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Dynamic rotation and stretch tensors from a dynamic polar decomposition

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ABSTRACT

The local rigid-body component of continuum deformation is typically characterized by the rotation tensor, obtained from the polar decomposition of the deformation gradient. Beyond its well-known merits, the polar rotation tensor also has a lesser known dynamical inconsistency: it does not satisfy the fundamental superposition principle of rigid-body rotations over adjacent time intervals. As a consequence, the polar rotation diverts from the observed mean material rotation of fibers in fluids, and introduces a purely kinematic memory effect into computed material rotation. Here we derive a generalized polar decomposition for linear processes that yields a unique, dynamically consistent rotation component, the dynamic rotation tensor, for the deformation gradient. The left dynamic stretch tensor is objective, and shares the principal strain values and axes with its classic polar counterpart. Unlike its classic polar counterpart, however, the dynamic stretch tensor evolves in time without spin. The dynamic rotation tensor further decomposes into a spatially constant mean rotation tensor and a dynamically consistent relative rotation tensor that is objective for planar deformations. We also obtain simple expressions for dynamic analogues of Cauchy's mean rotation angle that characterize a deforming body objectively.

1. Introduction

In continuum mechanics, the now classic procedure for isolating the rotational component of the deformation gradient is the polar decomposition. To describe this decomposition, we consider a deformation field $\mathbf{X}_{t_0}^t: \mathbf{x}(t_0) \mapsto \mathbf{x}(t)$ defined on a spatial domain $\mathcal{B}(t_0) \subset \mathbb{R}^3$ over the time interval $[t_0, t_1] \subset \mathbb{R}$. The trajectories $\mathbf{x}(t)$ depend on the initial time t_0 and the initial position $\mathbf{x}_0$, but this will be suppressed for notational simplicity. By the polar decomposition theorem, the deformation gradient $\mathbf{F}_\tau^t = \nabla \mathbf{X}_\tau^t$ (with $\tau, t \in [t_0, t_1]$) has unique left and right decompositions of the form

$$\mathbf{F}_\tau^t = \mathbf{R}_\tau^t \mathbf{U}_\tau^t = \mathbf{V}_\tau^t \mathbf{R}_\tau^t, \tag{1}$$

with a proper orthogonal matrix $\mathbf{R}_\tau^t$ and symmetric, positive definite matrices $\mathbf{U}_\tau^t$ and $\mathbf{V}_\tau^t$ (Truesdell and Noll, 1965). Although customarily suppressed in their notation, the rotation and stretch tensors do depend on the time interval $[\tau, t]$. We keep this dependence in our notation for later purposes. We also emphasize that we consider general non-autonomous deformation fields for which the velocity field $\dot{\mathbf{X}}_{t_0}^t$ may depend explicitly on the current time t, which therefore cannot be set to zero at

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arbitrary intermediate configurations for convenience.

In finite-strain theory, the polar rotation tensor $\mathbf{R}_\tau^t$ is interpreted as solid-body rotation, while $\mathbf{U}_\tau^t$ and $\mathbf{V}_\tau^t$ are referred to as right and left stretch tensors between the times τ and t (Truesdell and Noll, 1965). The tensor $\mathbf{R}_\tau^t$ is generally obtained from (1) after $\mathbf{U}_\tau^t$ is computed as the principal square root of the right Cauchy–Green strain tensor $\mathbf{C}_\tau^t = (\mathbf{F}_\tau^t)^T\mathbf{F}_\tau^t$.

As Boulanger and Hayes (2001) (see also Jaric et al., 2006) point out, there are in fact infinitely many possible rotation-stretch decompositions of the form $\mathbf{F}_\tau^t = \mathbf{\Omega\Delta}$, where $\mathbf{\Omega} \in SO(3)$ is a rotation and Δ is a non-degenerate tensor whose singular values and singular vectors coincide with the eigenvalues and eigenvectors of $\mathbf{C}_\tau^t$. Indeed, an infinity of such decompositions can be generated from any given one by selecting an arbitrary rotation $\mathbf{\Xi} \in SO(3)$ and letting

$$\mathbf{F}_\tau^t = \hat{\mathbf{\Omega}}\hat{\mathbf{\Delta}}, \quad \hat{\mathbf{\Omega}} = \mathbf{\Omega\Xi}, \quad \hat{\mathbf{\Delta}} = \mathbf{\Xi}^T\mathbf{\Delta}. \tag{2}$$

Out of these infinitely many rotation-stretch decompositions, the left polar decomposition in (1) is uniquely obtained by requiring $\mathbf{\Delta}$ to be symmetric and positive definite. This convenient choice has important advantages, but is by no means necessary for capturing the strain invariants of the deformation, given that $\mathbf{C}_s^t = \mathbf{\Delta}^T\mathbf{\Delta}$ is always the case, even for a non-symmetric choice of $\mathbf{\Delta}$. In addition, there is no a priori physical reason why the stretching component of the deformation gradient should be symmetric. In particular, requiring $\mathbf{\Delta} = \mathbf{\Delta}^T = \mathbf{U}_s^t$ does not render $\dot{\mathbf{\Delta}}\mathbf{\Delta}^{-1}$ symmetric. In other words, the evolution of $\mathbf{U}_\tau^t$ is not spin-free.

The main advantage of the polar decomposition (1) is an appealing geometric interpretation of the particular rotation generated by $\mathbf{R}_\tau^t$. Indeed, $\mathbf{R}_\tau^t$ rotates the eigenvectors of $\mathbf{C}_\tau^t$ into eigenvectors of the left Cauchy–Green strain tensor $\mathbf{B}_\tau^t = \mathbf{F}_\tau^t(\mathbf{F}_\tau^t)^T$, or equivalently, into eigenvectors of $\mathbf{C}_t^\tau$ (Truesdell and Noll, 1965). This property distinguishes $\mathbf{R}_\tau^t$ as a highly plausible geometric rotation component for the deformation gradient between the times τ and t. A further remarkable feature of the polar rotation tensor is that $\mathbf{R}_\tau^t$ represents, among all rotations, the closest fit to $\mathbf{F}_\tau^t$ in the Frobenius matrix norm (Grioli, 1940; Neff et al., 2014).

These geometric advantages of $\mathbf{R}_\tau^t$, relative to a fixed initial time τ and a fixed end time t, however, also come with a disadvantage for times evolving within $[\tau, t]$: polar rotations computed over adjacent time intervals are not additive. More precisely, for any two sub-intervals $[\tau, s]$ and $[s, t]$ within $[\tau, t]$, we have

$$\mathbf{R}_\tau^t \neq \mathbf{R}_s^t\mathbf{R}_\tau^s, \tag{3}$$

unless $\mathbf{U}_s^t\mathbf{V}_\tau^s = \mathbf{V}_\tau^s\mathbf{U}_s^t$ holds (Ito et al., 2004). $\mathbf{U}_s^t$ and $\mathbf{V}_\tau^s$, however, fail to commute even for the simplest deformations, such as planar rectilinear shear (cf. formula (44)). This implies, for instance, that $\mathbf{R}_\tau^t$ cannot be obtained from an incremental computation starting from an intermediate state of the body at time s. We refer to this feature of the polar rotation tensor, summarized in (3), as its *dynamical inconsistency* (see Fig. 1).

The dynamical inconsistency of $\mathbf{R}_\tau^t$ does not imply any flaw in the mathematics of polar decomposition. Neither does it detract from the usefulness of $\mathbf{R}_\tau^t$ in identifying a static rotational component of the deformation between two fixed configurations in a geometrically optimal sense. For configurations evolving in time, however, the polar decomposition is not an optimal tool: the polar rotation tensor does not represent a mean material rotation (cf. below), and the polar stretch tensor is not spin-free. As we shall see later (cf. Section 3), both these dynamical disadvantages stem from the relation (3), which may be well-known to experts, but is rarely, if ever, discussed in the literature. This has led some authors to erroneously assume dynamical consistency for $\mathbf{R}_\tau^t$ (see, e.g., Freed, 2008, 2010).

In contrast, most textbooks in fluid mechanics introduce a mean material rotation rate for a deforming volume element. This mean material rotation rate is defined by Cauchy (1841) as the average rotation rate of all material line elements emanating from the same point. Cauchy's mean rotation rate turns out to be one-half of the vorticity at that point (see, e.g., Batchelor, 1967; Tritton, 1988; Vallis, 2006). Two-dimensional experiments indeed confirm that small, rigid objects placed in a fluid rotate at a rate that is half of the local vorticity (Shapiro, 1961). There is, therefore, theoretical and experimental

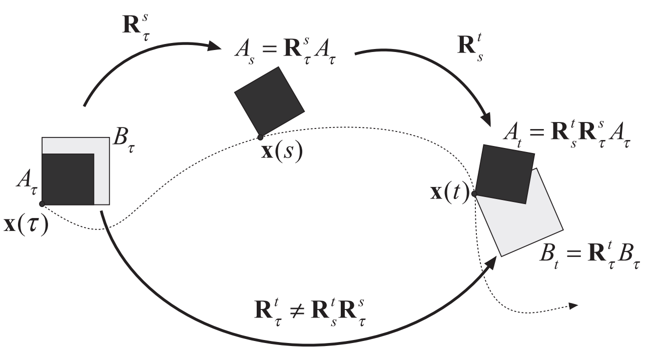

Fig. 1. The action of the polar rotations $\mathbf{R}_\tau^t$, $\mathbf{R}_\tau^s$ and $\mathbf{R}_\tau^t$, illustrated on two geometric volume elements A_τ and B_τ, based at the same initial point at time τ. The evolution of A_τ is shown incrementally under the subsequent polar rotations $\mathbf{R}_\tau^s$ and $\mathbf{R}_S^t$. The evolution of the volume B_τ (with initial orientation identical to that of A_τ) is shown under the polar rotation $\mathbf{R}_\tau^t$. All volume elements are non-material: they only serve to illustrate how orthogonal directions are rotated by the various polar rotations involved.

evidence for the existence of a well-defined and observable mean material rotation in continua that is free from the dynamical inconsistency (3). Yet a connection between this mean material rotation and the finite deformation gradient has not been established at the level of mathematical rigor offered by the polar decomposition theorem underlying formula (1).

Indeed, a close link between the experimentally observed mean material rotation in fluids (Shapiro, 1961) and the rotation tensor $\mathbf{R}_\tau^t$ is only known in the limit of infinitesimally short deformations. To state this, we need the spin tensor field $\mathbf{W}(\mathbf{x}, t)$ and the rate-of-strain tensor field $\mathbf{D}(\mathbf{x}, t)$, defined for a general velocity field $\mathbf{v}(\mathbf{x}, t)$, as

$$\mathbf{W} = \frac{1}{2}[\nabla\mathbf{v} - (\nabla\mathbf{v})^T], \quad \mathbf{D} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T], \tag{4}$$

with ∇ denoting the spatial gradient operation and the superscript T referring to transposition. With these ingredients, we have the relationship

$$\dot{\mathbf{R}}_\tau^t|_{t=\tau} = \mathbf{W}(\mathbf{x}(t), t), \quad \dot{\mathbf{U}}_\tau^t|_{t=\tau} = \dot{\mathbf{V}}_\tau^t|_{t=\tau} = \mathbf{D}(\mathbf{x}(t), t), \tag{5}$$

where the dot denotes differentiation with respect to t (Truesdell and Noll, 1965). Using the definition of the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, one therefore obtains from (5) the formula

$$\dot{\mathbf{R}}_\tau^t|_{t=\tau}\mathbf{e} = -\frac{1}{2}\boldsymbol{\omega}(\mathbf{x}(t), t) \times \mathbf{e}, \quad \forall\, \mathbf{e} \in \mathbb{R}^3 \tag{6}$$

for infinitesimally short deformations.

For deformations over a finite time interval $[\tau, t]$, the simple relationship between the polar rotation rate and the vorticity is lost. Only the more complex and less illuminating relationship

$$\dot{\mathbf{R}}_\tau^t = [\mathbf{W}(\mathbf{x}(t), t) - \frac{1}{2}\mathbf{R}_\tau^t\left[\dot{\mathbf{U}}_\tau^t\left(\mathbf{U}_\tau^t\right)^{-1} - \left(\mathbf{U}_\tau^t\right)^{-1}\dot{\mathbf{U}}_\tau^t\right]\left(\mathbf{R}_\tau^t\right)^T]\mathbf{R}_\tau^t \tag{7}$$

can be deduced (see, e.g., Truesdell and Rajagopal, 2009).

An unexpected property of formula (7): it gives no well-defined material rotation rate $\dot{\mathbf{R}}_\tau^t(\mathbf{R}_\tau^t)^T$ in a deforming continuum at a given location $\mathbf{x}$ and given time t. Rather, the current polar rotation rate at time t depends on the starting time τ of the observation (cf. Appendix A). This effect is not to be mistaken for the usual *implicit* dependence of kinematic tensors on the reference configuration, entering through the dependence of the tensor on the initial conditions of its governing differential equation. Rather, the effect arises from the *explicit* dependence of the differential equation (7) on the initial time τ through $\mathbf{U}_\tau^t$. In other words, polar rotations do not form a dynamical system (or process): they satisfy a nonlinear differential equation with memory (see Appendix A for details).

Here we develop an alternative to the classic polar decomposition which is free from these issues. Our *dynamic polar decomposition* (DPD) applies to general, non-autonomous linear processes, as opposed to single linear operators. When applied to the deformation gradient, the DPD yields a unique factorization $\mathbf{F}_\tau^t = \mathbf{O}_\tau^t\mathbf{M}_\tau^t$, with a *dynamic stretch tensor* $\mathbf{M}_\tau^t$ that is free from spin, and a *dynamic rotation tensor* $\mathbf{O}_\tau^t$ that is free from the dynamical inconsistency (3). We point out partial connections and analogies between these dynamic tensors and prior work by Epstein (1962), Noll (1955) and Rubinstein and Atluri (1983) in Remark 8 of Section 3.

The tensor $\mathbf{O}_\tau^t$ is, in fact, the only dynamically consistent rotation tensor out of the infinitely many possible ones in (2). Likewise, $\mathbf{M}_\tau^t$ is the only spin-free stretch tensor out of the infinitely many possible ones in (2). Unlike the tensor pair $(\mathbf{R}_\tau^t, \mathbf{U}_\tau^t)$, the dynamic tensor pair $(\mathbf{O}_\tau^t, \mathbf{M}_\tau^t)$ forms a dynamical system.

The dynamic rotation tensor reproduces Cauchy's mean material rotation rate, giving the rate $\dot{\mathbf{O}}_\tau^t(\mathbf{O}_\tau^t)^T = \mathbf{W}(\mathbf{x}(t), t)$ for both finite and infinitesimal deformations. This fills the prior mathematical gap between the deformation gradient and numerical algorithms that rotate the reference frame incrementally (but not infinitesimally) at the spin rate (Hughes and Winget, 1980; Rubinstein and Atluri, 1983) rather than at the polar rotation rate.

The dynamic rotation rate $\dot{\mathbf{O}}_\tau^t(\mathbf{O}_\tau^t)^T$ eliminates the discrepancy of the rotation rate formula (7) with Shapiro's experiments, Helmholtz's view on continuum rotation (Helmholtz, 1858), and Cauchy's local mean rotation rate over all material fibers. We also show that $\mathbf{O}_\tau^t$ admits a further factorization into a spatial mean rotation tensor and a dynamically consistent relative rotation tensor, the latter of which is objective for planar deformations. Finally, we introduce dynamically consistent (i.e., temporally additive) rotation angles that extend Cauchy's classic mean rotation, and illustrate all these concepts on two- and three-dimensional examples.

2. Dynamic polar decomposition (DPD)

Several generalizations of the classic polar decomposition to linear operators on various spaces are available (see, e.g., Douglas, 1966; Conway, 1990). These, however, invariably target single linear operators, as opposed to operator families, and hence exhibit the dynamic inconsistency (3).

The only polar decomposition developed specifically for time-dependent operator families appears to be the one by Munthe-Kaas et al. (2001) and Zanna and Munthe-Kaas (2001). This targets Lie groups, such as matrix–exponential solutions of linear autonomous systems of differential equations. The decomposition, however, is approximate and exists only for small enough $t - \tau$, i.e., for small deformations in our context. More importantly, the deformation gradient $\mathbf{F}_\tau^t$ is generally a two-parameter process (Dafermos, 1971), not a one-parameter Lie group, even if the underlying deformation field has a steady velocity field.

In order to modify the classic polar decomposition to one with dynamic consistency, we first recall the notion of a process. We formulate the original definition of a nonlinear process here specifically for linear systems. The definition for nonlinear processes can be recovered by replacing the product of two linear operators in Definition 1 with the composition of two general functions that depend on t and τ as parameters (cf. Appendix A).

Definition 1. A two-parameter family $\mathbf{T}_\tau^t: \mathbb{R}^n \to \mathbb{R}^n$, $\tau, t \in \mathbb{R}$, of linear operators is a *linear process* if it is continuously differentiable with respect to the parameters τ and t, and satisfies

$$\mathbf{T}_t^t = \mathbf{I}, \quad \mathbf{T}_\tau^t = \mathbf{T}_s^t \mathbf{T}_\tau^s, \quad \tau, s, t \in \mathbb{R}.$$

For any linear process, we can write

$$\dot{\mathbf{T}}_\tau^t = \frac{d}{d\sigma}\mathbf{T}_\tau^{t+\sigma}\Big|_{\sigma=0} = \frac{d}{d\sigma}\mathbf{T}_t^{t+\sigma}\mathbf{T}_\tau^t\Big|_{\sigma=0} = \frac{d}{d\sigma}\mathbf{T}_t^{t+\sigma}\Big|_{\sigma=0}\mathbf{T}_\tau^t. \tag{8}$$

Therefore, any linear process $\mathbf{T}_\tau^t$ is the unique solution of a non-autonomous linear initial value problem of the form

$$\dot{\mathbf{Z}} = \mathbf{A}(t)\mathbf{Z}, \quad \mathbf{Z}(\tau) = \mathbf{I}; \quad \mathbf{A}(t) := \frac{d}{d\sigma}\mathbf{T}_t^{t+\sigma}\Big|_{\sigma=0}. \tag{9}$$

Conversely, the solution of any non-autonomous linear initial value problem $\dot{\mathbf{Z}} = \mathbf{A}(t)\mathbf{Z}$, $\mathbf{Z}(\tau) = \mathbf{I}$ is a linear process by the basic properties of fundamental matrix solutions of linear differential equations (Arnold, 1978).

Example 1. The deformation gradient $\mathbf{F}_\tau^t$ arising from a velocity field $\mathbf{v}(\mathbf{x}, t)$ is a linear process, as it satisfies the equation of variations

$$\dot{\mathbf{Z}} = \nabla\mathbf{v}(\mathbf{x}(t), t)\mathbf{Z} \tag{10}$$

with initial condition $\mathbf{Z}(\tau) = \mathbf{I}$, along the trajectory $\mathbf{x}(t)$. If the velocity field $\mathbf{v}$ is irrotational ($\nabla \times \mathbf{v} \equiv 0$), then its spin tensor $\mathbf{W}$ vanishes, and hence $\dot{\mathbf{Z}}\mathbf{Z}^{-1} = \nabla\mathbf{v}(\mathbf{x}(t), t) = \mathbf{D}(\mathbf{x}(t), t)$ is a symmetric matrix. Similarly, if the velocity field generates purely rotational motion without Eulerian strain ($\mathbf{D} \equiv \mathbf{0}$), then $\dot{\mathbf{Z}}\mathbf{Z}^{-1} = \nabla\mathbf{v}(\mathbf{x}(t), t) = \mathbf{W}(\mathbf{x}(t), t)$ is a skew-symmetric matrix.

Motivated by Example 1, we introduce the following definitions for smooth, two-parameter families of operators:

Definition 2. Let $\mathrm{Skew}(n)$, $\mathrm{Sym}(n)$, and $SO(n)$ denote the set of skew-symmetric, symmetric and proper-orthogonal linear operators on $\mathbb{R}^n$, respectively. Also, let $\mathbf{T}_\tau^t: \mathbb{R}^n \to \mathbb{R}^n$ be a smooth, two-parameter family of linear operators. Then

(i) $\mathbf{T}_\tau^t$ is *rotational* if $\dot{\mathbf{T}}_\tau^t[\mathbf{T}_\tau^t]^{-1} \in \mathrm{Skew}(n)$ for all $\tau, t \in \mathbb{R}$, or, equivalently, $\mathbf{T}_\tau^t \in SO(n)$ for all $\tau, t \in \mathbb{R}$;
(ii) $\mathbf{T}_\tau^t$ is *irrotational* if $\dot{\mathbf{T}}_\tau^t[\mathbf{T}_\tau^t]^{-1} \in \mathrm{Sym}(n)$ for all $\tau, t \in \mathbb{R}$.

The equivalence of the two characterizations of time-dependent rotations in (i) of Definition 2 is broadly known, as discussed, e.g., by Epstein (1966). The concept of an irrotational linear operator family in (ii) of Definition 2 serves as a relaxation of the concept of symmetric operator families. Instead of requiring $\mathbf{T}_\tau^t$ to be symmetric, we only require $\dot{\mathbf{T}}_\tau^t[\mathbf{T}_\tau^t]^{-1}$ to be symmetric, which guarantees $\mathbf{T}_\tau^t$ to be the deformation field of a purely straining linear velocity field. We then obtain the following result on the decomposition of an arbitrary smooth linear process $\mathbf{T}_\tau^t$ into a rotational process and an irrotational linear transformation family.

Theorem 1 (*Dynamic polar decomposition (DPD)*). *Any linear process $\mathbf{T}_\tau^t: \mathbb{R}^n \to \mathbb{R}^n$ admits a unique decomposition of the form*

$$\mathbf{T}_\tau^t = \mathbf{O}_\tau^t\mathbf{M}_\tau^t = \mathbf{N}_\tau^t\mathbf{O}_\tau^t, \tag{11}$$

where $\mathbf{O}_\tau^t$ is an n-dimensional rotational process, while $\mathbf{M}_\tau^t$ and $(\mathbf{N}_\tau^t)^T$ are n-dimensional irrotational operator families that have the same singular values as $\mathbf{T}_\tau^t$. Furthermore, with the operators

$$\mathbf{A}^-(t) = \tfrac{1}{2}\left[\dot{\mathbf{T}}_\tau^t\mathbf{T}_t^\tau - \left(\mathbf{T}_t^\tau\right)^T\left(\dot{\mathbf{T}}_\tau^t\right)^T\right], \quad \mathbf{A}^+(t) = \tfrac{1}{2}\left[\dot{\mathbf{T}}_\tau^t\mathbf{T}_t^\tau + \left(\mathbf{T}_t^\tau\right)^T\left(\dot{\mathbf{T}}_\tau^t\right)^T\right], \tag{12}$$

the factors in the decomposition (11) satisfy the linear differential equations

$$\begin{aligned}&\dot{\mathbf{O}}_\tau^t = \mathbf{A}^-(t)\mathbf{O}_\tau^t, \quad \mathbf{O}_\tau^\tau = \mathbf{I},\\ &\dot{\mathbf{M}}_\tau^t = [\mathbf{O}_t^\tau \mathbf{A}^+(t)\mathbf{O}_\tau^t]\mathbf{M}_\tau^t, \quad \mathbf{M}_\tau^\tau = \mathbf{I},\\ &\frac{d}{d\tau}\left(\mathbf{N}_\tau^t\right)^T = -[\mathbf{O}_\tau^t \mathbf{A}^+(\tau)\mathbf{O}_t^\tau]\left(\mathbf{N}_\tau^t\right)^T, \quad \left(\mathbf{N}_t^t\right)^T = \mathbf{I}.\end{aligned} \tag{13}$$

Both $\mathbf{A}^-(t)$ and $\mathbf{A}^+(t)$ are independent of τ, and hence $\mathbf{A}^+(\tau)$ is independent of t.

Proof. See Appendix B.

Once the rotational process $\mathbf{O}_\tau^t$ and one of the two irrotational operator families, $\mathbf{M}_\tau^t$ and $\mathbf{N}_\tau^t$, are known, the other irrotational operator family can be computed from the relationship (11).

3. DPD of the deformation gradient

Theorem 1 implies that the linear process $\mathbf{F}_\tau^t$ (cf. Example 1) can uniquely be written as the product of left (and right) rotational and irrotational operator families. Out of the two versions of this decomposition, the left irrotational operator family also turns out to be objective, i.e., its invariants transform properly under Euclidean transformations of the form

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{y} + \mathbf{b}(t), \tag{14}$$

where the matrix $\mathbf{Q}(t) \in SO(3)$ and the vector $\mathbf{b}(t) \in \mathbb{R}^3$ are smooth functions of t (Truesdell and Noll, 1965). We summarize these results in more precise terms as follows, using notation already introduced in (4).

Theorem 2 (*DPD of the deformation gradient*). *For the deformation field $\mathbf{X}_{t_0}^t: \mathcal{B}(t_0) \subset \mathbb{R}^3 \to \mathcal{B}(t)$, with $t \in [t_0, t_1]$, consider a trajectory $\mathbf{x}(t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$. Then for any initial time of observation $\tau \in [t_0, t_1]$:*

(i) *The deformation gradient $\mathbf{F}_\tau^t(\mathbf{x}(\tau))$ admits a unique decomposition of the form*

$$\mathbf{F}_\tau^t = \mathbf{O}_\tau^t \mathbf{M}_\tau^t = \mathbf{N}_\tau^t \mathbf{O}_\tau^t, \tag{15}$$

where the dynamic rotation tensor $\mathbf{O}_\tau^t$ is a rotational linear process; the dynamic right stretch tensor $\mathbf{M}_\tau^t$, and the transpose of the dynamic left stretch tensor $\mathbf{N}_\tau^t$ are irrotational families of transformations.

(ii) *For any $\tau, t \in \mathbb{R}$, the dynamic stretch tensors $\mathbf{M}_\tau^t$ and $\mathbf{N}_\tau^t = (\mathbf{M}_t^\tau)^{-1}$ are nonsingular, and have the same singular values and principal axes of strain as $\mathbf{U}_\tau^t$ and $\mathbf{V}_\tau^t$ do.*

(iii) *The dynamic rotation tensor $\mathbf{O}_\tau^t$, and the dynamic stretch tensors $\mathbf{M}_\tau^t$ and $\mathbf{N}_\tau^t$ are solutions of the linear initial value problems*

$$\dot{\mathbf{O}}_\tau^t = \mathbf{W}(\mathbf{x}(t), t)\mathbf{O}_\tau^t, \quad \mathbf{O}_\tau^\tau = \mathbf{I}, \tag{16}$$

$$\dot{\mathbf{M}}_\tau^t = [\mathbf{O}_t^\tau \mathbf{D}\left(\mathbf{x}(t), t\right)\mathbf{O}_\tau^t]\mathbf{M}_\tau^t, \quad \mathbf{M}_\tau^\tau = \mathbf{I}, \tag{17}$$

$$\frac{d}{d\tau}\left(\mathbf{N}_\tau^t\right)^T = -[\mathbf{O}_\tau^t \mathbf{D}\left(\mathbf{x}(\tau), \tau\right)\mathbf{O}_t^\tau]\left(\mathbf{N}_\tau^t\right)^T, \quad \left(\mathbf{N}_t^t\right)^T = \mathbf{I}. \tag{18}$$

(iv) *The left dynamic stretch tensor $\mathbf{N}_\tau^t$ is objective (cf. Remark 4).*

Proof. See Appendix C.

Remark 1. A physical interpretation of the left DPD in statement (i) Theorem 2 is the following. The deformation gradient $\mathbf{F}_\tau^t$ can uniquely be written as a product of two other deformation gradients: one for a purely rotational (i.e., strainless) linear deformation field, and one for a purely straining (i.e., irrotational) linear deformation field. Specifically, $\mathbf{O}_\tau^t = \partial_{\mathbf{a}_\tau}\mathbf{a}(t)$ is the deformation gradient of the strainless linear deformation $\mathbf{a}_\tau \mapsto \mathbf{a}(t; \tau, \mathbf{a}_\tau)$ defined by

$$\dot{\mathbf{a}} = \mathbf{W}(\mathbf{x}(t), t)\mathbf{a}, \tag{19}$$

and $\mathbf{M}_\tau^t = \partial_{\mathbf{b}_\tau}\mathbf{b}(t)$ is the deformation gradient of the irrotational linear deformation $\mathbf{b}_\tau \mapsto \mathbf{b}(t; \tau, \mathbf{b}_\tau)$ defined by

$$\dot{\mathbf{b}} = \mathbf{O}_t^\tau \mathbf{D}(\mathbf{x}(t), t)\mathbf{O}_\tau^t \mathbf{b}. \tag{20}$$

A similar interpretation holds for the right DPD in statement (i) of Theorem 2.

Remark 2. Theorem 2 guarantees that the dynamic rotation tensor $\mathbf{O}_\tau^t$ is the fundamental matrix solution of the classical linear system of ODEs (16). As a consequence, $\mathbf{O}_\tau^t$ forms a linear process (or linear dynamical system), thereby satisfying the required dynamical consistency condition

$$\mathbf{O}_\tau^t = \mathbf{O}_s^t \mathbf{O}_\tau^s, \quad \forall\, \tau, s, t \in \mathbb{R}. \tag{21}$$

By construction (cf. the proof of Theorem 2), $\mathbf{O}_\tau^t$ is the unique dynamically consistent rotation tensor out of the infinitely many possible ones in (2).

Remark 3. The formula

$$\dot{\mathbf{U}}_\tau^t = \left[\mathbf{R}_\tau^t\right]^T [\nabla \mathbf{v}\left(\mathbf{x}(t), t\right) - \dot{\mathbf{R}}_\tau^t\left[\mathbf{R}_\tau^t\right]^T]\mathbf{U}_\tau^t \tag{22}$$

(see, e.g, Truesdell and Rajagopal, 2009) reveals that $\dot{\mathbf{U}}_\tau^t[\mathbf{U}_\tau^t]^{-1}$ is not symmetric, and hence the evolution of the polar stretch tensor is *not* free from spin. Therefore, the polar decomposition does not fully separate a purely spinning component from a non-spinning component in the deformation. In contrast, the dynamic polar decomposition separates a purely spinning part of the deformation gradient (cf. (19)) from a purely straining part with zero spin (cf. (20)). By construction (cf. the proof of Theorem 2), $\mathbf{M}_\tau^t$ is the unique spin-free stretch tensor out of the infinitely many possible ones in (2).

Remark 4. As seen from formulas (62)–(64) in the proof of Theorem 2, a general observer change (14) transforms the dynamic rotation and stretch tensors to the form

$$\tilde{\mathbf{O}}_\tau^t = \mathbf{Q}^T(t)\mathbf{O}_\tau^t\mathbf{Q}(\tau), \quad \tilde{\mathbf{M}}_\tau^t = \mathbf{Q}^T(\tau)\mathbf{M}_\tau^t\mathbf{Q}(\tau), \quad \tilde{\mathbf{N}}_\tau^t = \mathbf{Q}^T(t)\mathbf{N}_\tau^t\mathbf{Q}(t)$$

in the $\mathbf{y}$ coordinate frame. Thus, the left stretch tensor $\mathbf{N}_\tau^t$ is objective but the right stretch tensor $\mathbf{M}_\tau^t$ is not. Analogously, the left polar stretch tensor $\mathbf{V}_\tau^t$ is objective but the right polar stretch tensor $\mathbf{U}_\tau^t$ is not (cf. Truesdell and Rajagopal, 2009).

Remark 5. The relationship (11) gives

$$\mathbf{M}_\tau^t = \mathbf{O}_t^\tau\mathbf{N}_\tau^t\mathbf{O}_\tau^t = \left[\mathbf{O}_\tau^t\right]^{-1}\mathbf{N}_\tau^t\mathbf{O}_\tau^t = \left[\mathbf{O}_\tau^t\right]^T\mathbf{N}_\tau^t\mathbf{O}_\tau^t,$$

revealing that the right dynamic stretch tensor is just the representation of the left dynamic stretch tensor in a coordinate frame rotating under the action of $\mathbf{O}_\tau^t$. Similarly, Eq. (17) shows that the stretch rate tensor $\dot{\mathbf{M}}_\tau^t[\mathbf{M}_\tau^t]^{-1}$ is just the rate of strain tensor $\mathbf{D}$ represented in the same rotating frame.

Remark 6. The stretch tensors $\mathbf{M}_\tau^t$ and $\mathbf{N}_\tau^t$ are also fundamental matrix solutions, yet $\mathbf{M}_\tau^t \neq \mathbf{M}_s^t\mathbf{M}_\tau^s$ and $\mathbf{N}_\tau^t \neq \mathbf{N}_s^t\mathbf{N}_\tau^s$. This is because the linear systems of ODEs (17) and (18) are not of the classical type: they have right-hand sides depending explicitly on the initial time τ as well. As a consequence, their fundamental matrix solutions do not form processes. However, the nonlinear system of differential equations (16) and (17) has no explicit dependence on τ when posed for the dependent variable $\mathbf{H}_\tau^t = (\mathbf{O}_\tau^t, \mathbf{M}_\tau^t)$. As a consequence, the nonlinear process property

$$\mathbf{H}_\tau^t = \mathbf{H}_s^t \bigcirc \mathbf{H}_\tau^s$$

holds for this system of equations, and hence the pair $(\mathbf{O}_\tau^t, \mathbf{M}_\tau^t)$ forms a nonlinear dynamical system. This is not the case for the polar rotation-stretch pair $(\mathbf{R}_\tau^t, \mathbf{U}_\tau^t)$ (cf. Appendix A).

Remark 7. The DPD of the deformation gradient in Theorem 2 replaces the requirement of symmetry for the polar stretch tensors $\mathbf{U}_\tau^t$ and $\mathbf{V}_\tau^t$ with the requirement that the dynamic stretch tensors be deformations generated by purely straining velocity fields. As noted in statement (ii) of Theorem 2, $\mathbf{M}_\tau^t$ and $\mathbf{N}_\tau^t$ still have the same singular values and corresponding principal axes of strain as their polar equivalents. Thus, they continue to capture the same objective information about stretch encoded in the right and left Cauchy–Green strain tensors, $\mathbf{C}_\tau^t = (\mathbf{F}_\tau^t)^T\mathbf{F}_\tau^t$ and $\mathbf{B}_\tau^t = \mathbf{F}_\tau^t(\mathbf{F}_\tau^t)^T$.

Remark 8 (*Connections with prior work*). Without the claim of uniqueness, the first dynamic decomposition $\mathbf{F}_\tau^t = \mathbf{O}_\tau^t\mathbf{M}_\tau^t$ in (15) and the two equations (16) and (17) could also be obtained by first extending Theorem 1 of Epstein (1962) on linear differential equations to arbitrary initial times τ, and then applying this extension to the equation of variations (10). Also, the finite rotation family generated by Eq. (16) is just the one considered by Noll (1955, p. 27) to derive isotropy-based invariance condition for general class of (hygrosteric) constitutive laws. In that context, however, $\mathbf{O}_\tau^t$ was selected in an ad hoc fashion out of infinitely many possible rotations because of the simplicity of its associated rotation rate $\dot{\mathbf{O}}_\tau^t\mathbf{O}_t^\tau = \mathbf{W}$. Finally, Eq. (16) also appears formally in the work of Rubinstein and Atluri (1983) (see their Eq. (41)). They, however, propose this ODE merely as one generating a plausible rotating frame in which to study deformation, as opposed to one deduced from a systematic decomposition of the deformation gradient.

4. The relative rotation tensor

The left dynamic stretch tensor obtained from the DPD of the deformation gradient is objective, but the dynamic rotation tensor is not. This is due to the inherent dependence of rigid body rotation on the reference frame. For deforming bodies, however, there is a non-vanishing part of the dynamic rotation that deviates from the spatial mean rotation of the body. This relative rotation is not only dynamically consistent, but also turns out to be objective for planar deformations.

To state this result more formally for a deforming body $\mathcal{B}(t) = \mathbf{X}_\tau^t(\mathcal{B}(\tau))$, we denote the spatial mean of any quantity $(\cdot)(\mathbf{x}, t)$ defined on $\mathcal{B}(t)$ by

$$\overline{(\cdot)}(t) = \frac{1}{\mathrm{vol}(\mathcal{B}(t))}\int_{\mathcal{B}(t)} (\cdot)(\mathbf{x}, t)\, dV,$$

where vol( ) denotes the volume for three-dimensional bodies, and the area for two-dimensional bodies. Accordingly, dV refers to the volume or area element.

Theorem 3 (*Relative and mean rotation tensors*).

(i) *The dynamic rotation tensor* $\mathbf{O}_\tau^t$ *admits a unique decomposition of the form*

$$\mathbf{O}_\tau^t = \mathbf{\Phi}_\tau^t \mathbf{\Theta}_\tau^t = \mathbf{\Sigma}_\tau^t \mathbf{\Phi}_\tau^t, \tag{23}$$

where the relative rotation tensor $\mathbf{\Phi}_\tau^t$ *and the mean rotation tensors* $\mathbf{\Theta}_\tau^t$ *and* $\mathbf{\Sigma}_\tau^t$ *satisfy the initial value problems*

$$\dot{\mathbf{\Phi}}_\tau^t = [\mathbf{W}\big(x(t), t\big) - \bar{\mathbf{W}}\big(t\big)]\mathbf{\Phi}_\tau^t, \quad \mathbf{\Phi}_\tau^\tau = \mathbf{I}, \tag{24}$$

$$\dot{\mathbf{\Theta}}_\tau^t = [\mathbf{\Phi}_t^\tau \bar{\mathbf{W}}\big(t\big)\mathbf{\Phi}_\tau^t]\mathbf{\Theta}_\tau^t, \quad \mathbf{\Theta}_\tau^\tau = \mathbf{I}, \tag{25}$$

$$\frac{d}{d\tau}\big(\mathbf{\Sigma}_\tau^t\big)^T = [\mathbf{\Phi}_\tau^t \bar{\mathbf{W}}(\tau)\mathbf{\Phi}_t^\tau]\big(\mathbf{\Sigma}_\tau^t\big)^T, \quad \mathbf{\Sigma}_t^t = \mathbf{I}. \tag{26}$$

(ii) *The relative rotation tensor* $\mathbf{\Phi}_\tau^t$ *is a rotational process. For two-dimensional deformations,* $\mathbf{\Phi}_\tau^t$ *is also objective.*

Proof. See Appendix D.

The joint application of Theorems 2 and 3 gives four possible decompositions of the deformation gradient:

$$\mathbf{F}_\tau^t = \mathbf{\Phi}_\tau^t \mathbf{\Theta}_\tau^t \mathbf{M}_\tau^t = \mathbf{\Sigma}_\tau^t \mathbf{\Phi}_\tau^t \mathbf{M}_\tau^t = \mathbf{N}_\tau^t \mathbf{\Phi}_\tau^t \mathbf{\Theta}_\tau^t = \mathbf{N}_\tau^t \mathbf{\Sigma}_\tau^t \mathbf{\Phi}_\tau^t.$$

The relative rotation tensor $\mathbf{\Phi}_\tau^t$ is dynamically consistent and objective in two dimensions; the right and left mean rotation tensors, $\mathbf{\Theta}_\tau^t$ and $\mathbf{\Sigma}_\tau^t$, are frame-dependent rotational operator families. While the relative rotation tensor $\mathbf{\Phi}_\tau^t$ is generally not objective for three-dimensional deformations, it still remains frame-invariant under all rotations $\mathbf{Q}(t)$ whose rotation-rate tensor $\dot{\mathbf{Q}}^T(t)\mathbf{Q}(t)$ commutes with $\mathbf{\Phi}_\tau^t$ (cf. formula (78) of Appendix D).

Remark 9. From Eqs. (74), (76), (77) and (67) of Appendix D, we deduce the following transformation formulas for the rotation tensors featured in Theorem 3, under observer changes of the form (14)

$$\tilde{\mathbf{\Phi}}_\tau^t = \mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t \mathbf{P}(t), \quad \tilde{\mathbf{\Theta}}_\tau^t = \mathbf{P}^T(t)\mathbf{\Theta}_\tau^t \mathbf{Q}(\tau), \quad \tilde{\mathbf{\Sigma}}_\tau^t = \mathbf{Q}^T(t)\mathbf{\Sigma}_\tau^t \mathbf{Q}(\tau).$$

Here the rotation tensor $\mathbf{P}(t) \in SO(3)$ satisfies the linear initial value problem

$$\dot{\mathbf{P}}(t) = \mathbf{\Phi}_t^\tau \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t \mathbf{P}(t), \quad \mathbf{P}(\tau) = \mathbf{Q}(\tau).$$

5. Dynamically consistent angular velocity and mean rotation angles

5.1. Angular velocity from the dynamic rotation tensor

By Eqs. (5) and (16), the time-derivatives of the rotation tensor and the dynamic rotation tensor agree in the limit of infinitesimally short deformations, i.e.,

$$\dot{\mathbf{R}}_\tau^t|_{t=\tau} = \dot{\mathbf{O}}_\tau^t|_{t=\tau} = \mathbf{W}.$$

As noted in the Introduction, however, the polar rotation does not give a well-defined, history-independent angular velocity for finite deformations. At the same time, the dynamic rotation gives the same angular velocity (deduced from $\mathbf{W}$) both for infinitely short and for finite deformations. This angular velocity equals the mean rotation rate of material fibers in two dimensions (Cauchy, 1841). Here we show that the same equality holds for three-dimensional deformations as well.

Clearly, the rotation of an infinitesimal rigid sphere in a fluid differs from the rotation of infinitesimal material fibers in the fluid. Each such material fiber rotates with a different angular velocity, even in the simplest two-dimensional steady flows (see Examples 2 and 3 below). Nevertheless, for all two-dimensional deformations, Cauchy (1841) found that averaging the angular velocity over all material fibers emanating from the same point gives a mean angular velocity equal to $\frac{1}{2}\boldsymbol{\omega}$ (see also Truesdell, 1954). This justifies the use of small spherical tracers to infer the rate of local mean material rotation in two-dimensional continuum motion (see, e.g., the experiments of Shapiro, 1961 for fluids).

In three-dimensional continuum motion, the Maxey–Riley equations (Maxey, 1990) continue to predict $\frac{1}{2}\boldsymbol{\omega}$ as the angular velocity of small spherical particles. Experiments on three-dimensional turbulence confirm this result (see, e.g., Meyer et al., 2013). One would ideally need, however, an extension of Cauchy's fiber-averaged angular velocity argument from two to three-dimensions to justify equating the observed rotation rate of small rigid spheres with the local mean rate of material rotation.

The main challenge for such an extension is that a one-dimensional material element has no well-defined angular velocity in three dimensions. To see this, we let

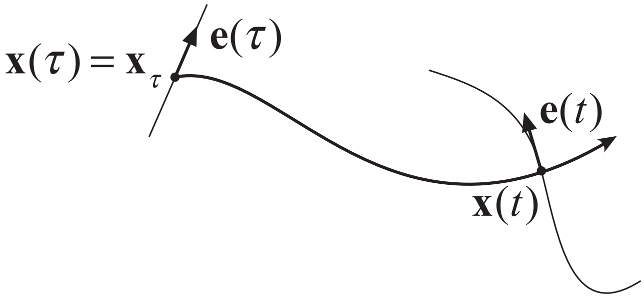

Fig. 2. The unit vector $\mathbf{e}(t)$ tangent to a material fiber evolving along the trajectory $\mathbf{x}(t)$.

$$\mathbf{e}(t) = \frac{\mathbf{F}_\tau^t \mathbf{e}(\tau)}{\left|\mathbf{F}_\tau^t \mathbf{e}(\tau)\right|}, \tag{27}$$

denote a unit vector tangent to a deforming material fiber along the trajectory $\mathbf{x}(t)$. This trajectory starts from the point $\mathbf{x}_\tau$ at time τ, as shown in Fig. 2.

There exists then an open half-plane $\mathcal{P}$ spanned by admissible angular velocity vectors $\boldsymbol{\nu}$ such that the instantaneous velocity $\dot{\mathbf{e}}$ of the evolving $\mathbf{e}(t)$ satisfies $\dot{\mathbf{e}} = \boldsymbol{\nu} \times \mathbf{e}$. The magnitudes of these admissible angular velocity vectors range from $|\dot{\mathbf{e}}|$ to infinity, depending on the angle they enclose with $\mathbf{e}$ (see Fig. 3). There is, therefore, no unique angular velocity for the evolving material fiber tangent to $\mathbf{e}$.

We can nevertheless extend Cauchy's mean rotation result to three-dimension using the following construct. Let us define the *minimal angular velocity* vector $\boldsymbol{\nu}_{\min}(\mathbf{x}, t, \mathbf{e})$ for the unit vector $\mathbf{e}$ as the admissible angular velocity in $\mathcal{P}$ with the smallest possible norm

$$\boldsymbol{\nu}_{\min}(\mathbf{x}, t, \mathbf{e}) = \mathbf{e} \times \dot{\mathbf{e}}. \tag{28}$$

We then define the *material-fiber-averaged angular velocity* $\boldsymbol{\nu}(\mathbf{x}, t)$ at the point $\mathbf{x}$ of a deforming body $\mathcal{B}(t)$ by the formula

$$\boldsymbol{\nu}(x, t) := 2\langle \boldsymbol{\nu}_{\min}(\mathbf{x}, t, \mathbf{e})\rangle_{\mathbf{e}\in S^2_{\mathbf{x}}}, \quad \mathbf{x} \in \mathcal{B}(t), \tag{29}$$

with the $\langle\cdot\rangle_{\mathbf{e}\in S^2_{\mathbf{x}}}$ operation referring to the mean over all vectors in the unit sphere $S^2_{\mathbf{x}}$ centered at the point $\mathbf{x}$. For a perfectly rigid body, we recover from formula (29) the unique angular velocity of the body as the fiber-averaged angular velocity (see Appendix E). For a general deformable continuum, $\boldsymbol{\nu}(\mathbf{x}, t)$ still turns out to be computable and equal to half of the vorticity.

Proposition 1 (*Fiber-averaged angular velocity in 3D*). *For a general three-dimensional deforming body $\mathcal{B}(t)$, the material-fiber-averaged angular velocity at a location $\mathbf{x} \in \mathcal{B}(t)$ at time t is given by*

$$\boldsymbol{\nu}(\mathbf{x}, t) \equiv \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}, t),$$

where $\boldsymbol{\omega}(\mathbf{x}, t)$ denotes the vorticity vector field of $\mathcal{B}(t)$.

Proof. See Appendix F.

Proposition 1 extends Cauchy's mean material rotation rate result to three dimensions. It supports the expectation that a self-consistent description of mean material rotation should yield an instantaneous angular velocity equal to half of the vorticity for any finite deformation, just as the dynamic rotation tensor $\mathbf{O}_\tau^t$ does.

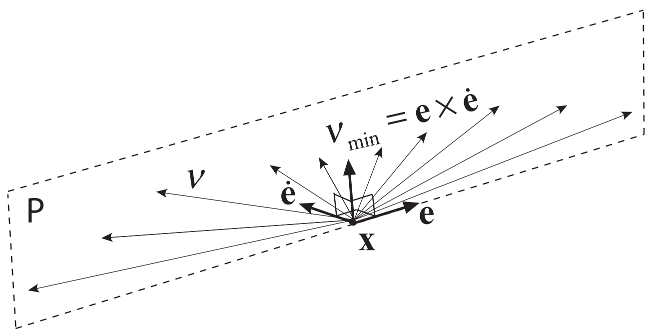

Fig. 3. The plane $\mathcal{P}$ of all admissible angular velocities $\boldsymbol{\nu}$ that generate the same velocity $\dot{\mathbf{e}}$ for a unit vector $\mathbf{e}$ tangent to a deforming material element at $\mathbf{x}$.

5.2. Dynamically consistent mean rotation angles

Cauchy (1841) measures the magnitude of finite continuum rotation locally by computing the rotation angle of initially co-planar line elements about the normal of their initial plane. This *mean rotation angle* obeys a complicated, coordinate-dependent formula (Truesdell, 1954) that remained unevaluated and largely unused for a long time.

Remarkably, Zheng and Hwang (1992) and Huang et al. (1996) succeeded in evaluating the integral in Cauchy's mean rotation angle for general planes, obtaining involved expressions defined on different angular domains. As an alternative measure of mean rotation, Novozhilov (1971) proposed to evaluate the spatial mean of the tangent of Cauchy's mean rotation angle, as opposed to the mean of the angle itself, over all initially co-planar material vectors. Invariant formulations of this idea appeared later in Truesdell and Toupin (1960) and de Oliveira et al. (2005). While simpler to evaluate, Novozhilov's version of the mean rotation angle suffers from singularities due to the use of the tangent function (de Oliveira et al., 2005). Finally, Marzano (1987) proposed the mean of the cosine of Cauchy's angle as a measure of mean rotation.

For all these mean rotation measures, the total rotation is not well-defined beyond a range of angles due to the inherent limitations of the inverse trigonometric functions used in their construction. A more important issue is, however, that even fully invariant formulations of the mean rotation angle concept (e.g., Martins and Podiu-Guiduigli, 1992; Zheng et al., 1994) extract the rotational component of a deformation gradient via polar decomposition between fixed initial and finite times. As a consequence, these mean rotations are *not material*: they inherit the dynamic inconsistency (3) of the rotation tensor.

When evaluated along a material trajectory $\mathbf{x}(t)$, with $\mathbf{x}(\tau) = \mathbf{x}_\tau$, any smooth unit vector field $\mathbf{g}(\mathbf{x}, t)$ defines a time-varying axis $\mathbf{g}(\mathbf{x}(t), t)$. For any smooth rotation family $\mathbf{Q}(s)$ defined along $\mathbf{x}(s)$ for $s \in [\tau, t]$, the total rotation angle α_τ^t with respect to the evolving axis $\mathbf{g}(\mathbf{x}(s), s)$ is equal to

$$\alpha_\tau^t(\mathbf{x}_\tau; \mathbf{g}) = \int_\tau^t \dot{\boldsymbol{q}}(s)\cdot\mathbf{g}(\mathbf{x}(s), s)\, ds,$$

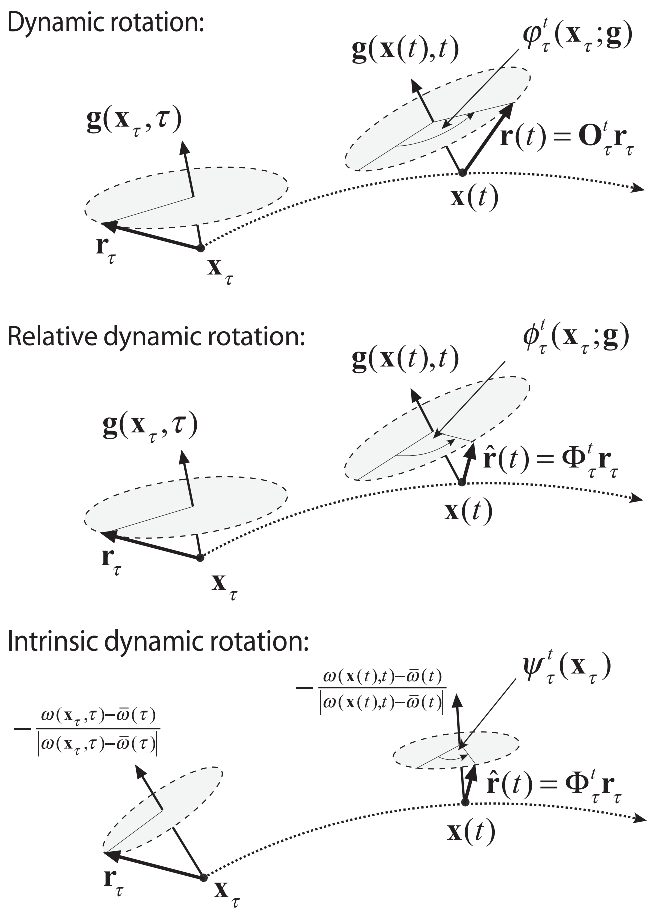

Fig. 4. The geometry of the dynamic rotation, relative dynamic rotation and intrinsic dynamic rotation obtained in Theorem 4. *Top*: a vector $\mathbf{r}_\tau$, based at the initial point $\mathbf{x}_\tau$, is rotated by the dynamic rotation tensor $\mathbf{O}_\tau^t$ into the vector $\mathbf{r}(t)$, spanning the dynamic rotation angle $\varphi_\tau^t(\mathbf{x}_\tau; \mathbf{g})$ around an a priori defined rotation axis family $\mathbf{g}$. *Middle*: the same initial vector $\mathbf{r}_\tau$ is now rotated by the relative rotation tensor $\mathbf{\Phi}_\tau^t$ into the vector $\hat{\mathbf{r}}(t)$, spanning the relative dynamic rotation angle $\phi_\tau^t(\mathbf{x}_\tau; \mathbf{g})$ around the axis family $\mathbf{g}$. *Bottom*: $\mathbf{r}_\tau$ is again rotated by the relative rotation tensor $\mathbf{\Phi}_\tau^t$ into the vector $\hat{\mathbf{r}}(t)$, spanning the intrinsic dynamic rotation angle $\psi_\tau^t(\mathbf{x}_\tau; \mathbf{g})$ around the intrinsically defined rotation axis family $-(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})/|\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}|$.

where the angular velocity vector $\dot{\boldsymbol{q}}(s)$ of $\mathbf{Q}(s)$ is defined by the relationship

$$\dot{\mathbf{Q}}(s)\mathbf{Q}^T(s)\mathbf{e} = \dot{\boldsymbol{q}}(s) \times \mathbf{e}, \quad \forall\ \mathbf{e} \in \mathbb{R}^3.$$

In line with our definition of dynamical consistency for rotation tensors, we say that the rotation angle α_τ^t with respect to the axis field $\mathbf{g}(\mathbf{x}, t)$ is *dynamically consistent* if it is additive along trajectories. Specifically, for all times $\tau, \sigma, t \in [t_0, t_1]$, the angle α_τ^t should satisfy

$$\alpha_\tau^t(\mathbf{x}_\tau; \mathbf{g}) = \alpha_\sigma^t(\mathbf{x}_\sigma; \mathbf{g}) + \alpha_\tau^\sigma(\mathbf{x}_\tau; \mathbf{g}) \tag{30}$$

for dynamical consistency. Note that the choice $\mathbf{Q}(t) = \mathbf{R}_\tau^t$ does not give a dynamically consistent angle by formula (3) (cf. Remark 12 in Appendix G). The dynamic polar decomposition, however, provides several dynamically consistent rotation angles, some of which are even objective. We keep the terminology used for Cauchy's angle, referring to these dynamically consistent rotation angles as mean rotation angles. This is because they represent single-valued, overall fits to a continuum of fiber rotation angles in a deforming volume element.

Theorem 4 (*Dynamically consistent mean rotation angles*).

(i) *The rotation angle generated by the dynamic rotation tensor $\mathbf{O}_\tau^t$ around the axis family $\mathbf{g}$ is given by the* ***dynamic rotation***

$$\varphi_\tau^t(\mathbf{x}_\tau; \mathbf{g}) = -\frac{1}{2}\int_\tau^t \boldsymbol{\omega}(\mathbf{x}(s), s)\cdot\mathbf{g}(\mathbf{x}(s), s)\, ds, \tag{31}$$

which is a dynamically consistent rotation angle.

(ii) *The rotation angle generated by the relative rotation tensor $\boldsymbol{\Phi}_\tau^t$ around the axis family $\mathbf{g}$ is given by the* ***relative dynamic rotation***

$$\phi_\tau^t(\mathbf{x}_\tau; \mathbf{g}) = -\frac{1}{2}\int_\tau^t [\boldsymbol{\omega}(\mathbf{x}(s), s) - \bar{\boldsymbol{\omega}}(s)]\cdot\mathbf{g}(\mathbf{x}(s), s)\, ds, \tag{32}$$

which is an objective and dynamically consistent rotation angle.

(iii) *The rotation angle generated by the relative rotation tensor $\boldsymbol{\Phi}_\tau^t$ around its own axis of rotation is given by the* ***intrinsic dynamic rotation***

$$\psi_\tau^t(\mathbf{x}_\tau) := \phi_\tau^t\left(\mathbf{x}_\tau; -\frac{\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}}{|\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}|}\right) = \frac{1}{2}\int_\tau^t \left|\boldsymbol{\omega}(\mathbf{x}(s), s) - \bar{\boldsymbol{\omega}}(s)\right| ds \tag{33}$$

which is an objective and dynamically consistent rotation angle.

Proof. See Appendix G.

Fig. 4 illustrates the geometry of the dynamically consistent mean rotation angles described in Theorem 4.

Remark 10. The intrinsic dynamic rotation ψ_τ^t measures the full angle swept by the relative rotation tensor along the evolving negative relative vorticity vector $-(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})$. This scalar measure is objective, even though the relative rotation tensor $\boldsymbol{\Phi}_\tau^t$ generating this angle is only objective in two dimensions. The intrinsic dynamic rotation rate

$$\dot{\psi}_\tau^t(\mathbf{x}) = \tfrac{1}{2}\left|\boldsymbol{\omega}(\mathbf{x}, t) - \bar{\boldsymbol{\omega}}(t)\right| \tag{34}$$

is also objective both in two- and three dimensions (cf. formula (87) in the proof of Theorem 4). The intrinsic dynamic rotation rate is, therefore, a viable candidate for inclusion is rotation-rate-dependent constitutive laws. In another context, it has already been used to define and detect rotationally coherent Eulerian vortices objectively in two-dimensional fluid flows (Haller et al., 2015).

Remark 11. The angle ψ_τ^t is always positive: its integrand generates a positive angular increment, even if the orientation of relative rotation changes in time due to a zero crossing of the relative vorticity. For instance, in the two-dimensional experiments of Shapiro (1961), $\varphi_\tau^t(\mathbf{x}_\tau; \mathbf{e}_3)$ gives precisely the observed net rotation of a small circular body placed in the fluid. In contrast, $\psi_\tau^t(\mathbf{x}_\tau)$ would report the total angle swept by the circular body relative to the total mean rotation of the fluid. Both measures are objective, as stated in Theorem 4.

6. Dynamic rotation and stretch in two dimensions

For the material deformation induced by a two-dimensional velocity field $\mathbf{v} = (v_1, v_2)^T$, the spin tensor is of the form

$$\mathbf{W}(\mathbf{x},t) = \begin{pmatrix} 0 & -\frac{1}{2}\omega_3(\mathbf{x},t) \\ \frac{1}{2}\omega_3(\mathbf{x},t) & 0 \end{pmatrix},$$

where $\omega_3 = \partial_{x_1}v_2 - \partial_{x_2}v_1$ is the plane-normal component of the vorticity. The initial value problem (16) can then be solved by direct integration to yield

$$\mathbf{O}_\tau^t = \begin{pmatrix} \cos\left[\frac{1}{2}\int_\tau^t \omega_3(\mathbf{x}(s),s)\,ds\right] & -\sin\left[\frac{1}{2}\int_\tau^t \omega_3(\mathbf{x}(s),s)\,ds\right] \\ \sin\left[\frac{1}{2}\int_\tau^t \omega_3(\mathbf{x}(s),\tau)\,ds\right] & \cos\left[\frac{1}{2}\int_\tau^t \omega_3(\mathbf{x}(s),s)\,ds\right] \end{pmatrix}, \tag{35}$$

whereas the remaining two equations (17) and (18) take the form

$$\dot{\mathbf{M}}_\tau^t = \left[\mathbf{O}_t^\tau \mathbf{D}\big(\mathbf{x}(t),t\big)\mathbf{O}_\tau^t\right]\mathbf{M}_\tau^t, \quad \mathbf{M}_\tau^\tau = \mathbf{I},$$
$$\frac{d}{d\tau}\left(\mathbf{N}_\tau^t\right)^T = -\left[\mathbf{O}_\tau^t \mathbf{D}\big(\mathbf{x}(\tau),\tau\big)\mathbf{O}_t^\tau\right]\left(\mathbf{N}_\tau^t\right)^T, \quad \left(\mathbf{N}_t^t\right)^T = \mathbf{I}, \tag{36}$$

generally solvable only numerically.

If the deformation gradient $\mathbf{F}_\tau^t$, however, is explicitly known, then using the self-consistency property (21), we obtain the solutions of (36) directly as

$$\mathbf{M}_\tau^t = \mathbf{O}_t^\tau \mathbf{F}_\tau^t = \begin{pmatrix} \cos\left[\frac{1}{2}\int_t^\tau \omega_3(\mathbf{x}(s),s)\,ds\right] & -\sin\left[\frac{1}{2}\int_t^\tau \omega_3(\mathbf{x}(s),s)\,ds\right] \\ \sin\left[\frac{1}{2}\int_t^\tau \omega_3(\mathbf{x}(s),s)\,ds\right] & \cos\left[\frac{1}{2}\int_t^\tau \omega_3(\mathbf{x}(s),s)\,ds\right] \end{pmatrix}\mathbf{F}_\tau^t,$$
$$\mathbf{N}_\tau^t = \mathbf{F}_\tau^t \mathbf{O}_t^\tau = \mathbf{F}_\tau^t \begin{pmatrix} \cos\left[\frac{1}{2}\int_t^\tau \omega_3(\mathbf{x}(s),s)\,ds\right] & -\sin\left[\frac{1}{2}\int_t^\tau \omega_3(\mathbf{x}(s),s)\,ds\right] \\ \sin\left[\frac{1}{2}\int_t^\tau \omega_3(\mathbf{x}(s),s)\,ds\right] & \cos\left[\frac{1}{2}\int_t^\tau \omega_3(\mathbf{x}(s),s)\,ds\right] \end{pmatrix}. \tag{37}$$

In the present two-dimensional context, we select the rotation axis $\mathbf{g}$ to be the unit normal $\mathbf{e}_3$ to the (x_1, x_2) plane. With this choice, the unique dynamically consistent, finite rigid-body rotation of the deformation field can be computed from (31) as

$$\varphi_\tau^t(\mathbf{x}_\tau;\mathbf{e}_3) = -\frac{1}{2}\int_\tau^t \omega_3(\mathbf{x}(s),s)\,ds. \tag{38}$$

The two-dimensional, objective expression for the relative dynamic rotation defined in (32) is

$$\phi_\tau^t(\mathbf{x}_\tau;\mathbf{e}_3) = -\frac{1}{2}\int_\tau^t \left[\omega_3(\mathbf{x}(s),s) - \bar{\omega}_3(s)\right]ds, \tag{39}$$

while the intrinsic dynamic rotation in

$$\psi_\tau^t(\mathbf{x}_\tau) = \frac{1}{2}\int_\tau^t \left|\omega_3(\mathbf{x}(s),s) - \bar{\omega}_3(s)\right| ds,$$

Below we evaluate the two-dimensional DPD formulas (35)–(37) and the dynamic rotation angle on the two examples of Bertrand (1873), which he thought proved the inability of vorticity to characterize material rotation rates correctly (cf. Truesdell and Rajagopal, 2009).

Example 2. *Simple planar shear*: Consider the incompressible velocity field $\mathbf{v}(\mathbf{x}) = (a(x_2), 0)^T$ for some continuously differentiable scalar function $a(x_2)$. The corresponding planar shear deformation gradient is

$$\mathbf{F}_0^t(\mathbf{x}_0) = \begin{pmatrix} 1 & a'(x_{20})t \\ 0 & 1 \end{pmatrix}. \tag{40}$$

The classic polar decomposition is generally prohibitive to calculate in the presence of parameters, even for the simple deformation gradient (40). From the calculations of Dienes (1979), however, we obtain

$$\mathbf{R}_0^t = \begin{pmatrix} \cos\left(-\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) & -\sin\left(-\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) \\ \sin\left(-\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) & \cos\left(-\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) \end{pmatrix},$$

$$\mathbf{U}_0^t = \begin{pmatrix} \cos\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) & \sin\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) \\ \sin\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) & a'(x_{20})t\,\sin\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) + \cos\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) \end{pmatrix},$$

$$\mathbf{V}_0^t = \begin{pmatrix} \dfrac{1 + \sin^2\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right)}{\cos\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right)} & \sin\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) \\ \sin\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) & \cos\left(\tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right]\right) \end{pmatrix}, \tag{41}$$

showing that the polar rotation angle $\beta(t, \tau)$ satisfies

$$\tan \beta(t, \tau) = -\tfrac{1}{2}a'(x_{20})\left(t - \tau\right). \tag{42}$$

The dynamic inconsistency (3) of polar rotations is already transparent in this simple example. Indeed, noting that

$$\beta(t, s) = \tan^{-1}\left[\frac{1}{2}a'(x_{20})\left(t - s\right)\right] + \tan^{-1}\left[\frac{1}{2}a'(x_{20})s\right] \neq \tan^{-1}\left[\frac{1}{2}a'(x_{20})t\right], \quad t \neq 0, \tag{43}$$

we obtain from (41) and (43) that

$$\mathbf{R}_s^t\mathbf{R}_0^s = \begin{pmatrix} \cos\left(\beta(t, s)\right) & \sin\left(\beta(t, s)\right) \\ -\sin\left(\beta(t, s)\right) & \cos\left(\beta(t, s)\right) \end{pmatrix} \neq \mathbf{R}_0^t, \quad \forall\, t \neq 0. \tag{44}$$

To compute the dynamic polar decomposition from Theorem 2, we first note that

$$\omega_3(\mathbf{x}(t)) = -\partial_{x_2}v_1(x_2(t)) \equiv -a'(x_{20}),$$

and hence the entries of the rate-of-strain tensor $\mathbf{D}(\mathbf{x}(t))$ satisfy

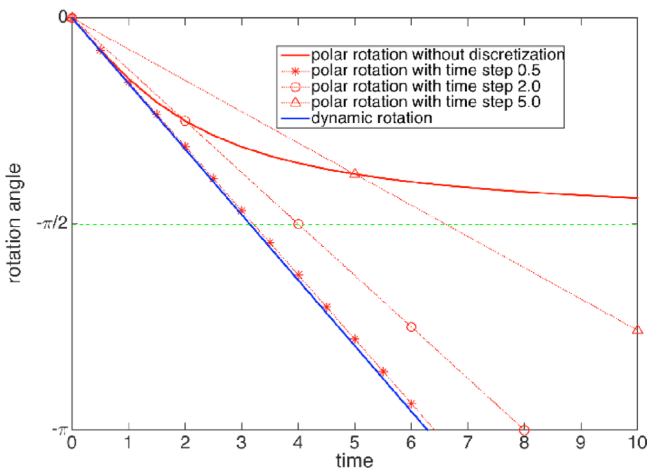

Fig. 5. The classic polar rotation angle (red) and the dynamic rotation angle (blue) as a function of time for the deformation gradient (40) describing planar, linear shear with $a(x_{20}) = x_{20}$. Also shown are the polar rotation angles computed from three different levels of discretization in time. At each time step, the polar rotation angle is incrementally recomputed, with the current time taken as the initial time in Eq. (42). The new rotational increment is then added to the rotation accumulated so far. With decreasing discretization step, the polar rotation angle computed in this incremental fashion necessarily converges to the dynamic rotation angle by formula (7). Both the polar and the dynamic rotation angles represent an overall assessment of the local rotation; individual material fibers all rotate by different angles. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

$$D_{11} = D_{22} = 0, \quad D_{12}(\mathbf{x}(t)) = D_{21}(\mathbf{x}(t)) \equiv \tfrac{1}{2}a'(x_{20}).$$

Therefore, formulas (35)–(37) give the dynamic polar decomposition factors

$$\mathbf{O}_0^t = \begin{pmatrix} \cos\left[-\frac{1}{2}a'(x_{20})t\right] & -\sin\left[-\frac{1}{2}a'(x_{20})t\right] \\ \sin\left[-\frac{1}{2}a'(x_{20})t\right] & \cos\left[-\frac{1}{2}a'(x_{20})t\right] \end{pmatrix},$$

$$\mathbf{M}_0^t = \begin{pmatrix} \cos\left[\frac{1}{2}a'(x_{20})t\right] & a'(x_{20})t\cos\left[\frac{1}{2}a'(x_{20})t\right] - \sin\left[\frac{1}{2}a'(x_{20})t\right] \\ \sin\left[\frac{1}{2}a'(x_{20})t\right] & a'(x_{20})t\sin\left[\frac{1}{2}a'(x_{20})t\right] + \cos\left[\frac{1}{2}a'(x_{20})t\right] \end{pmatrix},$$

$$\mathbf{N}_0^t = \begin{pmatrix} a'(x_{20})t\sin\left[\frac{1}{2}a'(x_{20})t\right] + \cos\left[\frac{1}{2}a'(x_{20})t\right] & a'(x_2)t\cos\left[\frac{1}{2}a'(x_{20})t\right] - \sin\left[\frac{1}{2}a'(x_{20})t\right] \\ \sin\left[\frac{1}{2}a'(x_{20})t\right] & \cos\left[\frac{1}{2}a'(x_{20})t\right] \end{pmatrix}.$$

Finally, by formula (38), the dynamic rotation is simply

$$\varphi_0^t(\mathbf{x}_0;\, \mathbf{e}_3) = -\tfrac{1}{2}a'(x_{20})t,$$

which we plot in Fig. 5 for comparison with the rotation angle generated by the rotation tensor $\mathbf{R}_0^t$ as a function of time. We also show in the figure the consequence of the lack of additivity for the polar rotation, as verified in (44). Indeed, computing the polar rotation angle as a superposition of finite sub-rotations, even from its analytic formula and hence without numerical error, will give differing results.

By formula (39), the relative dynamic rotation is

$$\phi_0^t(\mathbf{x}_0;\, \mathbf{e}_3) = -\tfrac{1}{2}\left[a'(x_{20}) - \overline{a'(x_{20})}\right]t,$$

with the overbar denoting spatial average over the domain of interest. Finally, the intrinsic dynamic rotation is

$$\psi_0^t(\mathbf{x}_0) = \tfrac{1}{2}\left|a'(x_{20}) - \overline{a'(x_{20})}\right|t$$

We conclude from Fig. 5 that generic material elements rotate at the well-defined mean rate $\dot{\varphi}_0^t(\mathbf{x}_0;\, \mathbf{e}_3) = -\frac{1}{2}a'(x_{20})$. This is at odds with the polar mean rotation rate which tends to zero over time.

At first sight, it is the decaying polar rotation rate that agrees with one's physical intuition. Indeed, as Flanagan and Taylor (1987) write about this example: "Clearly the body experiences rotations which diminish over time,…". By the end of any given finite deformation interval, the rotation of infinitely many material fibers indeed slows down. At the same time, however, the rotation of infinitely many other material fibers is accelerating. For instance, at any given time t, material fibers in vertical position are just reaching their maximal material rotation rate $-\frac{3}{2}a'(x_{20})$ (cf. formula (82)). Overall material fiber rotation, therefore, does not die out.

We show a more detailed sketch of the behavior of material fibers in Fig. 6. The frame is fixed to the trajectory in the middle, which then becomes a set of fixed points. At any given time, different material fibers rotate at different speeds; the lengths of the arcs illustrate the magnitudes of angular velocities for the corresponding material fibers. Only horizontal material fibers have zero angular velocity. The average material angular velocity is equal to $-\frac{1}{2}a'(x_{20})t$ by Cauchy's classic result (Cauchy, 1841) or by the restriction of our Proposition 1 to two dimensions. An infinitesimal, rigid circular tracer (shaded area) placed in the deformation field rotates precisely at this angular velocity. Most of this was already pointed out by Helmholtz (1868) in his response to Bertrand (1873), but his observations have apparently not been interpreted in the

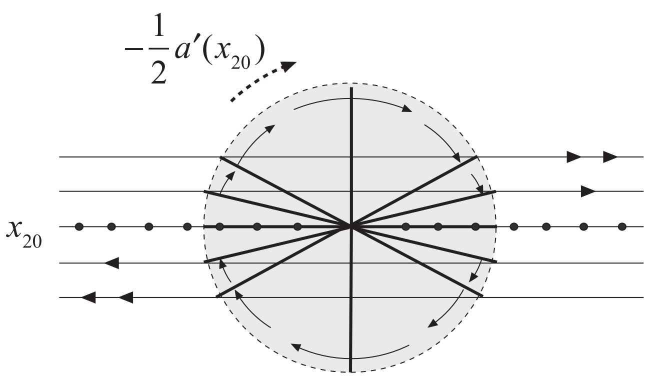

Fig. 6. Rotation of material lines in a parallel shear field.

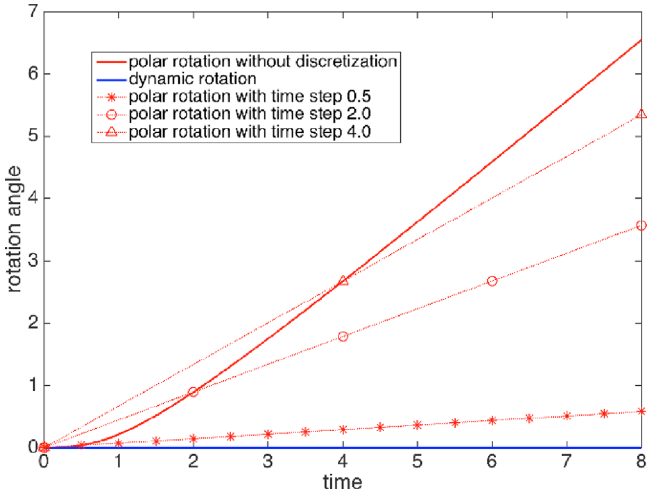

Fig. 7. The classic polar rotation angle (red) and the dynamic rotation angle (blue) as a function of time for the deformation gradient (45) of an irrotational vortex with $\alpha = 1$. Also shown are the exact polar rotation angles computed incrementally for three different levels of discretization in time, as in Fig. 5. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

context of polar rotations.

Example 3. *Irrotational vortex*: Consider the two-dimensional, circularly symmetric, incompressible velocity field $\mathbf{v}(\mathbf{x}) = \left(\frac{-x_2\alpha}{x_1^2+x_2^2}, \frac{\alpha x_1}{x_1^2+x_2^2}\right)^T$, where $\alpha \in \mathbb{R}$ is a parameter. By direct calculation, we obtain the vorticity and displacement fields

$$\omega_3 \equiv 0, \quad \mathbf{X}_0^t(x_0) = \begin{pmatrix} \cos\frac{\alpha t}{|\mathbf{x}_0|^2} & -\sin\frac{\alpha t}{|\mathbf{x}_0|^2} \\ \sin\frac{\alpha t}{|\mathbf{x}_0|^2} & \cos\frac{\alpha t}{|\mathbf{x}_0|^2} \end{pmatrix}\mathbf{x}_0,$$

as well as the deformation gradient

$$\mathbf{F}_0^t(x_0) = \begin{pmatrix} \cos\frac{\alpha t}{|\mathbf{x}_0|^2} + \frac{2x_{10}\alpha t\left(x_{10}\sin\frac{\alpha t}{|\mathbf{x}_0|^2} + x_{20}\cos\frac{\alpha t}{|\mathbf{x}_0|^2}\right)}{|\mathbf{x}_0|^4} & -\sin\frac{\alpha t}{|\mathbf{x}_0|^2} + \frac{2x_{20}\alpha t\left(x_{10}\sin\frac{\alpha t}{|\mathbf{x}_0|^2} + x_{20}\cos\frac{\alpha t}{|\mathbf{x}_0|^2}\right)}{|\mathbf{x}_0|^4} \\ \sin\frac{\alpha t}{|\mathbf{x}_0|^2} - \frac{2x_{10}\alpha t\left(x_{10}\cos\frac{\alpha t}{|\mathbf{x}_0|^2} - x_{20}\sin\frac{\alpha t}{|\mathbf{x}_0|^2}\right)}{|\mathbf{x}_0|^4} & \cos\frac{\alpha t}{|\mathbf{x}_0|^2} - \frac{2x_{20}\alpha t\left(x_{10}\cos\frac{\alpha t}{|\mathbf{x}_0|^2} - x_{20}\sin\frac{\alpha t}{|\mathbf{x}_0|^2}\right)}{|\mathbf{x}_0|^4} \end{pmatrix}. \tag{45}$$

We also obtain from (35) to (37) the dynamic polar decomposition factors

$$\mathbf{O}_0^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{M}_0^t = \mathbf{N}_0^t = \mathbf{F}_0^t(\mathbf{x}_0).$$

By formulas (38) and (39), the dynamic rotation, the relative dynamic rotation, and the intrinsic dynamic rotation all vanish

$$\varphi_0^t(\mathbf{x}_0;\, \mathbf{e}_3) = \phi_0^t(\mathbf{x}_0;\, \mathbf{e}_3) = \psi_0^t(\mathbf{x}_0) \equiv 0.$$

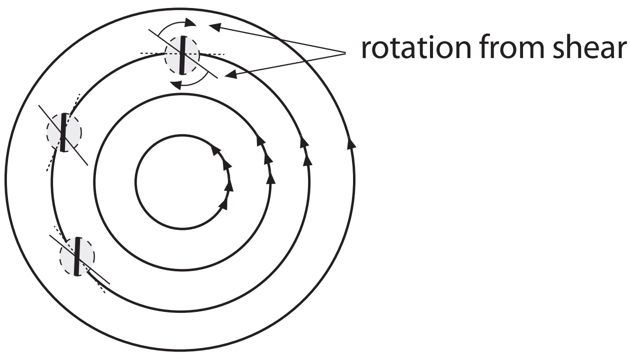

Fig. 8. Rotation of material line elements around an irrotational vortex.

We show this together with the numerically computed polar rotation angle in Fig. 7.

The vanishing dynamic rotation angle is consistent with the lack of rotation exhibited by circular tracers in irrotational vortex experiments (Shapiro, 1961). Fig. 8 illustrates the translation of such a tracer (shaded area). While exceptional material fibers tangent to trajectories rotate with the angular velocity of the trajectory, other fibers rotate in the opposite direction due to shear. The average material angular velocity is equal to zero by Cauchy's classic result, as well as by the restriction of our Proposition 1 to two dimensions. Again, these observations were already made by Helmholtz (1868) to Bertrand (1873), but have apparently not been evaluated relative to the rotation predicted by the polar decomposition (see, e.g., Dienes (1986), who mentions this example).

7. Dynamic rotation and stretch in three dimensions

For material deformation fields induced by three-dimensional velocity fields $\mathbf{v} = (v_1, v_2, v_3)^T$, the spin tensor can be written as

$$\mathbf{W}(\mathbf{x}, t) = \begin{pmatrix} 0 & -\frac{1}{2}\omega_3(\mathbf{x}, t) & \frac{1}{2}\omega_2(\mathbf{x}, t) \\ \frac{1}{2}\omega_3(\mathbf{x}, t) & 0 & -\frac{1}{2}\omega_1(\mathbf{x}, t) \\ -\frac{1}{2}\omega_2(\mathbf{x}, t) & \frac{1}{2}\omega_1(\mathbf{x}, t) & 0 \end{pmatrix},$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = \nabla \times \mathbf{v}$. The three-dimensional rotational process $\mathbf{O}_\tau^t$ is the normalized fundamental matrix solution of the non-autonomous, three-dimensional linear system of differential equations (16). At this level of generality, (16) must be solved numerically.

As in the two-dimensional case, if both the rotational process $\mathbf{O}_\tau^t$ and the deformation gradient $\mathbf{F}_\tau^t$ are known, then the remaining factors in the left and right DPD can be computed as

$$\mathbf{M}_\tau^t = \mathbf{O}_t^\tau \mathbf{F}_\tau^t, \quad \mathbf{N}_\tau^t = \mathbf{F}_\tau^t \mathbf{O}_t^\tau. \tag{46}$$

Finally, the dynamic rotation $\varphi_\tau^t(\mathbf{x}_\tau; \mathbf{g})$, its relative part $\phi_\tau^t(\mathbf{x}_\tau; \mathbf{g})$ and the intrinsic rotation $\psi_\tau^t(\mathbf{x}_\tau; \mathbf{g})$ obey the formulas (31)–(33) without simplification.

Example 4. *Three-dimensional, unsteady, parallel shear*: For a smooth, unsteady parallel shear field in three dimensions, the velocity field is in the general form

$$\mathbf{v}(\mathbf{x}, t) = \begin{pmatrix} v_1(x_3, t) \\ v_2(x_3, t) \\ v_3(t) \end{pmatrix},$$

where the velocity components are smooth functions of their arguments. The spin tensor and the deformation gradient can be obtained by direct calculation as

$$\mathbf{W}(\mathbf{x}(t), t) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\partial_{x_3} v_1(x_3(t), t) \\ 0 & 0 & \frac{1}{2}\partial_{x_3} v_2(x_3(t), t) \\ -\frac{1}{2}\partial_{x_3} v_1(x_3(t), t) & -\frac{1}{2}\partial_{x_3} v_2(x_3(t), t) & 0 \end{pmatrix},$$

$$\mathbf{F}_\tau^t(\mathbf{x}_0) = \begin{pmatrix} 1 & 0 & \int_\tau^t \partial_{x_3} v_1\Big(x_3(s), s\Big)\, ds \\ 0 & 1 & \int_\tau^t \partial_{x_3} v_2\Big(x_3(s), s\Big)\, ds \\ 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

The dynamic rotation tensor $\mathbf{O}_\tau^t$, therefore, satisfies the non-autonomous system of differential equations

$$\frac{d}{dt}\mathbf{O}_\tau^t = \begin{pmatrix} 0 & 0 & \frac{1}{2}\partial_{x_3} v_1(x_3(t), t) \\ 0 & 0 & \frac{1}{2}\partial_{x_3} v_2(x_3(t), t) \\ -\frac{1}{2}\partial_{x_3} v_1(x_3(t), t) & -\frac{1}{2}\partial_{x_3} v_2(x_3(t), t) & 0 \end{pmatrix}\mathbf{O}_\tau^t. \tag{48}$$

Without further assumptions, this non-autonomous system can only be solved numerically, or via an asymptotic Magnus-expansion (Magnus, 1954). For simplicity, we assume from now that $v_2(x_3, t) \equiv cv_1(x_3, t)$ for some constant $c \in \mathbb{R}$. In that case, the coefficient matrix of (48) commutes with its own integral, and hence the fundamental matrix solution of (48) is just the exponential of the integral of its coefficient matrix (Epstein, 1963). Indeed, we then have

$$\mathbf{O}_\tau^t = e^{(1/2)\int_\tau^t \partial_{x_3} v_1(x_3(\tau),\tau)\,d\tau} \exp\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & c \\ -1 & -c & 0 \end{pmatrix} = \frac{\exp\left[\frac{1}{2}\int_\tau^t \partial_{x_3} v_1\left(x_{30} + \int_\tau^\tau v_3(\sigma)\,d\sigma, s\right) ds\right]}{c^2+1}$$
$$\times \begin{pmatrix} \cos\left(c^2+1\right)+c^2 & c\left[\cos\left(c^2+1\right)-1\right] & \sqrt{c^2+1}\left(\sin\left(c^2+1\right)+c^2\right) \\ c\left[\cos\left(c^2+1\right)-1\right] & c^2\cos\left(c^2+1\right)+1 & c\sqrt{c^2+1}\,\sin\left(c^2+1\right) \\ -\sqrt{c^2+1}\left(\sin\left(c^2+1\right)+c^2\right) & -c\sqrt{c^2+1}\,\sin\left(c^2+1\right) & \cos\left(c^2+1\right) \end{pmatrix}. \tag{49}$$

Then, from formulas (46), (47) and (49) we obtain the left and right DPD factors $\mathbf{M}_\tau^t$ and $\mathbf{N}_\tau^t$ explicitly, which we omit here for brevity. With the vorticity vector

$$\boldsymbol{\omega} = \partial_{x_3} v_1 \begin{pmatrix} 1 \\ -c \\ 0 \end{pmatrix},$$

and with respect to a constant rotation axis defined by a unit vector $\mathbf{g} = (g_1, g_2, g_3)^T$, the frame-dependent dynamic rotation angle is of the form

$$\varphi_\tau^t(\mathbf{x}_\tau; \mathbf{g}) = \frac{1}{2}(g_1 - cg_2)\int_\tau^t \partial_{x_3} v_1\left(x_{30} + \int_\tau^s v_3(\sigma)\,d\sigma, s\right) ds.$$

In contrast, the (objective) relative dynamic rotation is given by

$$\phi_\tau^t(\mathbf{x}_\tau; \mathbf{g}) = \frac{1}{2}(g_1 - cg_2)\int_\tau^t \left[\partial_{x_3} v_1\left(x_{30} + \int_\tau^s v_3(\sigma)\,d\sigma, s\right) - \overline{\partial_{x_3} v_1}(s)\right] ds,$$

and the (objective) intrinsic dynamic rotation is given by

$$\psi_\tau^t(\mathbf{x}_\tau) = \frac{1}{2}\sqrt{c^2+1}\int_\tau^t \left|\partial_{x_3} v_1\left(x_{30} + \int_\tau^s v_3(\sigma)\,d\sigma, s\right) - \overline{\partial_{x_3} v_1}(s)\right| ds.$$

8. Conclusions

The classic polar decomposition of the deformation gradient is a broadly employed tool in analyzing continuum deformation. Given the deformation gradient, one obtains the polar rotation and stretch tensors from algorithms based on straightforward linear algebra. Beyond computational simplicity, polar rotation offers a powerful and rigorous tool to identify a static rotational component of the linearized deformation between fixed initial and final configurations.

Polar rotations computed over different time intervals, however, do not have the fundamental additivity property of solid-body rotations. As a consequence, polar rotation does not identify a mean material rotation for volume elements which is nevertheless experimentally observable in fluids (Shapiro, 1961). Polar rotation also suggests a mean angular velocity distribution that depends on the length of the observation period, introducing an irremovable memory effect into the deformation history on purely kinematic grounds (cf. Appendix A). Finally, the evolution of the polar stretch tensor is not free from spin. In summary, the static optimality of the polar decomposition between two fixed configurations also comes with dynamic sub-optimality for time-varying configurations.

To address these disadvantages, here we have extended the idea of polar decomposition from a single linear mapping between two fixed configurations to a time-dependent process. The resulting dynamic polar decomposition (DPD) yields unique left and right factorizations of $\mathbf{F}_\tau^t$ into the deformation gradient of a purely rotating (strainless) deformation and the deformation gradient of a purely straining (irrotational) deformation. The former deformation gradient, the dynamic rotation tensor, is a dynamically consistent rotation family. The latter deformation gradient, the (left) dynamic stretch tensor, is objective, just as its classic polar left stretch counterpart. The dynamic stretch tensors also reproduce the same Cauchy–Green strains and principal strain directions between any two configurations, as the classic polar stretch tensors do. Unlike the right polar stretch tensor, however, the right dynamic stretch tensor is spin-free.

The DPD provides a previously missing mathematical link between the deformation gradient and numerical algorithms that rotate the reference frame incrementally at the spin rate (Hughes and Winget, 1980; Rubinstein and Atluri, 1983). The dynamic rotation tensor arising from the DPD reproduces precisely the mean material rotation rate of volume elements, as defined by Cauchy (1841). This mean rotation rate is directly observable in two-dimensional fluids by placing a small spherical tracer in the flow (Shapiro, 1961). The same experiment cannot be carried out for solids. Any possible experiment in solids, however, with an ability to measure the average rotation rate of all fibers in a material volume element, necessarily has to return the rate obtained from the DPD (cf. Proposition 1).

The DPD also provides new dynamic rotation angles for volume elements. These angles represent dynamically consistent and simply computable alternatives to Cauchy's classic mean rotation angle, whose evaluation has been difficult using the classic polar decomposition (cf. Section 5.2). The dynamic rotation angles also enable the extension of polar-rotation-based material vortex detection in two-dimensional deformations (Farazmand and Haller, 2015) to DPD-based material vortex detection in three-dimensions (Haller et al., 2015).

On the computational side, the DPD cannot be obtained from simple linear algebraic manipulations on the single linear mapping $\mathbf{F}_\tau^t$, as is the case for the classic polar decomposition. Instead, one has to solve non-autonomous linear differential equations over the time interval $[\tau, t]$ to obtain the DPD of $\mathbf{F}_\tau^t$. On the upside, this also means that the dynamic rotation-stretch tensor pair $(\mathbf{O}_\tau^t, \mathbf{M}_\tau^t)$ together satisfies an explicit system of differential equations, i.e., form a dynamical system that is free from memory effects. This is not the case for their classic polar counterparts: $(\mathbf{R}_\tau^t, \mathbf{U}_\tau^t)$ satisfy an implicit, nonlinear system of differential equations, which does not define a dynamical system and has unavoidable memory effects (cf. Appendix A).

We believe that memory effects should enter models of the deformation process in a controlled fashion, through parameters in the constitutive equations, rather than in an uncontrolled and un-parametrized fashion, through the rotational kinematics. For this reason, we consider the intrinsic dynamic rotation rate $\dot{\psi}_\tau^t$, defined in (34), a viable candidate for inclusion in constitutive laws, given that it is simple, objective and memory-free. For two-dimensional deformations, the rotation rate $\dot{\boldsymbol{\Phi}}_\tau^t \boldsymbol{\Phi}_\tau^t = \mathbf{W} - \bar{\mathbf{W}}$ of the relative rotation tensor can also be used, as it is objective by Eq. (72).

Finally, we expect the DPD to be useful in experimental techniques producing time-resolved deformation with large strains. An example is the Digital Image Correlation (DIC) applied to granular materials, where the classic polar rotation tensor has been used so far to identify macroscopic rigid-body rotation components of the deformation field (see, e.g., Rechenmacher et al., 2011).

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Appendix A. Polar rotations do not form a dynamical system

We start by recalling the well-known temporal evolution of the deformation gradient. Let us fix a material trajectory $\mathbf{x}(t)$, with $\mathbf{x}(t_0) = \mathbf{x}_0$. The deformation gradient along this trajectory obeys the differential equation (cf. Example 1)

$$\dot{\mathbf{F}}_\tau^t = \nabla\mathbf{v}(\mathbf{x}(t), t)\mathbf{F}_\tau^t, \tag{50}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field associated with the deformation. The time $\tau \in [t_0, t_1]$ is arbitrary, labeling a reference configuration from which an observer follows the deformation gradient up to time $t \in [t_0, t_1]$. The solution $\mathbf{F}_\tau^t$ of the differential equation (50), therefore, depends *implicitly* on the start time τ of the observation, without τ entering the differential equation explicitly.

A deformation rate tensor (analogous to the polar rotation rate) can also be defined for the deformation gradient as

$$\dot{\mathbf{F}}_\tau^t\left(\mathbf{F}_\tau^t\right)^{-1} = \nabla\mathbf{v}(\mathbf{x}(t), t). \tag{51}$$

This gives a well-defined deformation rate at the point $\mathbf{x}(t)$ of the deformed configuration at time t, independent of the initial time τ at which the observer started monitoring the linearized deformation along the trajectory $\mathbf{x}(t)$. One may, in particular, select the start time of the observation as $t = \tau$ and obtain the same rate $\dot{\mathbf{F}}_t^t(\mathbf{F}_t^t)^{-1} = \nabla\mathbf{v}(\mathbf{x}(t), t)$.

We now show that this is not the case for the polar rotation rate. The differential equation for the polar rotation tensor along the trajectory $\mathbf{x}(t)$ is of the form

$$\dot{\mathbf{R}}_\tau^t = \left[\mathbf{W}(\mathbf{x}(t), t) - \frac{1}{2}\mathbf{R}_\tau^t\left[\dot{\mathbf{U}}_\tau^t\left(\mathbf{U}_\tau^t\right)^{-1} - \left(\mathbf{U}_\tau^t\right)^{-1}\dot{\mathbf{U}}_\tau^t\right]\left(\mathbf{R}_\tau^t\right)^T\right]\mathbf{R}_\tau^t. \tag{52}$$

This gives the instantaneous rotation rate at the point $\mathbf{x}(t)$, at time t, in the form

$$\dot{\mathbf{R}}_\tau^t\left(\mathbf{R}_\tau^t\right)^T = \mathbf{W}(\mathbf{x}(t), t) - \frac{1}{2}\mathbf{R}_\tau^t\left[\dot{\mathbf{U}}_\tau^t\left(\mathbf{U}_\tau^t\right)^{-1} - \left(\mathbf{U}_\tau^t\right)^{-1}\dot{\mathbf{U}}_\tau^t\right]\left(\mathbf{R}_\tau^t\right)^T \tag{53}$$

for the observer monitoring the infinitesimal deformation along $\mathbf{x}(t)$ from the initial time τ up to the present time t. Note that this rate depends *explicitly* on the initial time τ of observation through $\mathbf{U}_\tau^t$. In particular, for an observation starting at time $t=\tau$, we obtain $\dot{\mathbf{R}}_t^t(\mathbf{R}_t^t)^{-1}=\mathbf{W}(\mathbf{x}(t),t)$, which is quite different from (53) with $\tau\neq t$. Therefore, the instantaneous polar rotation rate at a given location and time is ill-defined when different start times for the observation are allowed.

There is, in fact, a deeper effect at play here. Rather than examining the rates $\dot{\mathbf{F}}_\tau^t(\mathbf{F}_\tau^t)^{-1}$ and $\dot{\mathbf{R}}_\tau^t(\mathbf{R}_\tau^t)^T$, let us simply examine if the derivatives $\dot{\mathbf{F}}_\tau^t$ and $\dot{\mathbf{R}}_\tau^t$ are independent of the observational history. Note that the derivative of $\mathbf{F}_\tau^t$ in (50) only depends on the current time t and $\mathbf{F}_\tau^t$ itself. Thus, in the language of differential equations, (50) is a non-autonomous dynamical system (or a *process*; cf. Dafermos, 1971) for the deformation gradient $\mathbf{F}_\tau^t$, with its future evolution fully determined by its present state. The defining properties of a process, spelled out for the tensor family $\mathbf{F}_\tau^t$, are

$$\mathbf{F}_\tau^t=\mathbf{F}_s^t\bigcirc\mathbf{F}_\tau^s,\quad \mathbf{F}_t^t=\mathbf{I},\quad \forall\, s,\tau,t\in[t_0,t_1],$$

with the circle denoting the composition of two functions. By the linearity of (50), $\mathbf{F}_\tau^t$ is actually a linear process, and hence we simply have $\mathbf{F}_s^t\bigcirc\mathbf{F}_\tau^s=\mathbf{F}_s^t\mathbf{F}_\tau^s$. The linearity of the dynamical system (50), however, plays no role in our current argument.[1]

In contrast, the derivative of $\mathbf{R}_\tau^t$ in (52) depends on the current time t, on the tensor $\mathbf{R}_\tau^t$ itself, as well as on the initial time τ of the observation through the quantity $\mathbf{U}_\tau^t$. As a consequence, the nonlinear differential equation (52) is *not* a dynamical system (or process), because its future evolution is not determined fully by its present state, and hence

$$\mathbf{R}_\tau^t\neq\mathbf{R}_s^t\bigcirc\mathbf{R}_\tau^s \tag{54}$$

holds. Thus, in addition to not being a linear process by (3), the polar rotation tensor also fails to be a nonlinear process by property (54). Instead, $\mathbf{R}_\tau^t$ satisfies a nonlinear differential equation with *memory*.

Even when considered together, the $(\mathbf{R}_\tau^t,\mathbf{U}_\tau^t)$ tensor pair does not satisfy an explicit system of differential equations. Rather, the pair satisfies a nonlinear implicit system of differential equations formed by (7)–(22) (albeit this system has no explicit dependence on τ). As a consequence, the pair $(\mathbf{R}_\tau^t,\mathbf{U}_\tau^t)$ generally does not form a nonlinear dynamical system (or nonlinear process) either, and hence displays explicit memory effects beyond the customary implicit dependence on the reference configuration.

Appendix B. Proof of Theorem 1

Substituting the decomposition in (11) into (12), and imposing the requirement that $\mathbf{O}_\tau^t$ is rotational and $\mathbf{M}_\tau^t$ is irrotational (cf. Definition 2), we obtain that

$$\begin{aligned}
\mathbf{A}^-(t)&=\dot{\mathbf{O}}_\tau^t\left[\mathbf{O}_\tau^t\right]^T+\frac{1}{2}\mathbf{O}_\tau^t\left[\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}-\left[\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}\right]^T\right]\left[\mathbf{O}_\tau^t\right]^T=\dot{\mathbf{O}}_\tau^t\left[\mathbf{O}_\tau^t\right]^T,\\
\mathbf{A}^+(t)&=\frac{1}{2}\left[\dot{\mathbf{O}}_\tau^t\left[\mathbf{O}_\tau^t\right]^T+\mathbf{O}_\tau^t\left[\dot{\mathbf{O}}_\tau^t\right]^T\right]+\frac{1}{2}\mathbf{O}_\tau^t\left[\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}+\left[\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}\right]^T\right]\left[\mathbf{O}_\tau^t\right]^T\\
&=\frac{1}{2}\frac{d}{dt}\left[\mathbf{O}_\tau^t\left[\mathbf{O}_\tau^t\right]^T\right]+\frac{1}{2}\mathbf{O}_\tau^t\left[\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}+\left[\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}\right]^T\right]\left[\mathbf{O}_\tau^t\right]^T=\frac{1}{2}\mathbf{O}_\tau^t\left[\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}+\left[\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}\right]^T\right]\left[\mathbf{O}_\tau^t\right]^T\\
&=\mathbf{O}_\tau^t\dot{\mathbf{M}}_\tau^t\left[\mathbf{M}_\tau^t\right]^{-1}\left[\mathbf{O}_\tau^t\right]^T.
\end{aligned}\tag{55}$$

Expressing the derivatives of $\mathbf{O}_\tau^t$ and $\mathbf{M}_\tau^t$ from (55) proves the first two equations in (69). We also note that

$$\left[\mathbf{T}_\tau^t\right]^T\mathbf{T}_\tau^t=\left[\mathbf{M}_\tau^t\right]^T\left[\mathbf{O}_\tau^t\right]^T\mathbf{O}_\tau^t\mathbf{M}_\tau^t=\left[\mathbf{M}_\tau^t\right]^T\mathbf{M}_\tau^t,$$

and hence $\mathbf{T}_\tau^t$ and $\mathbf{M}_\tau^t$ have the same singular values, as claimed.

Using the notation from Eq. (9) in Eq. (8), we can further write

$$\dot{\mathbf{T}}_\tau^t\mathbf{T}_t^\tau=\mathbf{A}(t)=\mathbf{A}^-(t)+\mathbf{A}^+(t),$$

with $\mathbf{A}^\pm(t)=\frac{1}{2}[\mathbf{A}(t)\pm\mathbf{A}^T(t)]$. Therefore, $\mathbf{A}^\pm(t)$ are indeed independent of τ and hence $\mathbf{A}^\pm(\tau)$ are independent of t, as already suggested by our notation.

From the now proven first equation of (69), we conclude that $\mathbf{O}_\tau^t$ is indeed a linear process, as the fundamental matrix solution of a classic non-autonomous system of linear ODEs (with no explicit dependence on the initial time τ). We also conclude that $\mathbf{M}_\tau^t$ is a two-parameter family of nonsingular operators, even though it is generally not a process. In particular, $\mathbf{M}_\tau^t$ does not form a process because the coefficient matrix of the second system of ODEs in (69) has explicit dependence on the initial time τ. As a consequence, we generally have

$$\left(\mathbf{M}_t^\tau\right)^{-1}\neq\mathbf{M}_\tau^t.$$

[1] An example of a nonlinear process is a general deformation field, satisfying the nonlinear differential equation $\dot{\mathbf{X}}_\tau^t=\mathbf{v}(\mathbf{X}_\tau^t,t)$. In this case, the nonlinear process properties take the form $\mathbf{X}_\tau^t=\mathbf{X}_s^t\bigcirc\mathbf{X}_\tau^s$ and $\mathbf{X}_t^t=\mathbf{I}$, for all $s,\tau,t\in[t_0,t_1]$. Here the function composition cannot be replaced by a simple product.

To prove the left-polar decomposition involving $\mathbf{N}_\tau^t$ in (11), we observe that

$$\mathbf{T}_\tau^t = \left[\mathbf{T}_t^\tau\right]^{-1} = \left[\mathbf{O}_t^\tau \mathbf{M}_t^\tau\right]^{-1} = \left(\mathbf{M}_t^\tau\right)^{-1}\left(\mathbf{O}_t^\tau\right)^{-1} = \left(\mathbf{M}_t^\tau\right)^{-1}\mathbf{O}_\tau^t,$$

thus setting

$$\mathbf{N}_\tau^t = \left(\mathbf{M}_t^\tau\right)^{-1}, \tag{56}$$

we conclude the existence of $\mathbf{N}_\tau^t$, as claimed. Interchanging the role of τ and t in the second equation of (69), we obtain the differential equation

$$\frac{d}{d\tau}\mathbf{M}_t^\tau = \left[\mathbf{O}_\tau^t \mathbf{A}^+(\tau)\mathbf{O}_t^\tau\right]\mathbf{M}_t^\tau, \quad \mathbf{M}_t^t = \mathbf{I}. \tag{57}$$

By formula (56), we have

$$\left(\frac{d}{d\tau}\mathbf{N}_\tau^t\right)\mathbf{M}_t^\tau + \mathbf{N}_\tau^t\left(\frac{d}{d\tau}\mathbf{M}_t^\tau\right) = 0,$$

which together with (57) yields

$$\frac{d}{d\tau}\mathbf{N}_\tau^t = -\mathbf{N}_\tau^t\left[\mathbf{O}_\tau^t \mathbf{A}^+(\tau)\mathbf{O}_t^\tau\right], \quad \mathbf{N}_t^t = \mathbf{I}. \tag{58}$$

Taking the transpose of the expressions involved in the initial value problem (58) proves the last equation in (69). Finally, the uniqueness of both decompositions in (11) follows from the uniqueness of the solutions of the initial value problems in (69).

Appendix C. Proof of Theorem 2

Statements (i)–(iii) follow by a direct application of Theorem 1 to the process $\mathbf{T}_\tau^t = \mathbf{F}_\tau^t$. To prove statement (iv), we apply the time-dependent coordinate change (14) to the expression $\mathbf{x}(t) = \mathbf{X}_\tau^t(\mathbf{x}_\tau)$ and obtain

$$\mathbf{y}(t) = \mathbf{Q}^T(t)\left[\mathbf{X}_\tau^t(\mathbf{Q}(\tau)\mathbf{y}_\tau + \mathbf{b}(\tau)) - \mathbf{b}(t)\right]. \tag{59}$$

Differentiation of this equation with respect to $\mathbf{y}_\tau$ yields the transformed deformation gradient $\tilde{\mathbf{F}}_\tau^t = \partial_{\mathbf{y}_\tau}\mathbf{y}(t)$ in the form

$$\tilde{\mathbf{F}}_\tau^t = \mathbf{Q}^T(t)\mathbf{F}_\tau^t\mathbf{Q}(\tau), \tag{60}$$

showing that the deformation gradient tensor is not objective (cf. Liu, 2004). Differentiating (60) with respect to time, and first subtracting then adding the transpose of the resulting equation, yields the transformed spin and rate-of-strain tensors

$$\tilde{\mathbf{W}}(\mathbf{y}, t) = \mathbf{Q}^T(t)\mathbf{W}(\mathbf{x}, t)\mathbf{Q}(t) - \mathbf{Q}^T(t)\dot{\mathbf{Q}}(t), \quad \tilde{\mathbf{D}}\left(\mathbf{y}, t\right) = \mathbf{Q}^T(t)\mathbf{D}(\mathbf{x}, t)\mathbf{Q}(t), \tag{61}$$

respectively, indicating that $\mathbf{W}$ is not objective but $\mathbf{D}$ is objective.

Using the decomposition $\mathbf{F}_\tau^t$ obtained from statement (i) in the original $\mathbf{x}$-frame, we factorize the transformed deformation gradient (60) as

$$\tilde{\mathbf{F}}_\tau^t = \mathbf{Q}^T(t)\mathbf{O}_\tau^t\mathbf{M}_\tau^t\mathbf{Q}(\tau) = \tilde{\mathbf{O}}_\tau^t\tilde{\mathbf{M}}_\tau^t, \quad \tilde{\mathbf{O}}_\tau^t = \mathbf{Q}^T(t)\mathbf{O}_\tau^t\mathbf{Q}(\tau), \quad \tilde{\mathbf{M}}_\tau^t = \mathbf{Q}^T(\tau)\mathbf{M}_\tau^t\mathbf{Q}(\tau). \tag{62}$$

We want to show that this factorization is in fact the unique DPD of the transformed deformation gradient $\tilde{\mathbf{F}}_\tau^t$.

To this end, note that

$$\begin{aligned}\dot{\tilde{\mathbf{O}}}_\tau^t &= \mathbf{Q}^T(t)\dot{\mathbf{O}}_\tau^t\mathbf{Q}(\tau) + \dot{\mathbf{Q}}^T(t)\mathbf{O}_\tau^t\mathbf{Q}(\tau) = \mathbf{Q}^T(t)\mathbf{W}(\mathbf{x}(t), t)\mathbf{O}_\tau^t\mathbf{Q}(\tau) + \dot{\mathbf{Q}}^T(t)\mathbf{O}_\tau^t\mathbf{Q}(\tau) = \tilde{\mathbf{W}}(\mathbf{y}(t), t)\tilde{\mathbf{O}}_\tau^t,\\ \dot{\tilde{\mathbf{M}}}_\tau^t &= \mathbf{Q}^T(\tau)\dot{\mathbf{M}}_\tau^t\mathbf{Q}(\tau) = \mathbf{Q}^T(\tau)\mathbf{O}_t^\tau\mathbf{D}(\mathbf{x}(t), t)\mathbf{O}_\tau^t\mathbf{M}_\tau^t\mathbf{Q}(\tau) = \left[\tilde{\mathbf{O}}_t^\tau\tilde{\mathbf{D}}\left(\mathbf{y}(t), t\right)\tilde{\mathbf{O}}_\tau^t\right]\tilde{\mathbf{M}}_\tau^t,\end{aligned} \tag{63}$$

where we have used the identity $\dot{\mathbf{Q}}^T\mathbf{Q} = -\mathbf{Q}^T\dot{\mathbf{Q}}$ and the formulas from (61). Therefore, by (63), $\tilde{\mathbf{O}}_\tau^t$ is a rotational process and $\tilde{\mathbf{M}}_\tau^t$ is an irrotational family of operators. By the uniqueness of the DPD, we conclude that (62) indeed represent the unique dynamic polar decomposition of the transformed deformation gradient $\tilde{\mathbf{F}}_\tau^t$. By the relation $\tilde{\mathbf{M}}_\tau^t = \mathbf{Q}^T(\tau)\mathbf{M}_\tau^t\mathbf{Q}(\tau)$, the transformed dynamic stretch tensor $\tilde{\mathbf{M}}_\tau^t$ is related to its original counterpart $\mathbf{M}_\tau^t$ through a similarity transformation, and hence all scalar invariants of $\mathbf{M}_\tau^t$ are preserved in the new frame. We do not, however, have $\tilde{\mathbf{M}}_\tau^t = \mathbf{Q}^T(t)\mathbf{M}_\tau^t\mathbf{Q}(t)$ and hence $\mathbf{M}_\tau^t$ is not objective. Finally, rewriting the transformed deformation gradient as

$$\tilde{\mathbf{F}}_\tau^t = \mathbf{Q}^T(t)\mathbf{N}_\tau^t\mathbf{O}_\tau^t\mathbf{Q}(\tau) = \tilde{\mathbf{N}}_\tau^t\tilde{\mathbf{O}}_\tau^t, \quad \tilde{\mathbf{N}}_\tau^t = \mathbf{Q}^T(t)\mathbf{N}_\tau^t\mathbf{Q}(t), \quad \tilde{\mathbf{O}}_\tau^t = \mathbf{Q}^T(t)\mathbf{O}_\tau^t\mathbf{Q}(\tau), \tag{64}$$

and repeating the rest of the above argument for the left dynamic stretch tensor $\mathbf{N}_\tau^t$ completes the proof of statement (iv). Note that $\mathbf{N}_\tau^t$ is objective by the second formula in (64).

Appendix D. Proof of Theorem 3

To prove the first decomposition in (23), we write the rotation tensor $\mathbf{O}_\tau^t$ in the form $\mathbf{O}_\tau^t = \mathbf{\Phi}_\tau^t \mathbf{\Theta}_\tau^t$, with $\mathbf{\Phi}_\tau^t, \mathbf{\Theta}_\tau^t \in SO(3)$, $\mathbf{\Phi}_\tau^\tau = \mathbf{\Theta}_\tau^\tau = \mathbf{I}$ yet to be determined. Differentiating this factorization with respect to t gives

$$\dot{\mathbf{O}}_\tau^t = \dot{\mathbf{\Phi}}_\tau^t \mathbf{\Theta}_\tau^t + \mathbf{\Phi}_\tau^t \dot{\mathbf{\Theta}}_\tau^t. \tag{65}$$

At the same time, we also rewrite the ODE (16) defining $\mathbf{O}_\tau^t$ in the form

$$\dot{\mathbf{O}}_\tau^t = \left[\mathbf{W}(\mathbf{x}(t), t) - \bar{\mathbf{W}}(t)\right]\mathbf{\Phi}_\tau^t \mathbf{\Theta}_\tau^t + \bar{\mathbf{W}}(t)\mathbf{\Phi}_\tau^t \mathbf{\Theta}_\tau^t. \tag{66}$$

Equating the first and second terms in the right-hand sides of (65) and (66) leads to the initial value problems (24) and (25) proving the uniqueness of the first decomposition in statement (i). The relative rotation tensor is a rotational process, given that it is the fundamental matrix solution of the classical system of ODEs (24), whose skew-symmetric right-hand side has no explicit dependence on the initial time τ. By (25), the mean-rotation tensor $\mathbf{\Theta}_\tau^t$ forms a rotational operator family. However, $\mathbf{\Theta}_\tau^t$ is generally not a linear process, given the explicit dependence of the right-hand side of (25) on the initial time τ.

To prove the second decomposition of $\mathbf{O}_\tau^t$ in (23), we observe that

$$\mathbf{O}_\tau^t = \left[\mathbf{O}_t^\tau\right]^T = \left[\mathbf{\Phi}_t^\tau \mathbf{\Theta}_t^\tau\right]^T = \left(\mathbf{\Theta}_t^\tau\right)^T \left(\mathbf{\Phi}_t^\tau\right)^T = \left(\mathbf{\Theta}_t^\tau\right)^T \mathbf{\Phi}_\tau^t,$$

thus setting

$$\mathbf{\Sigma}_\tau^t = \left(\mathbf{\Theta}_t^\tau\right)^T \in SO(3), \tag{67}$$

we recover the left mean rotation tensor $\mathbf{\Sigma}_\tau^t$, as claimed. Interchanging the role of τ and t in the second equation of (69), we find that

$$\frac{d}{d\tau}\mathbf{\Theta}_t^\tau = \left[\mathbf{\Phi}_\tau^t \bar{\mathbf{W}}(\tau)\mathbf{\Phi}_t^\tau\right]\mathbf{\Theta}_t^\tau, \quad \mathbf{\Theta}_t^t = \mathbf{I}, \tag{68}$$

thus, using formula (67), we obtain

$$\left(\frac{d}{d\tau}\mathbf{\Sigma}_\tau^t\right)\mathbf{\Theta}_t^\tau + \mathbf{\Sigma}_\tau^t\left(\frac{d}{d\tau}\mathbf{\Theta}_t^\tau\right) = 0.$$

This last equation together with (68) gives the initial value problem

$$\frac{d}{d\tau}\mathbf{\Sigma}_\tau^t = -\mathbf{\Sigma}_\tau^t\left[\mathbf{\Phi}_\tau^t \bar{\mathbf{W}}(\tau)\mathbf{\Phi}_t^\tau\right], \quad \mathbf{\Sigma}_t^t = \mathbf{I}. \tag{69}$$

Taking the transpose of (69) proves the last equation in (26). Again, the uniqueness of both decompositions in (23) follows from the uniqueness of solutions to (26). Finally, $\mathbf{\Sigma}_\tau^t$ is a rotational operator family, but not a process, as discussed already for $\mathbf{\Theta}_t^\tau$.

To prove the last statement of the theorem, we first change coordinates under a general Euclidean transformation (14), and use tilde, as in the proof of Theorem 2, to denote quantities in the $\mathbf{y}$ coordinate frame. We recall from formula (61) the form of the transformed vorticity tensor

$$\tilde{\mathbf{W}}(\mathbf{y}, t) = \mathbf{Q}^T(t)\mathbf{W}(\mathbf{x}, t)\mathbf{Q}(t) - \mathbf{Q}^T(t)\dot{\mathbf{Q}}(t). \tag{70}$$

Taking the spatial mean of both sides in Eq. (70) over the body $\mathcal{B}(t)$, and noting that the transformation (14) preserves the volume of $\mathcal{B}(t)$, we obtain

$$\bar{\tilde{\mathbf{W}}}(t) = \mathbf{Q}^T(t)\bar{\mathbf{W}}(t)\mathbf{Q}(t) - \mathbf{Q}^T(t)\dot{\mathbf{Q}}(t). \tag{71}$$

Subtracting (71) from (70) gives

$$\tilde{\mathbf{W}}(\mathbf{y}, t) - \bar{\tilde{\mathbf{W}}}(t) = \mathbf{Q}^T(t)\left[\mathbf{W}(\mathbf{x}, t) - \bar{\mathbf{W}}(t)\right]\mathbf{Q}(t). \tag{72}$$

Next, using the decomposition of $\mathbf{O}_\tau^t$ from statement (i) in the original $\mathbf{x}$-frame, we factorize the transformed dynamic rotation tensor $\tilde{\mathbf{O}}_\tau^t$ obtained in Eq. (62) as

$$\tilde{\mathbf{O}}_\tau^t = \tilde{\mathbf{\Phi}}_\tau^t \tilde{\mathbf{\Theta}}_\tau^t, \quad \tilde{\mathbf{\Phi}}_\tau^t = \mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t \mathbf{P}(t), \quad \tilde{\mathbf{\Theta}}_\tau^t = \mathbf{P}^{-1}(t)\mathbf{\Theta}_\tau^t \mathbf{Q}(\tau), \tag{73}$$

with the matrix $\mathbf{P}(t)$ to be determined in a way that (73) gives the unique relative-mean rotation decomposition of $\tilde{\mathbf{O}}_\tau^t$ in the $\mathbf{y}$ coordinate frame. Both $\tilde{\mathbf{\Phi}}_\tau^t$ and $\mathbf{\Phi}_\tau^t$, as well as $\tilde{\mathbf{\Theta}}_\tau^t$ and $\mathbf{\Theta}_\tau^t$, are equal to the identity matrix at time $t = \tau$, thus by (73), $\mathbf{P}(t)$ must necessarily satisfy

$$\mathbf{P}(\tau) = \mathbf{Q}(\tau). \tag{74}$$

To determine $\mathbf{P}(t)$, we differentiate the expression for $\tilde{\mathbf{\Phi}}_\tau^t$ in (73), then use (24) and (72) to obtain

$$\begin{aligned}\dot{\tilde{\mathbf{\Phi}}}_\tau^t &= \dot{\mathbf{Q}}^T(t)\mathbf{\Phi}_\tau^t\mathbf{P}(t) + \mathbf{Q}^T(t)\dot{\mathbf{\Phi}}_\tau^t\mathbf{P}(t) + \mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t\dot{\mathbf{P}}(t), = \dot{\mathbf{Q}}^T(t)\mathbf{\Phi}_\tau^t\mathbf{P}(t) + \mathbf{Q}^T(t)\Big[\mathbf{W}(\mathbf{x},t) - \bar{\mathbf{W}}(t)\Big]\mathbf{\Phi}_\tau^t\mathbf{P}(t) + \mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t\dot{\mathbf{P}}(t)\\ &= \Big[\tilde{\mathbf{W}}(\mathbf{y},t) - \bar{\tilde{\mathbf{W}}}(t)\Big]\tilde{\mathbf{\Phi}}_\tau^t + \dot{\mathbf{Q}}^T(t)\mathbf{\Phi}_\tau^t\mathbf{P}(t) + \mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t\dot{\mathbf{P}}(t).\end{aligned} \tag{75}$$

The transformed relative rotation tensor $\tilde{\mathbf{\Phi}}_\tau^t$ is defined by the equation $\dot{\tilde{\mathbf{\Phi}}}_\tau^t = [\tilde{\mathbf{W}}(\mathbf{y},t) - \bar{\tilde{\mathbf{W}}}(t)]\tilde{\mathbf{\Phi}}_\tau^t$ in the $\mathbf{y}$ coordinates, therefore (75) implies

$$\dot{\mathbf{Q}}^T(t)\mathbf{\Phi}_\tau^t\mathbf{P}(t) + \mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t\dot{\mathbf{P}}(t) = \mathbf{0},$$

or, equivalently,

$$\dot{\mathbf{P}}(t) = \mathbf{\Phi}_t^\tau\dot{\mathbf{Q}}(t)\mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t\mathbf{P}(t). \tag{76}$$

This linear system of differential equations has a skew-symmetric coefficient matrix, therefore $\mathbf{P}(t)$ is a proper orthogonal matrix, and hence

$$\mathbf{P}^{-1}(t) = \mathbf{P}^T(t). \tag{77}$$

For two-dimensional deformations, the skew-symmetric tensor $\dot{\mathbf{Q}}^T(t)\mathbf{Q}(t)$ is always a scalar multiple of a rotation tensor, and hence commutes with any other two-dimensional rotation tensor. Consequently, Eq. (76) can be re-written as

$$\dot{\mathbf{Q}}^T(t)\mathbf{\Phi}_\tau^t\mathbf{P}(t) + \mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t\dot{\mathbf{P}}(t) = \mathbf{\Phi}_\tau^t\Big[\dot{\mathbf{Q}}^T(t)\mathbf{P}(t) + \mathbf{Q}^T(t)\dot{\mathbf{P}}(t)\Big] = \mathbf{\Phi}_\tau^t\frac{d}{dt}\Big[\mathbf{Q}^T(t)\mathbf{P}(t)\Big] = \mathbf{0}, \tag{78}$$

implying that $\mathbf{Q}^T(t)\mathbf{P}(t)$ is a constant rotation. Therefore, by (74), we conclude from (78) for two-dimensional deformations that $\mathbf{P}(t) \equiv \mathbf{Q}(t)$. Thus formula (73) gives

$$\tilde{\mathbf{\Phi}}_\tau^t = \mathbf{Q}^T(t)\mathbf{\Phi}_\tau^t\mathbf{Q}(t),$$

proving statement (ii) of Theorem 3.

Appendix E. Fiber-averaged angular velocity of a rigid body

Consider a perfectly rigid body $\mathcal{R}(t)$, with a well-defined angular velocity vector $\boldsymbol{\nu}_{rigid}(t)$ (see Fig. 9a).

We seek to average $\boldsymbol{\nu}_{\min}(\mathbf{x}, t; \mathbf{e})$ over all vectors $\mathbf{e}(\phi, \psi)$ taken from the spherically parametrized unit sphere $S_{\mathbf{x}}^2$. Note the cancellation of the averaged vector in radial directions normal to $\boldsymbol{\nu}_{rigid}(t)$ due to the circular symmetry shown in Fig. 9b.

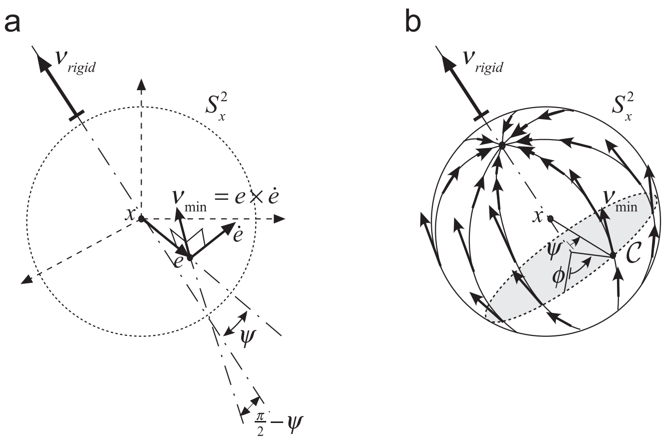

Fig. 9. (a) The geometry of the minimal admissible angular velocity $\boldsymbol{\nu}_{\min}(\mathbf{x}, t; \mathbf{e})$ at a point $\mathbf{x}$ and the actual angular velocity $\boldsymbol{\nu}_{rigid}(t)$ in case of an ideal rigid body motion. (b) The radial components of the vector field $\boldsymbol{\nu}_{\min}(\mathbf{x}, t; \mathbf{e})$ along the circle $\mathcal{C}$ average out to zero, and hence only the components normal to the plane of $\mathcal{C}$ contribute to the average $\langle\boldsymbol{\nu}_{\min}(\mathbf{x}, t, \mathbf{e})\rangle_{\mathbf{e}\in S_{\mathbf{x}}^2}$.

Further note from Fig. 9a that the projection of $\boldsymbol{\nu}_{\min}(\mathbf{x}, t; \mathbf{e})$ on the axis of rotation defined by $\boldsymbol{\nu}_{rigid}(t)$ is

$$\left|\boldsymbol{\nu}_{\min}(\mathbf{x}, t; \mathbf{e})\right| \sin\psi = \left|\boldsymbol{\nu}_{rigid}(t)\right| \sin^2\psi.$$

From these considerations, we obtain that the average of the vector field $\boldsymbol{\nu}_{\min}(\mathbf{x}, t; \mathbf{e})$ over $S^2_{\mathbf{x}}$ is

$$\langle\boldsymbol{\nu}_{\min}(\mathbf{x}, t, \mathbf{e})\rangle_{\mathbf{e}\in S^2_{\mathbf{x}}} = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{\pi}\int_0^{\pi}\boldsymbol{\nu}_{\min}\Big(\mathbf{x}, t; \mathbf{e}(\phi, \psi)\Big)\,d\psi\right]d\phi = \frac{1}{\pi}\int_0^{\pi}\boldsymbol{\nu}_{rigid}(t)\sin^2\psi\, d\psi \tag{79}$$

$$\langle\boldsymbol{\nu}_{\min}(\mathbf{x}, t, \mathbf{e})\rangle_{\mathbf{e}\in S^2_{\mathbf{x}}} = \frac{1}{2}\boldsymbol{\nu}_{rigid}(t). \tag{80}$$

Therefore, for the material fiber-averaged angular velocity $\boldsymbol{\nu}(t, \mathbf{x})$ defined in (29), we obtain

$$\boldsymbol{\nu}(t, \mathbf{x}) = \boldsymbol{\nu}_{rigid}(t) \tag{81}$$

in the case of a perfectly rigid body.

Appendix F. Proof of Proposition 1

In order to calculate the fiber-averaged angular velocity $\boldsymbol{\nu}(\mathbf{x}, t)$ defined in (29), we first need a general expression for the derivative $\dot{\mathbf{e}}(t)$ for an arbitrary unit vector $\mathbf{e}(t)$ tangent to an evolving material fiber. Differentiating the definition (27) of $\mathbf{e}(t)$ in time, and using Example 1, we obtain

$$\dot{\mathbf{e}} = \frac{\nabla\mathbf{v}(\mathbf{x}(t), t)\mathbf{F}^t_\tau\mathbf{e}(\tau) - \mathbf{F}^t_\tau\mathbf{e}(\tau)\frac{\left\langle\nabla\mathbf{v}(\mathbf{x}(t), t)\mathbf{F}^t_\tau\mathbf{e}(\tau), \mathbf{F}^t_\tau\mathbf{e}(\tau)\right\rangle}{\left|\mathbf{F}^t_\tau\mathbf{e}(\tau)\right|}}{\left|\mathbf{F}^t_\tau\mathbf{e}(\tau)\right|^2}.$$

Setting τ equal to t in this last equation and using formula (4) gives

$$\dot{\mathbf{e}} = \big[\mathbf{W} + \mathbf{D} - \langle\mathbf{e}, \mathbf{D}\mathbf{e}\rangle\mathbf{I}\big]\mathbf{e}. \tag{82}$$

This equation is broadly known (see, e.g., Chadwick, 1976), and has only been re-derived here for completeness and notational consistence.

Taking the cross product of both sides with $\mathbf{e}$ and using the definitions (28) and (29), we obtain from (82) the general expression

$$\boldsymbol{\nu} = 2\langle\mathbf{e}\times\mathbf{W}\mathbf{e}\rangle_{\mathbf{e}\in S^2_{\mathbf{x}}} + 2\langle\mathbf{e}\times\mathbf{D}\mathbf{e}\rangle_{\mathbf{e}\in S^2_{\mathbf{x}}} = \tfrac{1}{2}\boldsymbol{\omega} + 2\langle\mathbf{e}\times\mathbf{D}\mathbf{e}\rangle_{\mathbf{e}\in S^2_{\mathbf{x}}}, \tag{83}$$

where we have applied the relationship (81) to the rigid body rotation generated by the angular velocity tensor $\mathbf{W}(\mathbf{x}, t)$ with angular velocity $\boldsymbol{\nu}_{rigid}(t) = \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}, t)$.

Let $\{\mathbf{b}_i(\mathbf{x}, t)\}_{i=1}^3$ denote a positively oriented orthonormal basis for the rate-of-strain tensor $\mathbf{D}(\mathbf{x}, t)$, with corresponding eigenvalues $\sigma_1(\mathbf{x}, t) \le \sigma_2(\mathbf{x}, t) \le \sigma_3(\mathbf{x}, t)$. In this basis, the unit vector $\mathbf{e}$ has the classic spherical coordinate representation (cf. Fig. 9b)

$$\mathbf{e} = \cos\psi\cos\phi\mathbf{b}_1 + \cos\psi\sin\phi\mathbf{b}_2 + \sin\psi\mathbf{b}_3,$$

from which we obtain

$$\mathbf{e}\times\mathbf{D}\mathbf{e} = \tfrac{1}{2}\Big(\sigma_3 - \sigma_2\Big)\sin 2\psi\sin\phi\mathbf{b}_1 + \tfrac{1}{2}\Big(\sigma_2 - \sigma_3\Big)\sin 2\psi\cos\phi\mathbf{b}_2 + \tfrac{1}{2}\Big(\sigma_2 - \sigma_1\Big)\sin 2\phi\cos^2\phi\mathbf{b}_3.$$

This shows that

$$\langle\mathbf{e}\times\mathbf{D}\mathbf{e}\rangle_{\mathbf{e}\in S^2_{\mathbf{x}}} = 0,$$

thus formula (83) simplifies to $\boldsymbol{\nu}(\mathbf{x}, t) = \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}, t)$, proving the statement of Proposition 1.

Appendix G. Proof of Theorem 4

By Theorem 2, we have

$$\dot{\mathbf{O}}^t_\tau\big[\mathbf{O}^t_\tau\big]^T\mathbf{e} = \mathbf{W}(\mathbf{x}(t), t)\mathbf{e} = -\tfrac{1}{2}\boldsymbol{\omega}(\mathbf{x}(t), t)\times\mathbf{e},$$

$$\dot{\boldsymbol{\Phi}}^t_\tau\big[\boldsymbol{\Phi}^t_\tau\big]^T\mathbf{e} = \Big[\mathbf{W}(\mathbf{x}(t), t) - \bar{\mathbf{W}}(t)\Big]\mathbf{e} = -\tfrac{1}{2}\Big[\boldsymbol{\omega}(\mathbf{x}(t), t) - \bar{\boldsymbol{\omega}}(t)\Big]\times\mathbf{e}.$$

Therefore,

$$\varphi_\tau^t(\mathbf{x}_\tau;\,\mathbf{g}) = -\frac{1}{2}\int_\tau^t \omega(\mathbf{x}(s),\,s)\cdot\mathbf{g}(\mathbf{x}(s),\,s)\,ds = -\frac{1}{2}\int_\sigma^t \omega(\mathbf{x}(s),\,s)\cdot\mathbf{g}(\mathbf{x}(s),\,s)\,ds - \frac{1}{2}\int_\tau^\sigma \omega(\mathbf{x}(s),\,s)\cdot\mathbf{g}(\mathbf{x}(s),\,s)\,ds = \varphi_\sigma^t(\mathbf{x}_\sigma;\,\mathbf{g}) + \varphi_\tau^\sigma(\mathbf{x}_\tau;\,\mathbf{g}),$$

and similarly,

$$\phi_\tau^t(\mathbf{x}_\tau;\,\mathbf{g}) = \phi_\sigma^t(\mathbf{x}_\sigma;\,\mathbf{g}) + \phi_\tau^\sigma(\mathbf{x}_\tau;\,\mathbf{g}), \tag{84}$$

proving the dynamical consistency of φ_τ^t and ϕ_τ^t, and completing the proof of statement (i) of the theorem.

To complete the proof of statement (ii), we must prove the objectivity of the relative dynamic rotation $\varphi_\tau^t(\mathbf{x}_\tau;\,\mathbf{g})$ under a Euclidean frame change of the form (14). As is well known (see., e.g., Truesdell and Rajagopal, 2009), the transformed vorticity $\tilde{\omega}(\mathbf{y},\,t)$ is related to the original vorticity $\omega(\mathbf{x},\,t)$ through the formula

$$\omega(\mathbf{x},\,t) = \mathbf{Q}(t)\tilde{\omega}(\mathbf{y},\,t) + \dot{\mathbf{q}}(t), \tag{85}$$

where the vector $\dot{\mathbf{q}}$ is defined via the identity $\frac{1}{2}\dot{\mathbf{q}}\times\tilde{\mathbf{a}} = \dot{\mathbf{Q}}\mathbf{Q}^T\tilde{\mathbf{a}}$ for all $\tilde{\mathbf{a}}\in\mathbb{R}^3$, accounting for the additional vorticity introduced by the frame change. Taking the spatial means of both sides in (85) over the evolving continuum $\mathcal{B}(t)$ gives

$$\bar{\omega}(t) = \mathbf{Q}(t)\bar{\tilde{\omega}}(t) + \dot{\mathbf{q}}(t), \tag{86}$$

because the volume of $\mathcal{B}(t)$ remains constant under the Euclidean frame change (14). Subtracting (86) from (85), we obtain that

$$\omega(\mathbf{x},\,t) - \bar{\omega}(t) = \mathbf{Q}(t)\left[\tilde{\omega}(\mathbf{y},\,t) - \bar{\tilde{\omega}}(t)\right]. \tag{87}$$

The vector field $\mathbf{g}(\mathbf{x},\,t)$ is transformed under the frame change as

$$\tilde{\mathbf{g}}(\mathbf{y},\,t) = \mathbf{Q}^T(t)\mathbf{g}(\mathbf{x},\,t). \tag{88}$$

We observe that in the rotating frame, $\tilde{\mathbf{g}}$ is necessarily time-dependent, even if $\mathbf{g}$ was originally chosen as a time-independent constant direction. Using the formulas (87) and (88), we obtain that

$$\begin{aligned}\phi_\tau^t(\mathbf{x}_\tau;\,\mathbf{g}) &= -\frac{1}{2}\int_\tau^t \left[\omega(\mathbf{x}(s),\,s) - \bar{\omega}(s)\right]\cdot\mathbf{g}(\mathbf{x}(s),\,s)\,ds = -\frac{1}{2}\int_\tau^t \mathbf{Q}(s)\left[\tilde{\omega}(\mathbf{y}(s),\,s) - \bar{\tilde{\omega}}(s)\right]\cdot\mathbf{Q}(s)\mathbf{g}(\mathbf{x}(s),\,s)\,ds,\\ &= -\frac{1}{2}\int_\tau^t \left[\tilde{\omega}(\mathbf{y}(s),\,s) - \bar{\tilde{\omega}}(s)\right]\cdot\tilde{\mathbf{g}}(\mathbf{x}(s),\,s)\,ds = \phi_\tau^t(\mathbf{y}_\tau;\,\tilde{\mathbf{g}}),\end{aligned}$$

which completes the proof of statement (ii) of the theorem. Statement (iii) then follows by setting $\mathbf{g} = -(\omega - \bar{\omega})/|\omega - \bar{\omega}|$.

Remark 12. The argument leading to (84) would *not* work for the polar rotation angle. Indeed, the angular velocity $\dot{\mathbf{q}}_{polar}(t,\,\tau)$ of the polar rotations inherits explicit dependence on τ from $\dot{\mathbf{R}}_\tau^t(\mathbf{R}_\tau^t)^T$. As a consequence, for the polar rotation angle defined as

$$\gamma_\tau^t(\mathbf{x}_\tau;\,\mathbf{g}) = \int_\tau^t \dot{\mathbf{q}}_{polar}(s,\,\tau)\cdot\mathbf{g}(\mathbf{x}(s),\,s)\,ds,$$

we obtain

$$\begin{aligned}\gamma_\tau^t(\mathbf{x}_\tau;\,\mathbf{g}) &= \int_\sigma^t \dot{\mathbf{q}}_{polar}(s,\,\tau)\cdot\mathbf{g}(\mathbf{x}(s),\,s)\,ds + \int_\tau^\sigma \dot{\mathbf{q}}_{polar}(s,\,\tau)\cdot\mathbf{g}(\mathbf{x}(s),\,s)\,ds = \int_\sigma^t \dot{\mathbf{q}}_{polar}(s,\,\tau)\cdot\mathbf{g}(\mathbf{x}(s),\,s)\,ds\\ &\quad + \gamma_\tau^\sigma(\mathbf{x}_\tau;\,\mathbf{g}) \neq \gamma_\sigma^t(\mathbf{x}_\sigma;\,\mathbf{g}) + \gamma_\tau^\sigma(\mathbf{x}_\tau;\,\mathbf{g}),\end{aligned}$$

because we generally have $\dot{\mathbf{q}}_{polar}(s,\,\tau) \neq \dot{\mathbf{q}}_{polar}(s,\,\sigma)$ for $\tau \neq \sigma$.

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