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Periodic orbit: motion that repeats identically after a finite period of time
The Role of Periodic Orbits in Dynamics

**Periodic orbit**: motion that repeats identically after a finite period of time

Dynamical models

\[ \dot{x} = f(x) + \varepsilon g(x, t, \varepsilon), \quad 0 < \varepsilon \ll 1 \]

Can we predict existence and stability of periodic orbits of the perturbed system starting from those of the conservative system?
Consider $N$ coupled, periodically forced and damped oscillators for arbitrary motion amplitude. Some nonlinear phenomena
Motivations: the case of Mechanical Vibrations

Why would practitioners capitalize on analytical tools?

Computational speed-up for studies of the effect forcing & damping terms

Find isolas: identification is challenging from numerical continuation

Validate and extend experimental routines using the phase-lag quadrature

Available methods:
- Asymptotic expansions from an equilibrium
- LSM & SSM
- Energy-type arguments
- Melnikov methods

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Overview of the Classic Melnikov Method

- Or better, the Poincaré-Arnold-Melnikov method (1963)

- Originally: \( \dot{x} = JDH(x) + \epsilon g(x, t), \quad g(x, t + T) = g(x, t), \quad x \in \mathbb{R}^2 \)

\( \epsilon = 0 \)

If \( \mathcal{M}(s) \) has a transverse zero

Homoclinic Tangle

Chaotic attractor

Overview of the Classic Melnikov Method

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- Originally: \( \dot{x} = JDH(x) + \epsilon g(x, t), \quad g(x, t + T) = g(x, t), \quad x \in \mathbb{R}^2 \)

- Extended to integrable, low-dimensional hamiltonian systems

\[ \epsilon = 0 \]

\[ 0 < \epsilon \ll 1 \]

... not the case for structural problems in practical applications.

Mechanical system with $n$ degrees of freedom, whose conservative limit is defined by the Lagrangian $q \in \mathbb{R}^n$

$$L(q, \dot{q}) = \frac{1}{2} \langle \dot{q}, M(q)\dot{q} \rangle + \langle \dot{q}, G_1(q) \rangle + G_0(q) - V(q)$$

and its energy reads: $H(q, \dot{q}) = \frac{1}{2} \langle \dot{q}, M(q)\dot{q} \rangle - G_0(q) + V(q)$

Collecting any dissipative or active force in the small, time-periodic Lagrangian component $Q$ with frequency $\Omega$, the equations of motion are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \varepsilon Q(q, \dot{q}, t; \Omega, \varepsilon), \quad 0 \leq \varepsilon \ll 1$$
Periodic Orbits of Conservative Systems

- Present in almost all energy levels
- Generically, they exist in families \( \mathcal{NNM} \)
- Not structurally stable

Types of orbits in 1 parameter families:

- Regular periodic orbits

\[ h', \omega' \neq 0 \]

Periodic Orbits of Conservative Systems

- Present in almost all energy levels
- Generically, they exist in families \( NNMs \)
- Not structurally stable
- Types of orbits in 1 parameter families:
  - Regular periodic orbits
  - Folding periodic orbits

\[
\begin{align*}
  h' &\neq 0 \\
  \omega' &\neq 0 \\
  h' &\neq 0 \\
  \omega' &\neq 0 \\
\end{align*}
\]

\( \text{Muñoz-Almaraz, Freire, Galán, Doedel & Vanderbauwhede (2003)} \)
Periodic Orbits of Conservative Systems

- Present in almost all energy levels
- Generically, they exist in families \( \mathcal{NNMs} \)
- Not structurally stable
- Types of orbits in 1 parameter families:
  - Regular periodic orbits
  - Folding periodic orbits
  - Critical cases

Perturbation from the Conservative limit

- We look for subharmonic orbits of order $l \in \mathbb{N}$ in the forced-damped system.

- Pick a regular orbit $q_0(t)$ with period $\tau_0$ of the conservative backbone curve at $(\omega_0, h_0)$.

- Set $q(t) = q_0(t + s) + O(\varepsilon)$ as well as a resonance constraint to fix $\Omega$, either

  (a) **Exact resonance**: $m\Omega = l\omega_0$ with $m, l$ being relatively prime integers, or

  (b) **Near resonance**: $m\Omega = l\omega_0 + O(\varepsilon)$ and $H(q(0), \dot{q}(0)) = h_0$
Main Result: Existence

- Define the Melnikov function

\[ \mathcal{M}_{m:l}(s) = \int_{0}^{m\tau_{0}} \langle \dot{q}_{0}(t + s), Q(q_{0}(t + s), \dot{q}_{0}(t + s), t; l\omega_{0}/m, 0) \rangle dt \]

- If \( \mathcal{M}_{m:l}(s_{0}) = 0 \) \& \( \mathcal{M}'_{m:l}(s_{0}) \neq 0 \), the conservative limit \( q_{0}(s_{0} + t) \) persists for the weakly damped, periodically forced system

\[ \varepsilon = 0 \]

\[ 0 < \varepsilon \ll 1 \]

Main Result: Existence

- Define the Melnikov function

\[ M_{m:l}(s) = \int_0^{m\tau_0} \langle \dot{q}_0(t + s), Q(q_0(t + s), \dot{q}_0(t + s), t; l\omega_0/m, 0) \rangle dt \]

- If \( M_{m:l}(s_0) = 0 \) & \( M'_{m:l}(s_0) \neq 0 \), but the backbone curve has a fold at \((\omega_0, h_0)\), then \( q_0(s_0 + t) \) persists in any direction transverse to the folding direction

Main Result: Existence

- Define the Melnikov function

\[ M_{m:l}(s) = \int_0^{m\tau_0} \langle \dot{q}_0(t+s), Q(q_0(t+s), \dot{q}_0(t+s), t; l\omega_0/m,0) \rangle dt \]

- If \[ |M_{m:l}(s)| > 0 \], the conservative limit does not persist for the weakly damped, periodically forced system

- If the conservative periodic orbit \( q_0(t) \) is a critical orbit, the Melnikov function alone is not sufficient to predict the fate of the fate of \( q_0(t) \)

Towards Stability

- Write the system in Hamiltonian form

\[ p = \frac{\partial L}{\partial \dot{q}} = M(q)\dot{q} + G_1(q) \]
\[ x = (q, p) \]

- For the stability of a periodic orbit with period \( l\Omega \) we need to study the eigenvalues of the monodromy matrix \( X(l\Omega) \in \mathbb{R}^{n \times n} \)

\[ \dot{X} = JD^2H(x(t))X + \varepsilon D_xg(x(t), t; \Omega, \varepsilon) \]
\[ X(0) = I \]
\[ \Pi_0 = X_0(m\tau_0) \text{ is the solution at } \varepsilon = 0 \]

Towards Stability

The conservative limit has always at least 2 eigenvalues of $\Pi_0$ equal to $+1$. Possible configurations of the unperturbed spectrum
Towards Stability

- We consider a conservative limit that satisfies

- For each of the \( n \) couples of eigenvalues, define the nonlinear damping rate

\[
C_i = -\frac{1}{m\tau_0} \int_0^{m\tau_0} \text{trace} \left( S_i X_0^{-1}(t) D_x g(x_0(t), t; l\omega_0/m, 0) X_0(t) R_i \right) dt
\]

\( \text{span}(R_i) \) is the \( i \)-th eigenspace, \( S_i = (R_i^T J R_i)^{-1} R_i^T J \) and \( \dot{X}_0 = JD^2H(x_0(t))X_0, \ X_0(0) = I \)
Main Result: Stability

The forced-damped periodic orbit is **unstable** if

\[ \mathcal{M}'_{m:l}(s_0)\omega'_0 < 0 \quad \text{or} \quad \exists \, i \in \{1,...,n\} : C_i < 0 \]

The forced-damped periodic orbit is **asymptotically stable** if

\[ \mathcal{M}'_{m:l}(s_0)\omega'_0 > 0 \quad \text{and} \quad C_i > 0 \ \forall \, i \in \{1,...,n\} \]

Connection with Experimental Observations

- Assume that the nonlinear damping rates are positive

These predictions are obtained without any simulation of the forced-damped system.
Remarks

- The formula for the nonlinear damping rate is complex.

\[
C_i = - \frac{1}{m\tau_0} \int_0^{m\tau_0} \text{trace}\left( S_i X_0^{-1}(t) D_x g(x_0(t), t; l\omega_0/m, 0) X_0(t) R_i \right) dt
\]

- For \( n = 1 \), \( C_1 = - \frac{1}{m\tau_0} \int_0^{m\tau_0} \text{trace}\left( D_x g(x_0(t), t; l\omega_0/m, 0) \right) dt \)

- For \( Q = F(t) - \alpha M(q)p \), then \( C_i = \alpha \ \forall \ i \in \{1, \ldots, n\} \)

- Instability conditions can be formulated for other cases
Example: Subharmonics in a Gyro

\[ m_b \ddot{q} + 2G \dot{q} - m_b \Omega^2 q + DV(q) = \hat{Q}(q, \dot{q}, t) \]

\[ G = m_b \begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix}, \]

\[ V(q) = \frac{1}{2} \sum_{j=1}^{4} k_j (l_j(x, y) - l_0)^2, \]

\[ l_{1,3}(x, y) = \sqrt{(l_0 \pm x)^2 + y^2}, \]

\[ l_{2,4}(x, y) = \sqrt{x^2 + (l_0 \pm y)^2}, \]

\[ \Omega = 0.942, \ l_0 = 1, \ k_1 = 1, \ k_2 = 4.08, \ k_3 = 1.37, \ k_4 = 2.51 \]
Example: Subharmonics in a Gyro

\[ m_b \ddot{q} + 2G \dot{q} - m_b \Omega^2 q + DV(q) = \hat{Q}(q, \dot{q}, t) \]

\[ \hat{Q}(q, \dot{q}, t) = \varepsilon \left( Q_{d,\alpha}(q, \dot{q}) + Q_{d,\beta}(q, \dot{q}) + Q_f(t) \right) \]

- Damping linearly depending on the absolute velocities the mass \( m_b \) (e.g. air damping) \( \varepsilon Q_{d,\alpha}(q, \dot{q}) = -\varepsilon \alpha m_b (\dot{q} + m_b^{-1} Gq) \);

- Stiffness-proportional damping for the spring-damper elements, i.e. \( c_j = \varepsilon \beta k_j \) for \( j = 1, \ldots, 4 \) and \( \varepsilon Q_{d,\beta}(q, \dot{q}) = -\varepsilon \beta C(q) \dot{q} \),

\[
C(q) = \sum_{j=1}^{4} k_j \begin{bmatrix} \left( \partial_x l_j(x, y) \right)^2 & \partial_x l_j(x, y) \partial_y l_j(x, y) \\ \partial_x l_j(x, y) \partial_y l_j(x, y) & \left( \partial_y l_j(x, y) \right)^2 \end{bmatrix}
\]
Example: Subharmonics in a Gyro

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- Mono-harmonic forcing of frequency \( l\Omega \)

\[ \varepsilon Q_f(t) = \varepsilon \begin{pmatrix} \cos(l\Omega t) \\ -\sin(l\Omega t) \end{pmatrix}, \quad l \in \mathbb{N}. \]
Example: Subharmonics in a Gyro

- Equations of motion in Hamiltonian form
  \[
  \dot{q} = -Gq + p, \\
  \dot{p} = -DV(q) - Gp + \varepsilon\left(Q_f(t) - \alpha p - \beta C(q)(p - Gq)\right).
  \]

- Conservative limit:
  - Linearized frequencies at the equilibrium 0.92513 and 3.1431
  - Focus on the first NNM
Example: Subharmonics in a Gyro

- Equations of motion in Hamiltonian form
  \[ \dot{q} = -Gq + p, \]
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- Conservative limit:
  - Linearized frequencies at the equilibrium
    0.92513 and 3.1431
  - Focus on the first NNM
  - Set \( l = 3 \)
Example: Subharmonics in a Gyro

- Analysis of the two separate damping mechanisms

- The Melnikov function is $\mathcal{M}_{1:3}(s) = 1.4402 \cos(3\Omega s) - 1.1553$ for $\alpha = 0.76376$, $\beta = 0$ and $\alpha = 0$, $\beta = 0.32$

Simulations with COCO at $\varepsilon = 0.01$
We have a framework to study eventual **singular behaviors** when varying a parameter $\kappa$.

We focus on quadratic zeros, defined as:

\[
\mathcal{M}_{m:l}(s_0, \kappa_0) = D_s \mathcal{M}_{m:l}(s_0, \kappa_0) = 0 \quad D_{ss} \mathcal{M}_{m:l}(s_0, \kappa_0) \neq 0
\]

The simplest case is the one of **limit point** (codim. 0)

\[
D_{\kappa} \mathcal{M}_{m:l}(s_0, \kappa_0) \neq 0
\]

Detection of maximal responses.
A Zoo of Bifurcations

Defining conditions: \[ \mathcal{M}_{m:l}(s_0, \omega_0) = D_s \mathcal{M}_{m:l}(s_0, \omega_0) = D_\omega \mathcal{M}_{m:l}(s_0, \omega_0) = 0 \]

### Isola Center
- **No solution**
- **Single solution**
- **Closed isola**

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### Simple Bifurcation
- **Bottleneck**
- **Node singularity**
- **Break-up**

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Example: Parametric Forcing and Isolas

\[ \dot{q} = p, \]
\[ \dot{p}_1 = -k(q_1 - q_2) - k/3q_1 - aq_1^2 - bq_1^3 - \varepsilon \alpha p_1, \]
\[ \dot{p}_2 = -k(q_2 - q_1) - k(q_2 - q_3) - \varepsilon \alpha p_2, \]
\[ \dot{p}_3 = -k(q_3 - q_2) + \varepsilon (q_3 f(t; \Omega) - \alpha p_3), \]

\[ q, p \in \mathbb{R}^3 \]
\[ k = 1, \]
\[ a = -0.5, \]
\[ b = 1, \]

Approximation of a square-wave up to the 6-th harmonic

\[ f(t; \Omega) = \frac{4}{\pi} \sum_{j=1}^{3} \frac{1}{2j - 1} \sin((2j - 1)\Omega t), \]
Example: Parametric Forcing and Isolas

- Assume a 1:1 resonance and evaluate $\mathcal{M}_{1:1}$ along the family

$\frac{\pi}{2}$ $\pi$ $\frac{3\pi}{2}$ $\frac{5\pi}{2}$

Distance of the frequency response from the backbone curve

Loci of zeros of the Melnikov function as function of the frequency and the phase shift of the orbit

$\epsilon = 0.0025$
Experimental Applications

Testing for backbone curve extraction

**Phase-lag quadrature criterion:** forcing is exactly balancing the damping if the phase lag between forcing and measurement is 90°

This was show for: synchronous motions and linear damping

Experimental Applications

Testing for backbone curve extraction

Phase-lag quadrature criterion: forcing is exactly balancing the damping if the phase lag between forcing and measurement is 90°

Using our Melnikov analysis one can show that this is valid, when forcing is mono-harmonic, for arbitrary motions and damping shapes, but only for co-located measurements!
Summary and Future Directions

- An energy balance is sufficient to establish the existence of weakly forced-damped vibrations from the conservative limit, while their stability can be studied with nonlinear damping rates.

- These analytical results matches with available ones for single-degree-of-freedom oscillators and with real life observations.

- Our approach offers significant advantages both for numerical and experimental studies.

- What about the survival of tori?