta \rightarrow 0$  limit gives

$$|\alpha_t^t|^2 = \lim_{\delta \rightarrow 0} [-\langle \mathbf{e}_v(t), \dot{\mathbf{e}}_v(t + \delta) \rangle] = |\dot{\mathbf{e}}_v(t)|^2, \quad (22)$$

where the last equality follows from differentiating the identity  $\langle \mathbf{e}(t), \mathbf{e}(t) \rangle \equiv 1$  twice with respect to  $t$ . Formula (22) shows that  $\lim_{\delta \rightarrow 0} |\alpha_t^{t+\delta}|$  indeed exists and is bounded, as assumed. Using this formula in (20) then gives

$$\bar{\alpha}_0^{t_N}(\mathbf{x}_0, \mathbf{v}_0) = \frac{1}{t_N - t_0} \int_{t_0}^{t_N} |\alpha_t^t| dt, \quad (23)$$

showing that  $\bar{\alpha}_0^{t_N}(\mathbf{x}_0, \mathbf{v}_0)$  is the time-average of  $|\alpha_t^t|$ . If only discretized trajectory data  $\{\mathbf{x}(t_i)\}_{i=1}^N$  are available, then a discretized approximation of (23) leads to the following result.

**Theorem 2.** Under assumptions (A1) and (A2), the trajectory angular velocity, defined as

$$\overline{\text{TRA}}_0^{t_N}(\mathbf{x}_0) = \frac{1}{t_N - t_0} \sum_{i=0}^{N-1} \cos^{-1} \frac{\langle \dot{\mathbf{x}}(t_i), \dot{\mathbf{x}}(t_{i+1}) \rangle}{|\dot{\mathbf{x}}(t_i)| |\dot{\mathbf{x}}(t_{i+1})|} \quad (24)$$

for all  $\dot{\mathbf{x}}(t_i) \neq \mathbf{0}$ ,  $i = 0, \dots, N$ , is a quasi-objective measure of total trajectory rotation.

Indeed, the  $i$ th term of the sum in (24) measures the angle between the velocity vectors  $\dot{\mathbf{x}}(t_i)$  and  $\dot{\mathbf{x}}(t_{i+1})$ , returning always a positive value to account for the modulus sign in (23). As already noted, the trajectory velocities  $\dot{\mathbf{x}}(t_i)$  can be computed from finite differencing or spline interpolation performed on the subsequent trajectory positions  $\mathbf{x}(t_i)$ .

We make the following remarks regarding Theorem 2:

1. For the objective autonomous vector fields mentioned in Remark 3 after Theorem 1, the unit vector field  $\mathbf{e}_v(t)$  is already objective and hence one does not need to use its modified version,  $\mathbf{e}_v(t)$ . As a consequence, assumption (A2) is no longer necessary for such vector fields and  $\overline{\text{TRA}}$  is objective without further assumptions (see Ref. 26 for examples of such vector fields). For any  $\mathbf{v}(\mathbf{x})$ , the net rotation speed  $\alpha_0^{t_N}$  defined in (9) is also an objective vector field and can be approximated from trajectories as

$$\text{TRA}_0^{t_N}(\mathbf{x}_0) = \frac{1}{t_N - t_0} \cos^{-1} \frac{\langle \dot{\mathbf{x}}(t_0), \dot{\mathbf{x}}(t_N) \rangle}{|\dot{\mathbf{x}}(t_0)| |\dot{\mathbf{x}}(t_N)|} \quad (25)$$

in both two and three dimensions (see Sec. V C for an application).

2. Assumption (A2) will automatically hold for any trajectory with nonzero acceleration if  $\bar{\boldsymbol{\omega}}(t) \approx \mathbf{0}$ . The mean vorticity  $\bar{\boldsymbol{\omega}}(t)$  has

been found to vanish, up to a very good approximation, in large enough ocean regions.<sup>1,7,23</sup>

3. By their construction, the  $\overline{\text{TRA}}$  and TRA fields are expected to reproduce features of the PRA field but without the computational challenges of the PRA field. Indeed, unlike the PRA, the  $\overline{\text{TRA}}$  and TRA can be computed even for a single trajectory.
4. For  $N = 1$ , closed level curves of the  $\overline{\text{TRA}}$  yield single-trajectory-based approximation for elliptic OECs,<sup>45</sup> the instantaneous limits of elliptic LCSs.

In order to verify assumption (A2), one can compute  $\ddot{\mathbf{x}}(t)$  by finite differencing as we already noted in Sec. III A. In addition, one needs an estimate for the averaged vorticity  $\bar{\omega}(t)$ , which may either be generally known for the underlying velocity field or needs to be estimated from the available trajectory data (see, e.g., Ref. 41).

#### IV. QUASI-OBJECTIVE COHERENCE MEASURES FOR UNSTEADY FLOWS

We now extend the results of Sec. III to unsteady flows. The idea is to consider trajectory motion in the extended phase space of positions and times, in which trajectories are always governed by a steady velocity field. Each trajectory is a material curve in the extended phase space. One may, therefore, track the relative stretching and rotation of the tangent vectors of such curves using the measures we introduced in Sec. III for steady flows. Apart from the work of Ref. 5 on an extension of the Q-criterion, coherent structures have apparently not been studied in the extended phase space.

The quantities  $\mathbf{x}$ ,  $t$ , and  $\mathbf{v}$  generally have physical units, but our upcoming discussion requires non-dimensionalized variables. To this end, we select a dimensional characteristic length  $L > 0$  and a dimensional characteristic time  $T > 0$  for the flow defined by (1) and introduce the new quantities

$$\mathbf{y} := \frac{\mathbf{x}}{L}, \quad \tau := \tau_0 + \frac{t - t_0}{T}, \quad v_0 := \frac{L}{T}, \quad (26)$$

$$\mathbf{u}(\mathbf{y}, \tau) := \mathbf{v}(L\mathbf{y}, t_0 + T(\tau - \tau_0)),$$

where  $\mathbf{y}$  is a non-dimensional position variable,  $\tau$  is a non-dimensional time,  $v_0$  is a characteristic (dimensional) velocity for the velocity field  $\mathbf{v}(\mathbf{x}, t)$ , and  $\mathbf{u}(\mathbf{y}, \tau)$  is the non-dimensionalized velocity field. If the variables in (1) were non-dimensional to begin with, then one can simply set  $L = T = v_0 = 1$ ,  $\tau = t$ ,  $\mathbf{y} = \mathbf{x}$ , and  $\mathbf{u} \equiv \mathbf{v}$  in all our formulas below.

The non-dimensionalized trajectories satisfy the differential equation

$$\mathbf{y}' = \mathbf{u}(\mathbf{y}, \tau), \quad (27)$$

with  $\mathbf{y} \in V \subset \mathbb{R}^n$  and  $\tau \in [\tau_0, \tau_N]$ . We introduce the extended phase space

$$\mathcal{V} = \left\{ \mathbf{Y} = \begin{pmatrix} \mathbf{y} \\ z \end{pmatrix} : \mathbf{y} \in V, z \in \mathbb{R} \right\},$$

in which (1) becomes an autonomous dynamical system of the form

$$\mathbf{Y}' = \mathbf{U}(\mathbf{Y}), \quad \mathbf{U}(\mathbf{Y}) := \begin{pmatrix} \mathbf{u}(\mathbf{y}, z) \\ 1 \end{pmatrix}, \quad \mathbf{Y} \in \mathcal{V}. \quad (28)$$

Note that without non-dimensionalization, different components of the extended quantities  $\mathbf{Y}$  and  $\mathbf{U}$  would have different physical units.

The autonomous dynamical system (28) is of the general form (1) with  $n = 3$  (planar flows) or  $n = 4$  (three-dimensional flows). An extended material curve in this system is of the form

$$\Gamma(\tau; s) = \begin{pmatrix} \boldsymbol{\gamma}(\tau; s) \\ \gamma_z(s) \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{\tau_0}^{\tau}(\boldsymbol{\gamma}(\tau_0; s)) \\ s \end{pmatrix}. \quad (29)$$

The initial tangent vector  $\Xi_0 = \partial_s \Gamma(\tau_0; s)$  to this material curve evolves under the extended equation of variations

$$\Xi' = \partial_{\mathbf{Y}} \mathbf{U}(\mathbf{Y}(\tau)) \Xi. \quad (30)$$

The relationship between the extended material tangent vector  $\Xi(\tau)$  and the tangent vector  $\xi(t)$  of a material curve  $\boldsymbol{\gamma}(t; s)$  advected by the original velocity field  $\mathbf{v}(\mathbf{x}, t)$  is

$$\Xi(\tau) = \begin{pmatrix} \frac{T}{L} \xi(t) \\ 1 \end{pmatrix} = \frac{1}{v_0} \begin{pmatrix} \xi(t) \\ v_0 \end{pmatrix}, \quad (31)$$

with  $v_0$  denoting the characteristic velocity introduced in the non-dimensionalization (26).

We can apply the general Lagrangian stretching and rotation measures introduced in Sec. II to track the stretching and rotation of  $\Xi(\tau)$ . As we have seen, these measures,  $\lambda_{\tau_0}^{\tau N}(\mathbf{Y}_0, \Xi_0)$ ,  $\bar{\lambda}_{\tau_0}^{\tau N}(\mathbf{Y}_0, \Xi_0)$ , and  $\bar{\alpha}_{\tau_0}^{\tau N}(\mathbf{Y}_0, \Xi_0)$ , will remain invariant under all time-dependent, formally extended Euclidean transformations

$$\mathbf{Y} = \mathcal{Q}(\tau) \tilde{\mathbf{Y}} + \mathbf{B}(\tau), \quad \mathcal{Q}(\tau) = \mathcal{Q}^{-T}(\tau) \in SO(n + 1) \quad (32)$$

of the extended phase space. In particular, the extended stretching and rotation measures for  $\Xi(\tau)$  remain invariant under the physically relevant subset of the transformations (32),

$$\mathcal{Q}(\tau) = \begin{pmatrix} \mathbf{Q}(\tau) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \quad \mathbf{B}(\tau) = \begin{pmatrix} \mathbf{b}(\tau) \\ 0 \end{pmatrix},$$

which represent physical observer changes for the non-dimensionalized system (27).

Assumption (A1) is always satisfied for the autonomous extended velocity field (28). To evaluate assumption (A2), we denote the normalized extended velocity vector and the extended averaged spin tensor, respectively, by

$$\mathbf{E}_U := \frac{\mathbf{U}}{|\mathbf{U}|}, \quad \bar{\mathbf{W}} := \begin{pmatrix} \bar{\mathbf{W}}_{\mathbf{u}} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} = -\bar{\mathbf{W}}^T,$$

$$\mathbf{W}_{\mathbf{u}} = \frac{1}{2} [\partial_{\mathbf{y}} \mathbf{u} - (\partial_{\mathbf{y}} \mathbf{u})^T]$$

with bar denoting spatial averaging over the domain  $V$ . In terms of these quantities, assumption (A2) applied to the extended system takes the form  $|\bar{\mathbf{W}} \mathbf{E}_U| \ll |\mathbf{E}_U|$ , or, equivalently,  $|\bar{\mathbf{W}} \mathbf{e}_{\mathbf{v}}| \ll |\dot{\mathbf{e}}_{\mathbf{v}}|$ , which is just the original form of assumption (A2) in dimensional coordinates. Theorems 1 and 2 then become applicable in the present setting, without the requirement (A1), but have to be stated for the extended Lagrangian velocity

$$\mathbf{Y}'(\tau_i) = \begin{pmatrix} \frac{T}{L} \mathbf{v}(\mathbf{x}(t_i), t_i) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{v_0} \mathbf{v}(\mathbf{x}(t_i), t_i) \\ 1 \end{pmatrix}.$$

Substituting these fields into formulas (19) and (24), we can summarize the main results for unsteady flows as follows.

**Theorem 3.** (i) *The extended trajectory stretching exponents (TSEs), defined as*

$$\begin{aligned} \text{TSE}_{t_0}^{t_N}(\mathbf{x}_0; \mathbf{v}_0) &= \frac{1}{t_N - t_0} \log \sqrt{\frac{|\dot{\mathbf{x}}(t_N)|^2 + v_0^2}{|\dot{\mathbf{x}}(t_0)|^2 + v_0^2}}, \\ \overline{\text{TSE}}_{t_0}^{t_N}(\mathbf{x}_0; \mathbf{v}_0) &= \frac{1}{t_N - t_0} \sum_{i=0}^{N-1} \left| \log \sqrt{\frac{|\dot{\mathbf{x}}(t_{i+1})|^2 + v_0^2}{|\dot{\mathbf{x}}(t_i)|^2 + v_0^2}} \right|, \end{aligned} \quad (33)$$

are objective measures of trajectory stretching and hyperbolicity strength in the extended phase space. (ii) *Under assumption (A2), the extended trajectory angular velocity (TRA), defined as*

$$\overline{\text{TRA}}_{t_0}^{t_N}(\mathbf{x}_0; \mathbf{v}_0) = \frac{1}{t_N - t_0} \sum_{i=0}^{N-1} \cos^{-1} \frac{\langle \dot{\mathbf{x}}(t_i), \dot{\mathbf{x}}(t_{i+1}) \rangle + v_0^2}{\sqrt{(|\dot{\mathbf{x}}(t_i)|^2 + v_0^2)(|\dot{\mathbf{x}}(t_{i+1})|^2 + v_0^2)}}, \quad (34)$$

is a quasi-objective measure of total trajectory rotation in the extended phase space.

The TSE, TSE, and TRA defined in Theorem 3 differ from their counterparts defined in Theorems 1 and 2. This difference only arises because the same underlying stretching and rotation metrics are evaluated on the velocity field  $\mathbf{v}$  in Theorems 1 and 2, whereas they are evaluated on the extended velocity field  $(\mathbf{v}, 1)$  in Theorem 3. Setting the characteristic velocity  $v_0$  to zero in the formulas of Theorem 3, one recovers the formulas of Theorems 1 and 2, which, however, are only valid under assumption (A1).

We stress that the TSE measures defined in Theorem 3 are objective without any particular assumption but in the extended phase space. This means that they always approximate an objective measure of material stretching for the material curve defined by the extended trajectory  $(\mathbf{x}(t), t)$ . The TSE measures, however, are not objective in the original phase space of the  $\mathbf{x}$  variable, and they are only quasi-objective in that phase space under assumption (A1), as we discussed in Theorem 1. In contrast, the extended TRA measure defined in Theorem 3 is quasi-objective in the extended phase under assumption (A2) and quasi-objective in the original phase space under assumptions (A1)–(A2), as we discussed in Theorem 2.

## V. EXAMPLES

### A. Two-dimensional, unsteady ocean surface velocity (AVISO) data set

We first evaluate the proposed quasi-objective, single-particle LCS diagnostics in a two-dimensional, satellite-altimetry-derived ocean-surface current product (AVISO)<sup>16</sup> that has been the focus of several coherent structure studies (see, e.g., Refs. 2, 6, 18, and 25). The geostrophic component  $\mathbf{v} = (v_1, v_2)$  of ocean currents is calculated from the sea-surface height anomaly via the formulas

$$fv_2 = \frac{1}{\rho} \frac{\partial p}{\partial x}, \quad fv_1 = -\frac{1}{\rho} \frac{\partial p}{\partial y}, \quad \frac{1}{\rho} \frac{\partial p}{\partial z} = -g, \quad (35)$$

where  $\rho$  is the density of water,  $p$  is the pressure as calculated from the sea-surface height,  $g$  is the constant of gravity, and  $f$  is the Coriolis parameter. A global daily-gridded version of this data is freely

available from the Copernicus Marine Environment Monitoring Service.

We first verify the minimal differences between the objective continuous and the quasi-objective discrete formulations of the stretching and rotation metrics developed in Sec. IV. Our analysis focuses on the Agulhas leakage region

$$U = \{(x, y) \in [-2.5^\circ, 5^\circ] \times [-40^\circ, -30^\circ]\}, \quad (36)$$

where strong hyperbolic and elliptic coherent structures have been previously identified. Our default rectangular initial grid for generating trajectories in  $U$  will contain  $250 \times 250$  points. Using an integration time of 25 days, we find the characteristic velocity of  $U$ ,

$$v_0 = \frac{1}{NM} \sum_{j=1}^M \sum_{i=0}^{N-1} |\mathbf{v}(\mathbf{x}_j, t_i)| = 0.2 \text{ m s}^{-1}, \quad (37)$$

where  $j$  is an index over all initial positions  $\mathbf{x}_0 \in U$ .

As already noted in Remark 4 after Theorem 2, condition (A2) is expected to hold for surface currents on large enough ocean domains. We nevertheless verify (A2) on trajectory data generated by the AVISO velocity field. Along individual trajectories originating in  $U$ , we compute the ratio

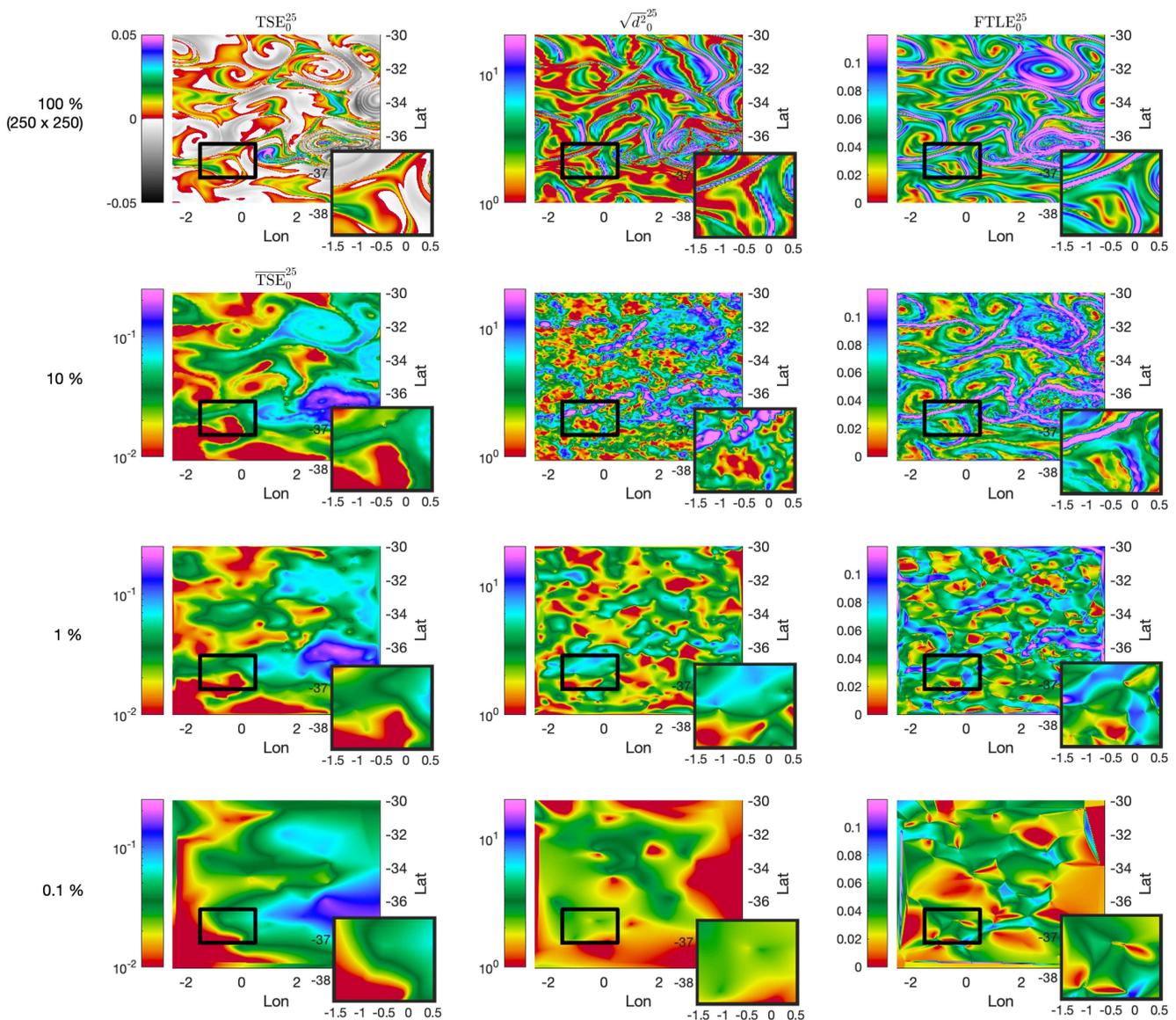
$$\delta_{A2}(\mathbf{x}_0, t_0, t_N) = \frac{\int_{t_0}^{t_1} \left| \frac{1}{2} \bar{\boldsymbol{\omega}}(t) \times \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \right| dt}{\int_{t_0}^{t_1} \left| \frac{d}{dt} \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \right| dt}, \quad (38)$$

which reflects the trajectory-averages of the quantities featured in assumption (A2). If  $\delta_{A2} \ll 1$  holds, then (A2) is satisfied on average along the trajectory data serving as a basis of our analysis. We find that on the domain  $U$  defined in (36),  $\delta_{A2}(\mathbf{x}_0, 0, 25) < 0.01$  for 98.9% of all initial positions and  $\delta_{A2}(\mathbf{x}_0, 0, 25) < 0.1$  for all fluid particles. In comparison, on the smaller subdomain  $U' = [2^\circ, 4^\circ] \times [-33^\circ, -31^\circ]$ , highlighted as an inset in Fig. 2, the relation  $\delta_{A2}(\mathbf{x}_0, 0, 25) < 0.1$  holds for only 55% of the initial positions and  $\delta_{A2}(\mathbf{x}_0, 0, 25) < 0.01$  holds for only 0.3% of them. This is because this domain contains only a single coherent structure with substantial average vorticity that does not represent the near-zero average vorticity of the background flow surrounding the coherent structures.

Ridges of the forward-time FTLE provide an effective diagnostic for identifying repelling LCSs,<sup>27</sup> but their computation relies on spatial differentiation of the flow map over a regular grid, which cannot be carried out from randomly placed single trajectories. For a fair comparison, we will restrict our calculations of the FTLE to the same grid resolution as our underlying flow; i.e., we do not interpolate to a refined grid for spatial differentiation. Another objective LCS diagnostic we will consider in our comparison is the single point (squared) relative dispersion<sup>20</sup>

$$d^2(\mathbf{x}_{i_0}, t_0, t) = \frac{|\mathbf{x}_i(t) - \mathbf{x}_{-i}(t)|^2}{|\mathbf{x}_{i_0} - \mathbf{x}_{-i_0}|^2}, \quad (39)$$

where  $\mathbf{x}_i$  and  $\mathbf{x}_{-i}$  are trajectory pairs with initial conditions  $\mathbf{x}_{i_0}$  and  $\mathbf{x}_{-i_0}$  that are initially close at time  $t_0$  (e.g.,  $|\mathbf{x}_{i_0} - \mathbf{x}_{-i_0}| = r_0 \ll 1$ ). While dividing by the initial distance makes no difference for a uniform grid of initial conditions, we still include this normalization to allow for small deviations in  $r_0$  under a random subsampling



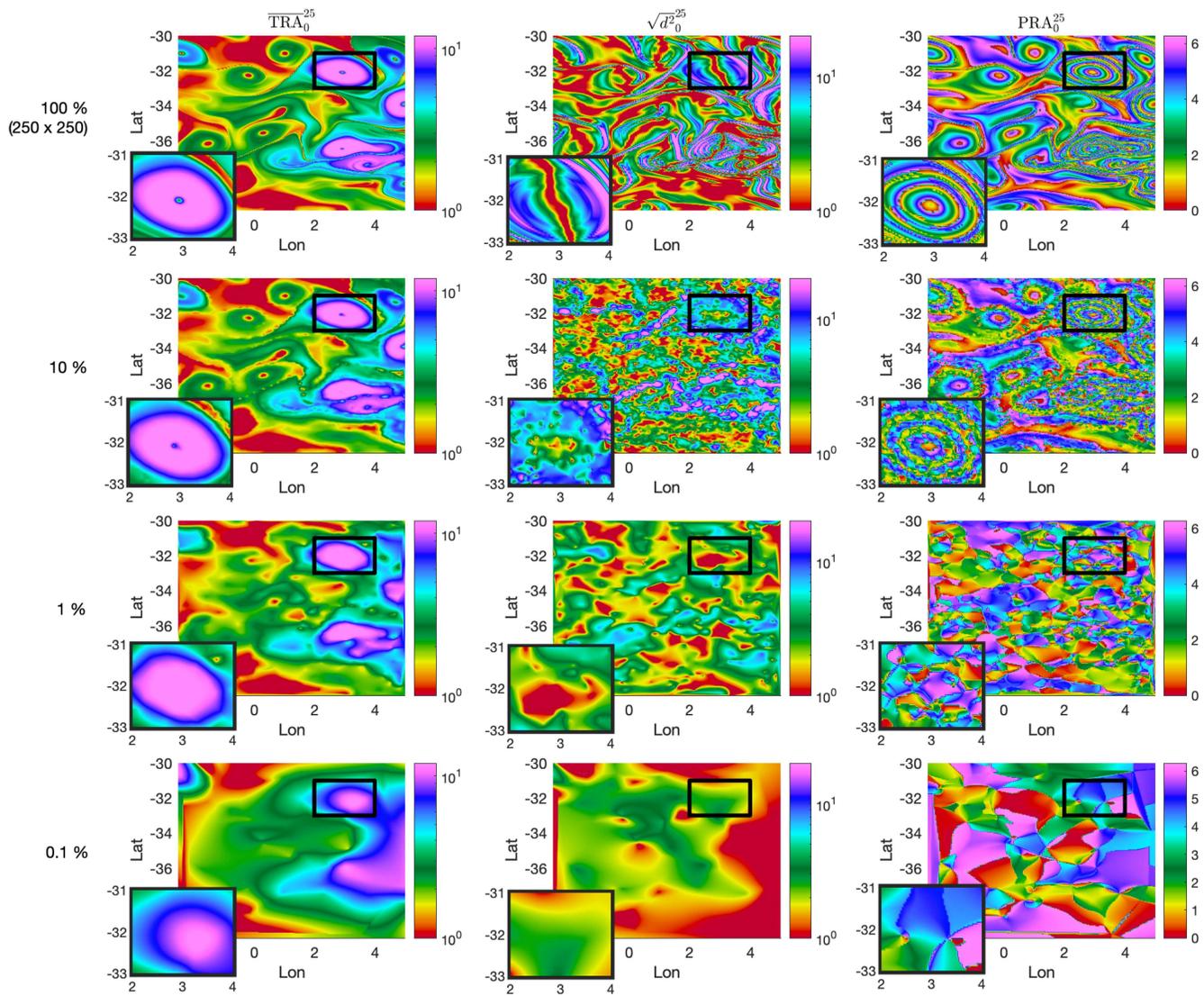
**FIG. 1.** Stretching metric comparisons for AVISO ocean surface current fields. The top row shows computations at full resolution with the lower rows having progressively reduced resolution by randomly subsampling with fewer trajectories.

of the initial conditions that mimics real-life sparse trajectory data. We note that the temporal behavior of ensemble averages of relative dispersion is a commonly studied statistic in oceanography.<sup>30</sup>

To highlight the significant advantages of TSE and TRA (as defined in Theorem 3) on sparse and non-uniform data, we perform a progressive random subsampling of the grid of initial positions to degrade the resolution of trajectories in a manner typical of what is found in experimental data. Computations of  $d^2$  can accommodate this subsampling if we choose an  $r_0$  value for which every initial position  $\mathbf{x}_{i0}$  has a suitable match  $\mathbf{x}_{-i0}$ . The FTLE, however, practically

requires a regular grid of initial conditions for the spatial differentiation of the flow map. To this end, we mimic the standard FTLE computation over an irregular grid by creating a  $C^1$ -interpolant of final positions and then computing the deformation gradient from those interpolations. Alternative methods for unstructured meshes have been suggested for FTLE but are not the focus of this study (see, e.g., Ref. 31).

The first row of Fig. 1 shows the full-resolution computation of  $TSE_0^{25}(\mathbf{x}_0; 0.2)$ ,  $d^2(\mathbf{x}_0; 0, 25)$ , and  $FTLE_0^{25}(\mathbf{x}_0)$ , with an inset zoom of a front that separates two recirculation regions in the flow. Successive

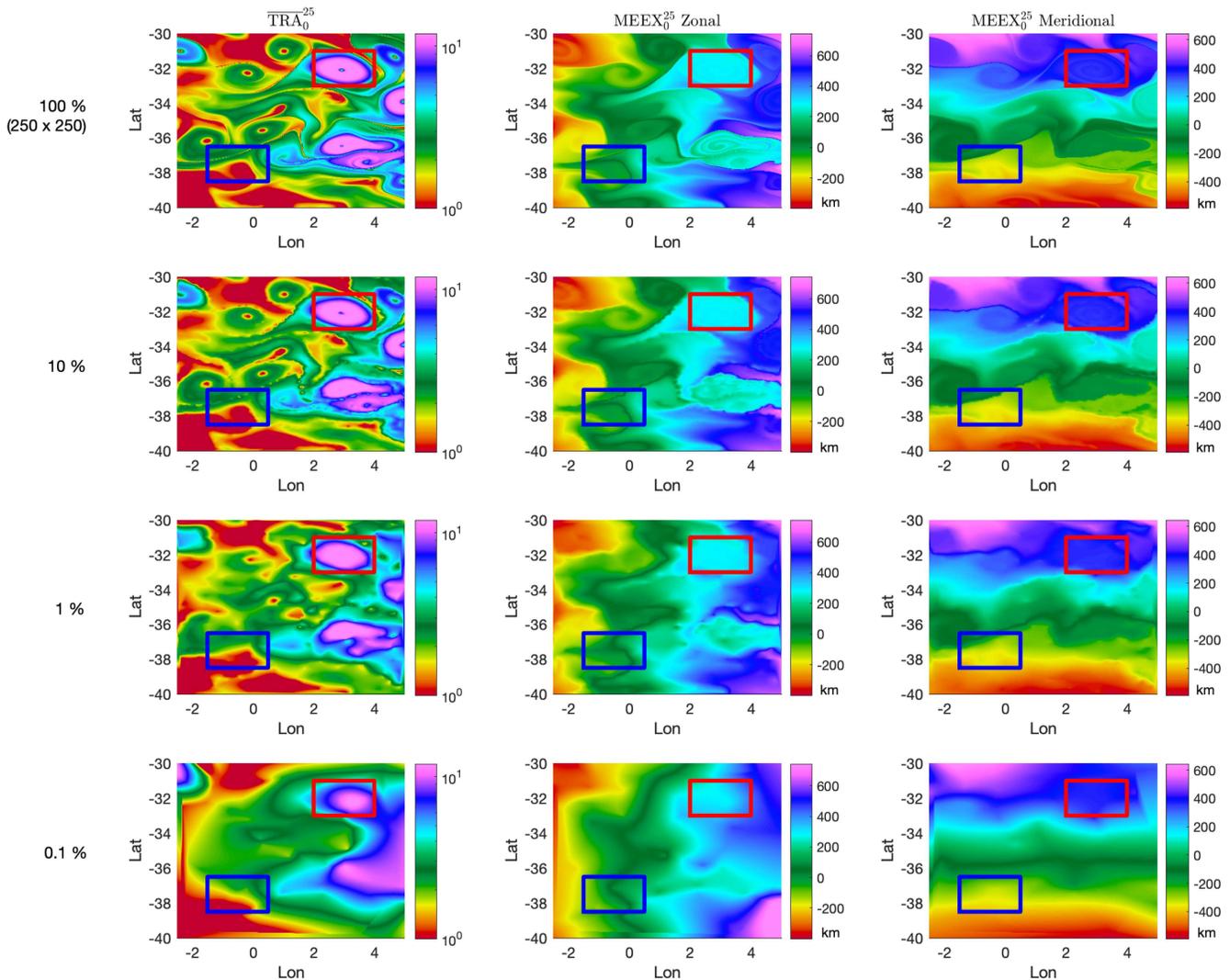


**FIG. 2.** Rotation metric comparisons for AVISO ocean surface current fields. The top row shows computations at full resolution with the lower rows having progressively reduced resolution by randomly subsampling with fewer trajectories. We display  $\sqrt{d^2}$  here for clarity.

rows gradually degrade from 10% to 0.1% of the full resolution. In these rows, we replace the stretching diagnostic  $TSE_0^{25}(\mathbf{x}_0; 0.2)$  with the hyperbolicity strength diagnostic  $\overline{TSE}_0^{25}(\mathbf{x}_0; 0.2)$ , which is a more robust single-trajectory indicator for sparse data. The highlighted repelling LCS is evident in the top row as a ridge in all three diagnostics. The quality of FTLE quickly degrades as interpolations between final positions of particles oversimplify the flow dynamics. The FTLE ridge of interest is still evident at the 10% subsampling, but it disappears below that resolution along with many other previously distinguishable features. The relative dispersion  $d^2$  does a good job of approximating FTLE ridges at full resolution and

provides a hint at some circulation patterns at lower resolutions but displays no clear separating feature reminiscent of a ridge at any degree of subsampling.

As TSE values at a given location do not rely on any nearby neighbors, their pointwise values remain unchanged and their features are more resilient under subsampling in comparison with those of multi-trajectory objective LCS metrics. Boundaries of multiple vortical features in the flow, such as those shown in the inset, can be tracked through multiple reductions in resolution as transitions between blue elliptic features and the surrounding green turbulent regions. In fact, at 0.1% of the full resolution, there is still evidence



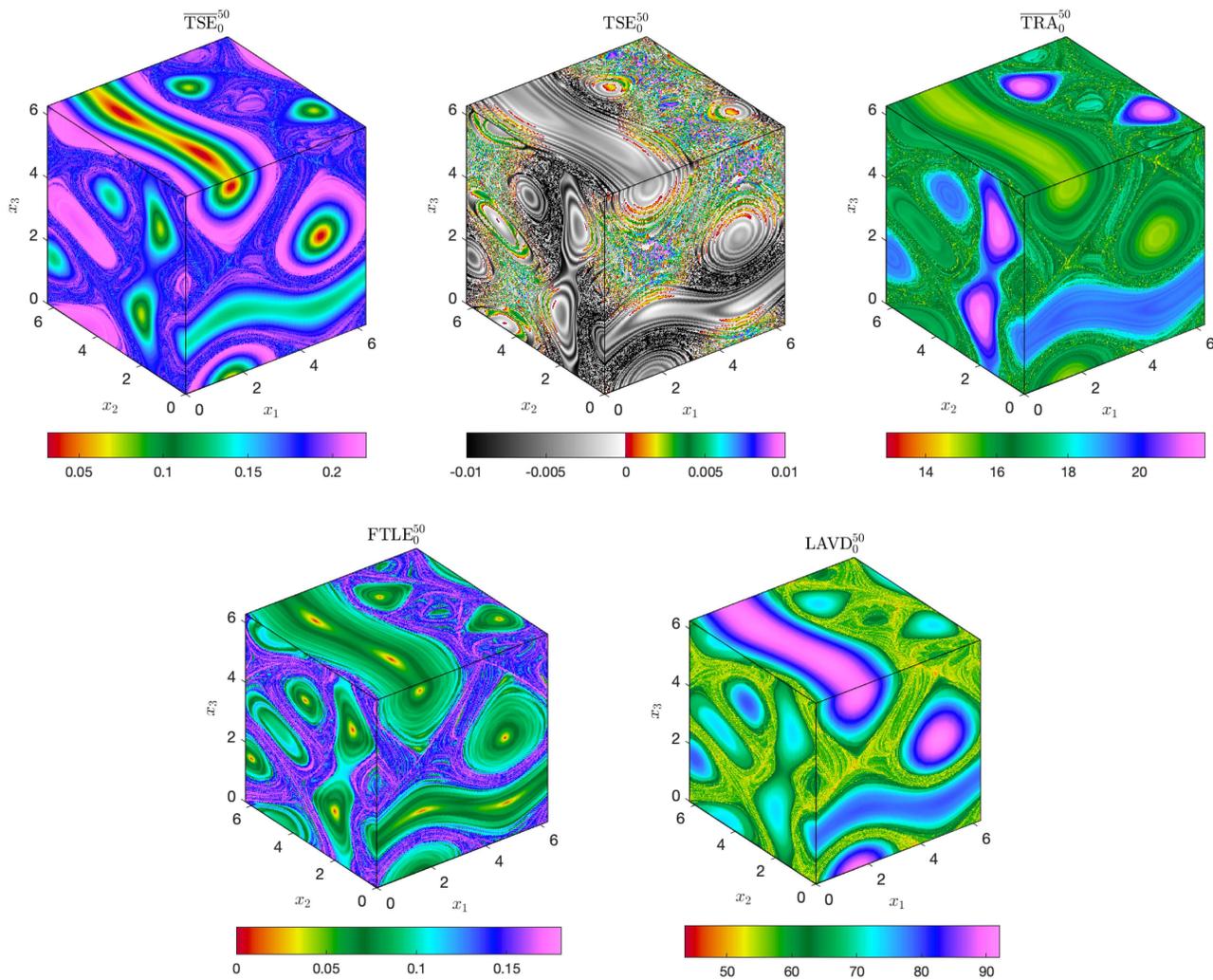
**FIG. 3.** Comparison of  $\overline{\text{TRA}}_0^{25}(\mathbf{x}_0; 0.2)$  with the  $\text{MEEEX}_0^{25}(\mathbf{x}_0)$  diagnostic fields proposed in Ref. 37, calculated in the zonal and meridional directions for the Agulhas leakage. Blue boxes indicate the location of frontal features from Fig. 1, and red boxes highlight the vortex from Fig. 2.

of several of the original major Lagrangian eddies in blue, as well as the front in our inset.

Next, we evaluate the ability of the  $\overline{\text{TRA}}$ , as defined in Theorem 3, to identify vortical features. We apply the same method of random subsampling to compare  $\overline{\text{TRA}}_0^{25}$  to the polar rotation angle,  $\text{PRA}_0^{25}$ , whose features (i.e., level curves) are objective in two dimensions.<sup>12</sup> In order to compute the flow map for the purposes of calculating PRA from sparse data, we employ the same  $C^1$  interpolant on final trajectory positions that we used for computing  $\text{FTLE}_0^{25}$ . No objective, sparse-data rotation-specific metric is available, which prompts us to use  $d^2$  instead in our comparison of rotation metrics. We do expect  $d^2$  to highlight

elliptic LCSs as coherent Lagrangian vortices, as observed first by Ref. 40.

Figure 2 focuses on the same time period and the Agulhas region as Fig. 1. The inset in each plot is a zoom on a region with a vortex previously identified by exact mathematical methods as a black-hole-type elliptic LCS (see, e.g., Ref. 24). In the first row of plots, at full resolution, all three metrics [ $\overline{\text{TRA}}_0^{25}(\mathbf{x}_0; 0.2)$ ,  $d^2(\mathbf{x}_{10}, 0, 25)$  and  $\text{PRA}_0^{25}(\mathbf{x}_0)$ ] highlight this vortex, as well as several other elliptic LCSs in the flow. The degree of detail provided by the objective, multi-trajectory diagnostic, PRA, is not matched by the trajectory rotation metric  $\overline{\text{TRA}}_0^{25}$ , which nevertheless correctly highlights the vortex.



**FIG. 4.** Elliptic and hyperbolic LCSs in the unsteady ABC flow (41). The plots compare quasi-objective, single-trajectory metrics [ $TSE_0^{50}(\mathbf{x}_0; 1)$  and  $\overline{TRA}_0^{50}(\mathbf{x}_0; 1)$ ] with objective LCS metrics [ $FTLE_0^{50}(\mathbf{x}_0)$  and  $LAVD_0^{50}(\mathbf{x}_0)$ ] that require multiple neighboring trajectories or detailed knowledge of the velocity field.

The insensitivity of  $\overline{TRA}_0^{25}(\mathbf{x}_0; 0.2)$  to small details becomes an advantage at lower resolutions, as shown by subsequent rows of Fig. 2. Indeed,  $\overline{TRA}_0^{25}(\mathbf{x}_0; 0.2)$  retains much of its structure with progressive random subsamplings. As a result, it is still possible to recognize three of its maxima on the right-hand-side, corresponding to multiple vortices, even at the lowest resolution. Some strong rotational features can be tracked below 10% resolution in the PRA fields, but the elliptic LCS quickly becomes indistinguishable. Similarly, distinguishing coherent structures in the  $\sqrt{d^2}$  plots becomes unfeasible quickly under progressive subsampling.

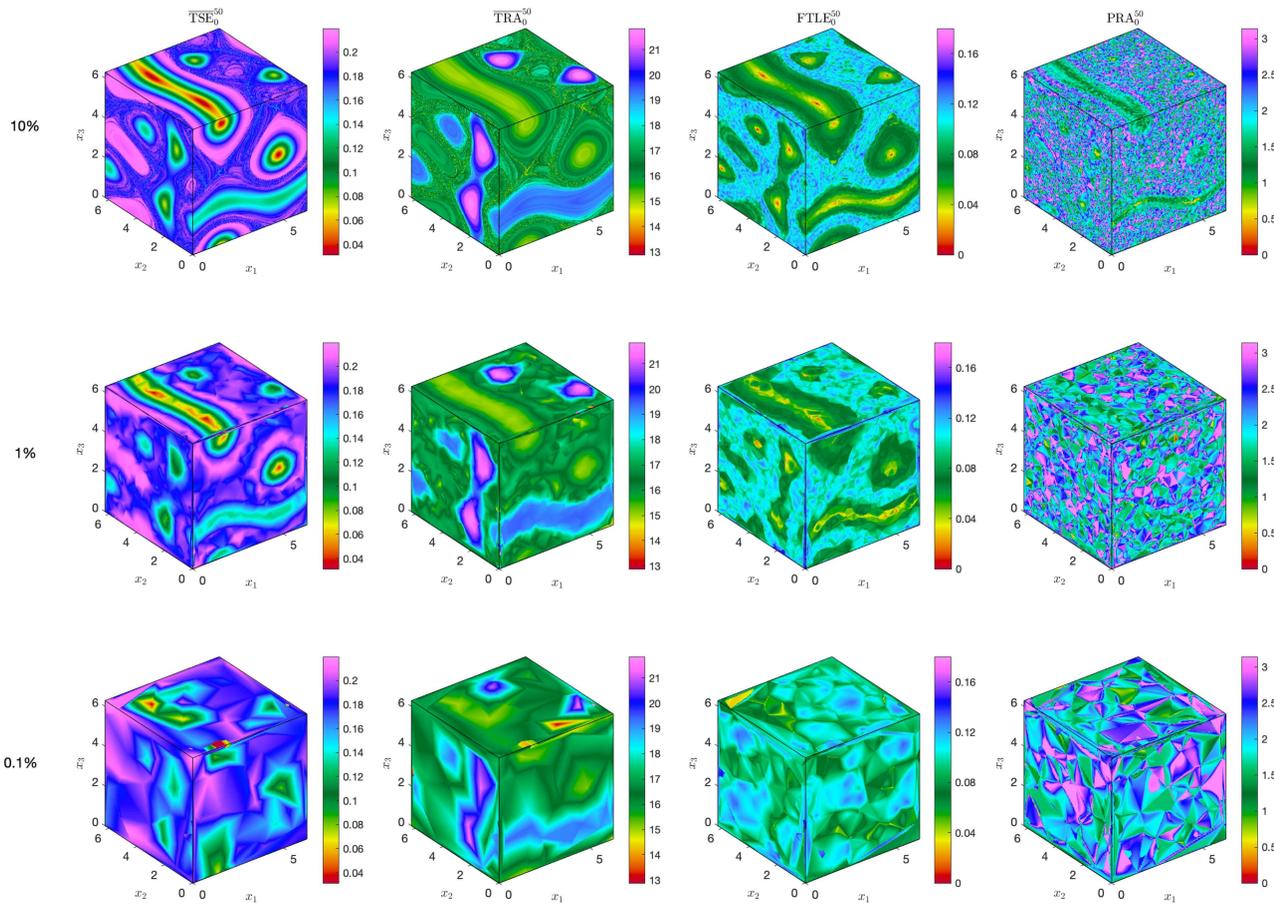
For a comparison with a non-objective diagnostic suggested for sparse geophysical data, we consider the mean of extrema extents (MEEEX) diagnostic proposed by Ref. 37. As we noted in Sec. I,

this diagnostic has no known relation to material stretching or rotation, but its evaluation is compellingly simple. Indeed, for a particle trajectory initiated at  $\mathbf{x}_0 \in U$ , the  $MEEEX_{t_0}^{t_f}(\mathbf{x}_0)$  field is simply defined as

$$MEEEX_{t_0}^{t_f}(\mathbf{x}_0) = \frac{\min_{t \in [t_0, t_f]} \phi(\mathbf{x}(t)) + \max_{t \in [t_0, t_f]} \phi(\mathbf{x}(t))}{2}, \quad (40)$$

where  $[t_0, t_f]$  is the observation interval and  $\phi(\mathbf{x})$  is a scalar-valued observable. The fluid particle zonal (longitudinal) or meridional (latitudinal) extent (measured in km) was suggested in Ref. 37 as a suitable  $\phi$  for sparse ocean drifter data.

The left column of Fig. 3 shows  $\overline{TRA}_0^{25}(\mathbf{x}_0; 0.2)$  calculated over the same spatial-temporal domain as Fig. 2, with the same degrees



**FIG. 5.** Elliptic and hyperbolic LCS in the randomly subsampled unsteady ABC flow (41). The plots compare quasi-objective single-trajectory metrics [ $\overline{\text{TSE}}_0^{50}(\mathbf{x}_0; 1)$  and  $\overline{\text{TRA}}_0^{50}(\mathbf{x}_0; 1)$ ] with LCS metrics [ $\text{FTLE}_0^{50}(\mathbf{x}_0)$  and  $\text{PRA}_0^{50}(\mathbf{x}_0)$ ], whose computation requires multiple neighboring trajectories. The initial condition grid for trajectories is randomized, and its density is gradually decreased to 0.1% of its initial value.

of data degradation. The middle column shows  $\text{MEEEX}_0^{25}(\mathbf{x}_0)$  with  $\phi$  equal to the zonal distance of fluid particles from the prime meridian, and the right column shows  $\text{MEEEX}_0^{25}(\mathbf{x}_0)$  with  $\phi$  equal to the meridional distance from the  $-35^\circ$  line of latitude. As expected from Ref. 37, the middle column corresponds with the best performing version of MEEEX for full-resolution current data in the Agulhas region. The blue boxes in each subplot correspond with the front highlighted in the previous TSE comparison (Fig. 1), and the red boxes correspond with the vortex in Fig. 2.

Overall, the MEEEX diagnostics show correlations with the Lagrangian features revealed by objective diagnostics in Fig. 2 and confirmed by particle advection in various earlier studies of the same data set.<sup>2,6,18,25</sup> This is to be expected, as any generic observable evaluated on trajectory positions will be influenced by LCSs, as demonstrated by Ref. 19 on several examples. The details of the MEEEX patterns and their closeness to LCSs, however, depend on the choice of  $\phi(\mathbf{x})$ .

Under subsampling of the data, the hyperbolic and elliptic structures persist in the  $\overline{\text{TRA}}_0^{25}(\mathbf{x}_0; 0.2)$  field, and several patterns

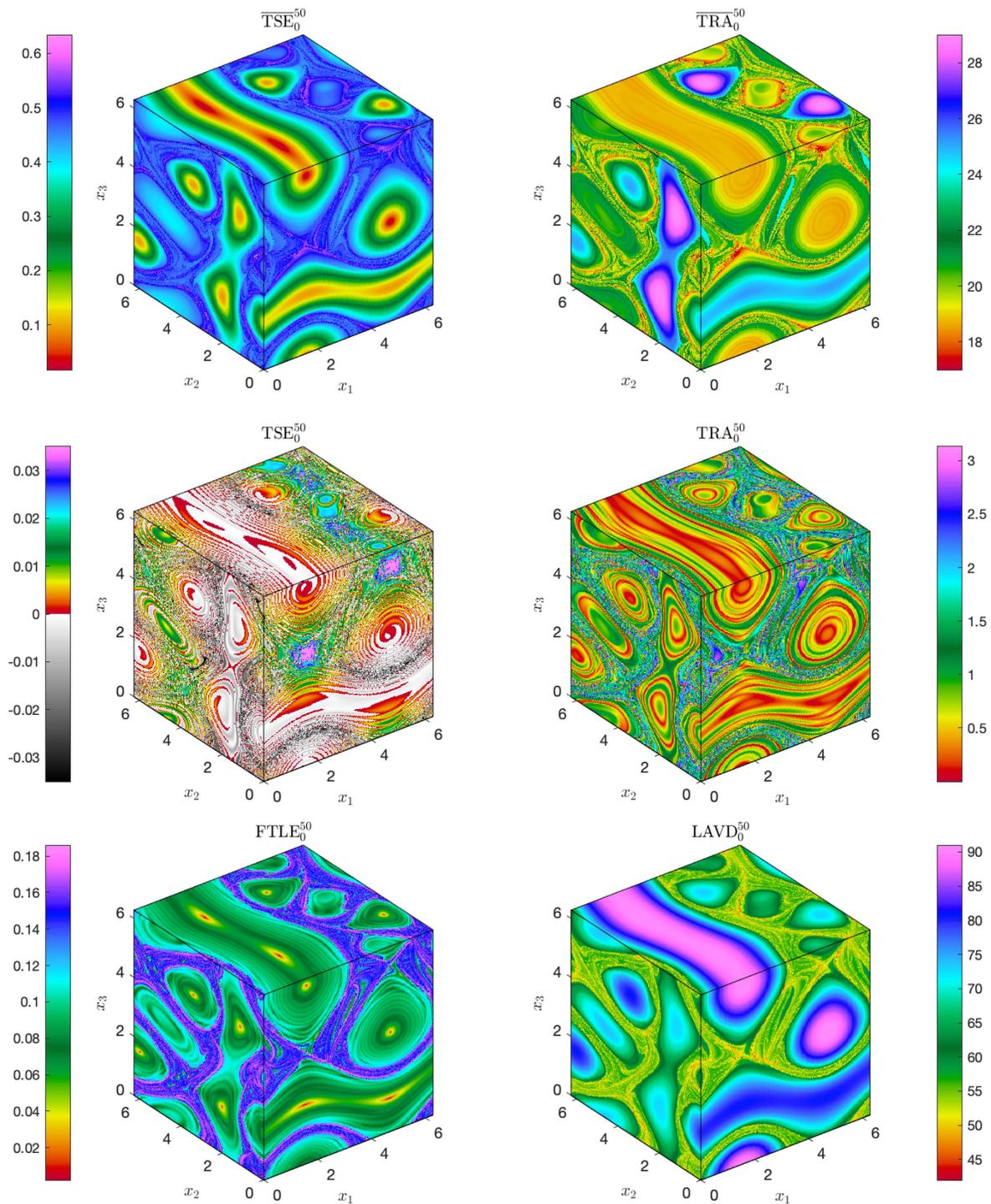
in  $\text{MEEEX}_0^{25}(\mathbf{x}_0)$  are also resilient. However, while patterns suggesting elliptic LCSs are initially visible in the full-resolution zonal and meridional  $\text{MEEEX}_0^{25}$  fields, they tend to disappear in the degraded data. The mixing behavior across the front in the blue box is also difficult to interpret in the  $\text{MEEEX}_0^{25}$  fields, with no indication of a major transport barrier at or below 1%. Combining interpretations of the two  $\text{MEEEX}_0^{25}$  fields presents a further challenge due to the seemingly contradictory patterns that arise from the different choices of spatial directions for the observable  $\phi(\mathbf{x})$ .

### B. Unsteady ABC flow

As a second example, we consider the three-dimensional velocity field

$$\mathbf{v}(\mathbf{x}, t) = e^{-\nu t} \begin{pmatrix} A \sin x_3 + C \cos x_2 \\ B \sin x_1 + A \cos x_3 \\ C \sin x_2 + B \cos x_1 \end{pmatrix}, \quad (41)$$

a viscous, unsteady version of the steady ABC flow,<sup>9</sup> defined on the triply periodic spatial domain  $U = [0, 2\pi]^3$ . As in Ref. 12, we use



**FIG. 6.** Elliptic and hyperbolic barriers of instantaneous linear-momentum transport in the unsteady ABC flow (41). The plots compare the objective single-trajectory metrics  $\overline{TSE}_0^{50}(\mathbf{x}_0)$  and  $\overline{TRA}_0^{50}(\mathbf{x}_0)$ ; see Remarks 3 and 1, after Theorems 1 and 2, respectively] with the objective LCS metrics  $FTLE_0^{50}(\mathbf{x}_0)$  and  $PRA_0^{50}(\mathbf{x}_0)$ , whose computation requires multiple neighboring trajectories or detailed knowledge of the velocity field.

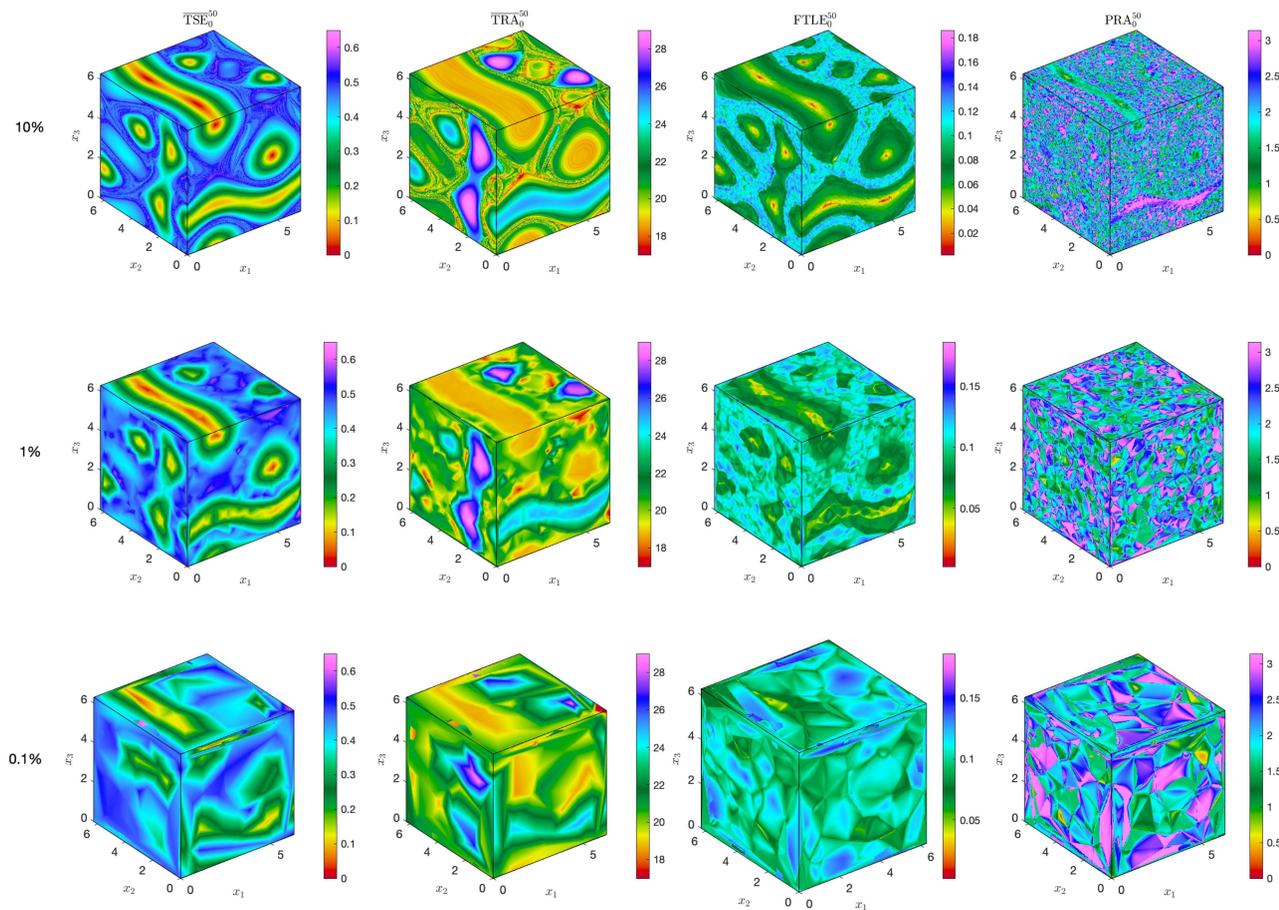


FIG. 7. Same as Fig. 6 but under subsampling of the unsteady ABC flow (41). The initial conditions are random subsamplings of the full resolution grid, and their density is gradually decreased to 0.1% of its initial value.

the parameter values  $A = 1$ ,  $B = \sqrt{2/3}$ ,  $C = \sqrt{1/3}$ , with the added viscosity value  $\nu = 0.01$ .

The unsteadiness of the flow necessitates the use of Theorem 3. By the triple periodicity of  $\mathbf{v}(\mathbf{x}, t)$  on  $U$ , we also have  $\bar{\omega}(t) \equiv \mathbf{0}$  and hence condition (A2) is also satisfied on  $U$ . We conclude that both TSE metrics are objective in the extended phase space, and the  $\overline{\text{TRA}}(\mathbf{x}_0; v_0)$  metric is quasi-objective in the extended phase space in the frame of (41). As the ABC flow is a dimensionless set of equations, we use  $v_0 = 1$ .

In Fig. 4, we show plots of the  $\overline{\text{TSE}}_0^{50}(\mathbf{x}_0; 1)$ ,  $\overline{\text{TSE}}_0^{50}(\mathbf{x}_0; 1)$ , and  $\overline{\text{TRA}}_0^{50}(\mathbf{x}_0; 1)$  metrics computed from Theorem 3 over trajectories launched from an initial grid of  $200 \times 200 \times 200$  points, integrated from time  $t_0 = 0$  to  $t_N = 50$ . Values of the FTLE and LAVD are also included for comparison. (We did not compute PRA for this example as level sets of the PRA are not objective in three dimensions.) We conclude that the quasi-objective, single-trajectory diagnostics ( $\overline{\text{TSE}}_0^{50}$ ,  $\overline{\text{TSE}}_0^{50}$ ,  $\overline{\text{TRA}}_0^{50}$ ) perform just as well as the multi-trajectory, objective metric, the  $\text{FTLE}_0^{50}$ , and the single-trajectory, objective, but velocity-field-reliant metric, the  $\text{LAVD}_0^{50}$ .

To evaluate the performance of the quasi-objective metrics under a random degradation of the initial grid, we employ the same random subsampling method as in Sec. V A. Even though level sets of the PRA are not objective in three-dimensions, we include PRA here in our comparison. This is because there is no systematic or widely accepted procedure for accurately approximating vorticity from sparse Lagrangian trajectories for LAVD calculations.

Figure 5 shows that  $\overline{\text{TSE}}_0^{50}$  and  $\overline{\text{TRA}}_0^{50}$  continue to provide information on dominant flow features even as trajectory data become substantially sparse. Specifically, as we keep the bounds on the colormaps constant, the separation between rotational regions remains clear from  $\overline{\text{TSE}}_0^{50}$  and  $\overline{\text{TRA}}_0^{50}$  when compared with  $\text{FTLE}_0^{50}$  and  $\text{PRA}_0^{50}$ , down to a random selection of trajectories consisting of 0.1% of the original computation grid. We conclude that even in three-dimensional unsteady flows, it is possible to observe a consistent range of stretching and rotation rates with single-trajectory, quasi-objective metrics from very sparse and randomly positioned data. Furthermore, interpolating between randomly oriented TSE and TRA metrics provides a diagnostic that highlights time-varying

coherent structures to an extent that is unreachable by prior LCS methods.

### C. Momentum barriers in the unsteady ABC flow

As our third example, we consider the linear-momentum barrier field equations, as defined in Ref. 26, for the unsteady ABC flow (41). As shown in Ref. 26, at a time  $t$ , the instantaneous barriers to momentum transport are invariant manifolds of the autonomous dynamical system

$$\mathbf{x}' = \mathbf{w}(\mathbf{x}; t) = \nu\rho\Delta\mathbf{v}(\mathbf{x}, t), \quad (42)$$

with prime denoting differentiation with respect to a fictitious time (barrier time) parameterizing the curves forming the barrier surface. At time  $t = 0$ , Eq. (42) simplifies to  $\mathbf{x}' = -\nu\rho\nabla(\mathbf{x}, 0)$  in the case of the unsteady ABC flow defined in (41). This autonomous dynamical system provides a steady flow field on which we can compute coherent structure diagnostics to highlight instantaneous barriers to the transport of momentum in the underlying fluid flow. This allows us to compare the steady versions of the (now objective) single-trajectory metrics  $\overline{\text{TSE}}_0^{50}(\mathbf{x}_0)$ ,  $\text{TSE}_0^{50}(\mathbf{x}_0)$ , and  $\overline{\text{TRA}}_0^{50}(\mathbf{x}_0)$ , as defined in Theorems 1 and 2, with the objective LCS metrics  $\text{FTLE}_0^{50}(\mathbf{x}_0)$  and  $\text{LAVD}_0^{50}(\mathbf{x}_0)$ . As these active barrier vector fields are objective,<sup>26</sup> we can now also include a comparison with objective  $\text{TRA}_0^{50}(\mathbf{x}_0)$ , as defined in (25) [see Remarks 3 and 1 after Theorems 1 and 2, respectively, on the objectivity of these metrics for  $\mathbf{w}(\mathbf{x}; t)$ ]. In our comparison, we will use the same parameters as in Sec. V B for the flow (41).

Figure 6 shows the ability of the single-trajectory metrics to accurately represent the elliptic and hyperbolic structures in the active barrier vector field (42) for trajectories with initial positions on the same rectangular grid in the domain  $U$  as in Sec. V B. When compared with  $\text{FTLE}_0^{50}$ , we find that the steady version of  $\text{TSE}_0^{50}$  provides a great degree of detail, highlighting hyperbolic regions of the flow as before. In addition,  $\overline{\text{TRA}}_0^{50}$  is able to provide a comparable amount of detail for elliptic features. As we degrade the resolution with the same random subsampling of initial positions, Fig. 7 shows that  $\overline{\text{TSE}}_0^{50}$  and  $\overline{\text{TRA}}_0^{50}$  are again robust and able to highlight hyperbolic and elliptic transport barriers down to 0.1% of the original resolution. Calculations of  $\text{FTLE}_0^{50}$  and  $\text{PRA}_0^{50}$  from sparse trajectory data on the steady barrier field perform comparably to the unsteady flow example, showing a quick deterioration in quality and interpretability.

## VI. CONCLUSIONS

Coherent structures, such as fronts, jets, and vortices, are best viewed as material objects. Indeed, these structures are most reliably identified in experiments and observations from their signatures in material tracer fields. By definition, such material coherent structures are indifferent to the observer and hence can only be self-consistently identified from objective quantities. In contrast, flows of practical relevance are mostly known from trajectory data, which is inherently non-objective. Specifically, all features of trajectories (including their length, curvature, velocity, and acceleration) are dependent on the observer. The question, therefore, naturally arises:

How can one still use trajectory data to extract reliable information from material coherent structures in a flow?

We have addressed this question here by introducing the notion of quasi-objectivity. We call a scalar-, vector-, or tensor-field quasi-objective under a condition if the field approximates another, objective field in all frames in which that condition is satisfied. We have derived quasi-objective measures of trajectory stretching ( $\text{TSE}$  and  $\overline{\text{TSE}}$ ) that approximate objective material stretching exponents in frames where the flow is steady, i.e., satisfies condition (A1). We have also derived quasi-objective measures of trajectory rotation ( $\text{TRA}$  and  $\overline{\text{TRA}}$ ) that approximate objective material rotation under conditions (A1) and (A2). The latter condition requires trajectory accelerations to dominate the angular acceleration induced by the spatial mean vorticity. Viewing general unsteady flows as steady flows in their extended phase space (space-time), we have also obtained objective TSE metrics and quasi-objective TRA metrics under assumption (A2) in the extended phase space. These extended metrics track the stretching and rotation of tangent vectors to fluid trajectories in space-time.

We have tested these quasi-objective diagnostics on two-dimensional mesoscale ocean surface velocity data (AVISO) obtained from satellite altimetry. Condition (A2) is clearly satisfied in regions large enough to contain several mesoscale eddies. Accordingly, we have found the quasi-objective metrics to perform very well. As a further test case, we have considered an unsteady version of the classic ABC flow,<sup>26</sup> which is an exact solution of the three-dimensional unsteady Navier–Stokes equation. This flow satisfies condition (A2) in its triply periodic domain of definition. We have also considered the steady version, which satisfies assumption (A1).

On trajectories generated by these example velocity fields from sufficiently dense grids, the trajectory stretching exponent (TSE) faithfully reproduced features of the FTLE field. This is notable because TSE is a single-trajectory-based diagnostic whose pointwise value is independent from the number and location of other trajectories in the data set. For sparse and irregularly spaced trajectory data, the trajectory hyperbolicity strength ( $\overline{\text{TSE}}$ ) returned distinct indications of coherent structures even when FTLE and relative dispersion computations gave no meaningful results. Similarly, we have found that on well-resolved trajectory data from objective vector fields, the trajectory rotation angle (TRA) accurately reproduced details of the PRA field, even though TRA is a single-trajectory-based diagnostic. For sparse and irregularly spaced trajectory data, the cancellation-free total rotation angle ( $\overline{\text{TRA}}$ ) gave a clear indication of coherent structures even when PRA computations were no longer feasible. All of these suggest great potential for  $\overline{\text{TSE}}$  and  $\overline{\text{TRA}}$  in visualizing coherent structures in 3D particle tracking velocimetry experiments, in which tracer concentrations are generally low (see, e.g., Refs. 28 and 42). In contrast, TSE and (for objective vector fields, such as the barrier equations in Ref. 26) the TRA are useful in reducing computational costs relative to multi-trajectory LCS diagnostics when trajectories from a well-resolved initial grid are available.

Further applications of the quasi-objective diagnostic involve the detection of barriers to the transport of dynamically active vector fields, such as the momentum and the vorticity.<sup>26</sup> These barriers are invariant manifolds of appropriate steady, volume-preserving,

and objective vector fields, the barrier vector fields. The computation of these barrier vector fields requires two spatial differentiations of the underlying velocity field, and hence, their analysis via FTLE and PRA is numerically challenging given that the latter two methods involve further differentiation. In this setting, condition (A1) is always satisfied, and (A2) can be waived without any loss of generality, as we have pointed out. Consequently, the use of TSE and TRA promises to bring a major improvement in computational efficiency for the detection of active transport barriers in three-dimensional unsteady flows.

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## DATA AVAILABILITY

The AVISO geostrophic current velocity product used in this study, “Global Ocean Gridded L4 Sea Surface Heights and Derived Variables Reprocessed,” is openly available in Copernicus Marine Environment Monitoring Service at <http://marine.copernicus.eu>, Ref. 16.

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